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A SHORT NOTE ON THE ORTHOGONALITY IN FUZZY METRIC SPACES

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Abstract. The purpose of this paper is to introduce and discuss the concept of orthogonality in the fuzzy metric spaces. At last we introduce and discuss the concept of orthogonality in the fuzzy normed spaces, and obtain some results on orthogonality in fuzzy normed spaces similar to orthogonality in normed spaces.

1. Introduction and Preliminaries

It is well known that the conception of fuzzy sets, firstly defined by Zadeh in 1965. Fuzzy set theory provides us with a framework which is wider than that of classical set theory. Various mathematical structures, whose features emphasize the effects of ordered structure, can be developed on the theory. The theory of fuzzy sets has become an area of active research for the last forty years. On the other hand, the notion of fuzzyness has a wide application in many areas of science and engineering, chaos control [3], nonlinear dynamical systems, etc. In physics, for example, the fuzzy structure of spacetime is followed by the fact that in strong quantum gravity regime spacetime points are determined in a fuzzy manner and therefore the impossibility of determining position of particles gives a fuzzy structure.

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Fuzzy metric spaces is one such branch, combining ordered structure with topological structures. Fuzzy metric spaces and fuzzy normed spaces are studied by many authors [2, 4, 6, 11] from different point of view. In [4] George and Veeramani modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8], and obtained a Hausdorff topology for this kind of fuzzy metric spaces. It was proved in [2] that the topology induced by a fuzzy metric space in George and Veeramani's sense is actually metrizable. In [1], the authors considered complete probabilistic metrizable (topological) space and prove that any G_δ set in a complete probabilistic metric spaces is a topologically complete probabilistic metrizable space. In [12], some fixed point theorems in intuitionistic fuzzy metric spaces are proved.

In Quantum mechanics one deals with measurable observable which denotes by Hermition operators. These operators satisfy some eigenvalue problems $< H|\psi_n >=< E_n|\psi_n >$. The eigenvalues are read and the eigenvectors are orthogonal, i.e., $\{< \psi_n|\psi_n >= \delta_{m,n}\}$.

A concept of orthogonality on normed spaces was introduced by Birchhoff. Our main motive of this paper is to establish the orthogonality in fuzzy metric spaces.

For easy underestanding of the material incorporated in this paper, we recall some basic definitions and results.

According to [10] a binary operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $\star$ satisfies the following conditions:

(1) $\star$ is commutative and associative;

(2) $\star$ is continuous;

(3) $a \star 1 = a$ for every $a \in [0,1]$;

(4) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

In fact a binary operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t- norm if $([0,1], \star)$ is a topological monoid with unit 1 such that $a \star b \leq c \star d$ whenever $a \leq c$

and $b \leq d$, $(a, b, c, d \in [0, 1])$.

According to [5], a fuzzy metric space is an ordered triple $(X, M, \star)$ such that X is a (nonempty) set, $\star$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X,\ s, t > 0$:

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, -) :\ (0, \infty) \longrightarrow [0, 1]$ is continuous.

If $(X, M, \star)$ is a fuzzy metric space, we will say that $(M, \star)$ is a fuzzy metric on X. A simple but useful fact is that $M(x, y, -)$ is nondecreasing for all $x, y \in X$ [13]. Let (X, d) be a metric space. Denote by $a.b$ the usual multiplication for all $a, b \in [0, 1]$, and let M_d be the function defined on $X \times X \times (0, \infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and $t > 0$. Then, $(X, M_d, \cdot)$ is a fuzzy metric space. This M_d is called a standard fuzzy metric induced by d.

George and Veeramani proved [5] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$, where $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for every r, with $0 < r < 1$, and $t > 0$. They proved that (X, τ_M) is a Hausdorff first countable topological space. Moreover, if (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the induced fuzzy metric M_d.

In this note, we want to define the orthogonality in fuzzy metric spaces and in fuzzy normed linear spaces. In Section 2, after introducing the orthogonality, we obtain some results and theorems about orthogonality. In Section 3, we want to introduce the concept of orthogonality in fuzzy normed spaces.

2. The orthogonality in fuzzy metric spaces

In this section, we state and prove the orthogonality in fuzzy metric spaces.

Definition 2.1. Let $(X, M, \star)$ be a fuzzy metric space. For $a_0 \in X$ and $x, y \in X$, $t > 0$, x is said to be a_0-fuzzy orthogonal to y and is denoted by $x \perp^{a_0} y$ if and only if $M(x, a_0, t) \geq M(x, y, t)$ for all t. Let G_1 and G_2 be subsets of X. Then, $G_1 \perp^{a_0} G_2$ if and only if for all $g_1 \in G_1, g_2 \in G_2$, $g_1 \perp^{a_0} g_2$. If $x \perp^{a_0} y$, then it is not necessary $y \perp^{a_0} x$.

Suppose that G is a subset of X. We know that a point $y_0 \in G$ is said to be a t-best approximation for $x \in X$ if and only if $M(y_0, x, t) \geq M(y, x, t)$ for all $y \in G$ (see [11]). The set of all t-best approximations of $x \in X$ in G is denoted by ${P^t}_G(x)$. In other words,

$$ {P^t}_G(x) = \{g_0 \in G \ : \ x \perp^{g_0} y \ \forall y \in G\}. $$

Theorem 2.2. *Let (X, d) be a metric space. In the standard fuzzy metric space $(X, M_d, .)$, for $x, y, a_0 \in X$, $x \perp^{a_0} y$ if and only if $d(x, a_0) \leq d(x, y)$.*

Proof. We have

$$ \begin{aligned} x \perp^{a_0} y &\iff M_d(a_0, x, t) \geq M_d(x, y, t) \\ &\iff \frac{t}{t + d(a_0, x)} \geq \frac{t}{t + d(x, y)} \\ &\iff d(a_0, x) \leq d(x, y). \blacksquare \end{aligned} $$

Theorem 2.3. *Let $(X, M, \star)$ be a fuzzy metric space and $a_0, x, y, z \in X$. If $x \perp^z y$ and $x \perp^y z$, then $M(x, y, t) = M(x, z, t)$.*

Proof. Since $x\perp^z y$ and $x\perp^y z$, therefore $M(x,y,t) \leq M(x,z,t)$ and $M(x,z,t) \leq M(x,y,t)$, Then, $M(x,y,t) = M(x,z,t)$. ■

Theorem 2.4. *Let $(X, M, \star)$ be a fuzzy metric