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BOUNDEDNESS OF MARCINKIEWICZ INTEGRALS ON HERZ SPACES WITH VARIABLE EXPONENT

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ABSTRACT. In this paper, the authors obtain some boundedness for Marcinkiewicz integrals and their commutators on Herz spaces with variable exponent.

1. Introduction

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The L^p spaces with variable exponent are a special case of Musielak-Orlicz spaces.

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For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by $L^{p(\cdot)}_{\rm loc}(\Omega) := \{f : f \in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be set of $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$. In addition, we denote the Lebesgue measure and characteristic function of measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A respectively.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. ([3]) *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_\Omega |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. ([1]) *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_1 and δ_2 are the same as in Lemma 1.2.

Lemma 1.3. ([1]) *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Firstly we recall the definition of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1. *(*[1]*)* Let $\alpha \in \mathbb{R}, 0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

Suppose that S^{n-1} denote the unit sphere in $\mathbb{R}^n (n \ge 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta \le 1$ be homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein [5] introduced the Marcinkiewicz integral related to the Littlewood-Paley g function on $\mathbb{R}^n$ as follows

$$\mu(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It was shown that μ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator generated by the Marcinkiewicz integral μ and b is defined by

$$[b,\mu](f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes

Lemma 1.4. ([2]) *Let k be a positive integer and B be a ball in R^n. Then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$,*

$$\frac{1}{C}\|b\|_*^k \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*^k,$$

$$\|(b - b_{B_i})^k \chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(j-i)^k \|b\|_*^k \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

There are two lemmas for μ and $[b,\mu]$ respectively.

Lemma 1.5. ([6]) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant C such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|\mu(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 1.6. ([6]) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|[b, \mu](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Inspired by [1-6], we obtain some boundedness for Marcinkiewicz integrals and their commutators on Herz spaces with variable exponent.

2. Boundedness of Marcinkiewicz integral operators

In this section we will prove the boundedness on Herz spaces with variable exponent for Marcinkiewicz integral operators μ.

Theorem 2.1. *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < p \leq \infty$ and $-n\delta_1 < \alpha < n\delta_2$. Then μ is bounded on $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Proof It suffices to prove the homogeneous case. The non-homogeneous case can be proved in the same way. We suppose $0 < p < \infty$, since the proof of the case $p = \infty$ is easier. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and we write $f(x) = \sum_{j=-\infty}^{\infty} f\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$
\begin{aligned}
\|\mu(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\mu(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|\mu(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\|\mu(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|\mu(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&=: CE_1 + CE_2 + CE_3.
\end{aligned} \tag{2.1}
$$

By Lemma 1.5, we know that μ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$. So we have

$$
E_2 \le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
$$

Now we estimate E_1. We consider

$$
\begin{aligned}
|\mu(f_j)(x)| &\le \left(\int_0^{|x|}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\quad + \left(\int_{|x|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&=: F_1 + F_2.
\end{aligned}
$$

Note that $x \in A_k$, $y \in A_j$ and $j \le k-2$. So we know that $|x-y| \sim |x|$, and by mean value theorem we have

$$
\left|\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\right| \le \frac{|y|}{|x-y|^3}. \tag{2.2}
$$

Since Ω is bounded, by (2.2), the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned} F_1 &\le C \int_{\mathbb{R}^n} \frac{|f_j(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|}^{|x|} \frac{dt}{t^3} \right)^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|f_j(y)|}{|x-y|^{n-1}} \left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right|^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|f_j(y)|}{|x-y|^{n-1}} \frac{|y|^{1/2}}{|x-y|^{3/2}} dy \\ &\le C \frac{2^{j/2}}{|x|^{n+1/2}} \int_{A_j} |f(y)| dy \\ &\le C 2^{(j-k)/2} 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Similarly, we consider F_2. Noting that $|x-y| \sim |x|$, by the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned} F_2 &\le C \int_{\mathbb{R}^n} \frac{|f_j(y)|}{|x-y|^{n-1}} \left(\int_{|x|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|f_j(y)|}{|x-y|^n} dy \\ &\le C 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

So we have

$$|\mu(f_j)(x)| \le C 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 1.2 and Lemma 1.3 we have

$$\begin{aligned} \|\mu(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C 2^{-kn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \left(|B_k| \|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)} \right) \\ &\le C \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\ &\le C 2^{(j-k)n\delta_2} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} E_1 &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &= C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}. \end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} E_1 &\le C\Bigg\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\alpha)p/2}\right) \\ &\quad \times\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\alpha)p'/2}\right)^{p/p'}\Bigg\}^{1/p} \\ &\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\ &= C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\ &\le C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &= C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{2.3}$$

If $0 < p \le 1$, then we have

$$\begin{aligned} E_1 &\le C\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &= C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p}\right)\right\}^{1/p} \\ &\le C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{2.4}$$

Let us now estimate E_3. Note that $x \in A_k$, $y \in A_j$ and $j \geq k+2$, so we have $|x-y| \sim |y|$. We consider

$$\begin{aligned}|\mu(f_j)(x)| &\leq \left(\int_0^{|y|}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\&\quad+\left(\int_{|y|}^{\infty}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\&=: G_1+G_2.\end{aligned}$$

Similar to the estimate for F_1, we get

$$G_1 \leq C2^{(k-j)/2}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

Similar to the estimate for F_2, we get

$$G_2 \leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So we have

$$|\mu(f_j)(x)| \leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 1.2 and Lemma 1.3 we have

$$\begin{aligned}\|\mu(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(|B_j|\|\chi_{B_j}\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leq C\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\&\leq C2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}E_3 &\leq C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\&=C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.\end{aligned}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1+\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
E_3 &\le C\Bigg\{\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\Bigg)\\
&\quad\times\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\alpha)p'/2}\Bigg)^{p/p'}\Bigg\}^{1/p}\\
&\le C\Bigg\{\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\Bigg)\Bigg\}^{1/p}\\
&= C\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p/2}\Bigg)\Bigg\}^{1/p}\\
&\le C\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/p}\\
&= C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.5}
$$

If $0<p\le 1$, then we have

$$
\begin{aligned}
E_3 &\le C\Bigg\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha p}2^{(k-j)(n\delta_1+\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/p}\\
&= C\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p}\Bigg)\Bigg\}^{1/p}\\
&\le C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.6}
$$

Therefore, by (2.1), (2.3)-(2.6) we complete the proof of Theorem 2.1.

3. Boundedness of the commutators of Marcinkiewicz integral operators

In this section we will prove the boundedness on Herz spaces with variable exponent for the commutators of Marcinkiewicz integral operators $[b,\mu]$.

Theorem 3.1. *Suppose* $b\in \mathrm{BMO}(\mathrm{R}^{\mathrm{n}})$, $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<p\le\infty$ *and* $-n\delta_1<\alpha<n\delta_2$. *Then* $[b,\mu]$ *is bounded on* $\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof Similar to Theorem 2.1, we only prove homogeneous case and still suppose $0 < p < \infty$. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and we write $f(x) = \sum_{j=-\infty}^{\infty} f\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$
\begin{aligned}
\|[b,\mu](f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|[b,\mu](f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&\leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} \|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\quad + C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{k+1} \|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\quad + C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k+2}^{\infty} \|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&=: CU_1 + CU_2 + CU_3.
\end{aligned} \tag{3.1}
$$

By Lemma 1.6, we know that $[b,\mu]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$. So we have

$$
U_2 \leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
$$

Now we estimate U_1. We consider

$$
\begin{aligned}
|[b,\mu](f_j)(x)| &\leq \left(\int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_{|x|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&=: V_1 + V_2.
\end{aligned}
$$

Note that $x \in A_k$, $y \in A_j$ and $j \leq k-2$, and we know that $|x-y| \sim |x|$. Since Ω is bounded, by (2.2), the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned} V_1 &\leq C\int_{\mathbb{R}^n}\frac{|b(x)-b(y)||f_j(y)|}{|x-y|^{n-1}}\left(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\right)^{1/2}dy \\ &\leq C\int_{\mathbb{R}^n}\frac{|b(x)-b(y)||f_j(y)|}{|x-y|^{n-1}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}dy \\ &\leq C\int_{\mathbb{R}^n}\frac{|b(x)-b(y)||f_j(y)|}{|x-y|^{n-1}}\frac{|y|^{1/2}}{|x-y|^{3/2}}dy \\ &\leq C\frac{2^{j/2}}{|x|^{n+1/2}}\int_{A_j}|b(x)-b(y)||f(y)|dy \\ &\leq C2^{(j-k)/2}2^{-kn}\left\{|b(x)-b_{B_j}|\int_{A_j}|f_j(y)|dy+\int_{A_j}|b_{B_j}-b(y)||f_j(y)|dy\right\} \\ &\leq C2^{(j-k)/2}2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \end{aligned}$$

Similarly, we consider V_2. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned} V_2 &\leq C\int_{\mathbb{R}^n}\frac{|b(x)-b(y)||f_j(y)|}{|x-y|^{n-1}}\left(\int_{|x|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy \\ &\leq C\int_{\mathbb{R}^n}\frac{|b(x)-b(y)||f_j(y)|}{|x-y|^{n}}dy \\ &\leq C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \end{aligned}$$

So we have

$$|[b,\mu](f_j)(x)|\leq C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}.$$

By Lemma 1.2, Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}
&\|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\Big\{\|(b-b_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\} \\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\Big\{(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|b\|_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\} \\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\le C(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\le C2^{(j-k)n\delta_2}(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
U_1 &\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&= C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} U_1 &\le C\|b\|_* \bigg\{ \sum_{k=-\infty}^{\infty} \bigg(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\alpha)p/2} \bigg) \\ &\quad \times \bigg(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\alpha)p'/2} (k-j)^{p'} \bigg)^{p/p'} \bigg\}^{1/p} \\ &\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\alpha)p/2} \right) \right\}^{1/p} \\ &= C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p/2} \right) \right\}^{1/p} \\ &\le C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\ &= C\|b\|_* \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.2}$$

If $0 < p \le 1$, then we have

$$\begin{aligned} U_1 &\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{j\alpha p} 2^{(j-k)(n\delta_2-\alpha)p} (k-j)^p \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\ &= C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p} (k-j)^p \right) \right\}^{1/p} \\ &\le C\|b\|_* \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.3}$$

Let us now estimate U_3. Note that $x \in A_k$, $y \in A_j$ and $j \ge k+2$, so we have $|x-y| \sim |y|$. We consider

$$\begin{aligned} |[b,\mu](f_j)(x)| &\le \left(\int_0^{|y|} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad + \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &=: W_1 + W_2. \end{aligned}$$

Similar to the estimate for V_1, we get

$$W_1 \le C2^{(k-j)/2}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big\}.$$

Similar to the estimate for V_2, we get

$$W_2 \le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big\}.$$

So we have

$$|[b,\mu](f_j)(x)| \le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big\}.$$

By Lemma 1.2, Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}
&\|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\Big\{\|(b-b_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}\\
&\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\Big\{\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+(j-k)\|b\|_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}\\
&\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(j-k)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\le C(j-k)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\
&\le C2^{(k-j)n\delta_1}(j-k)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
U_3 &\le C\|b\|_*\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}(j-k)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&= C\|b\|_*\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}(j-k)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_1 + \alpha > 0$, by the Hölder inequality we have

$$
\begin{aligned}
U_3 &\le C\|b\|_* \Bigg\{ \sum_{k=-\infty}^{\infty} \Bigg(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(n\delta_1+\alpha)p/2} \Bigg) \\
&\quad \times \Bigg(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\alpha)p'/2} (j-k)^{p'} \Bigg)^{p/p'} \Bigg\}^{1/p} \\
&\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(n\delta_1+\alpha)p/2} \right) \right\}^{1/p} \\
&= C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha)p/2} \right) \right\}^{1/p} \\
&\le C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&= C\|b\|_* \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}
$$

If $0 < p \le 1$, then we have

$$
\begin{aligned}
U_3 &\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} 2^{j\alpha p} 2^{(k-j)(n\delta_1+\alpha)p} (j-k)^p \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&= C\|b\|_* \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha)p} (j-k)^p \right) \right\}^{1/p} \\
&\le C\|b\|_* \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.5}
$$

Therefore, by (3.1)-(3.5) we complete the proof of Theorem 3.1.

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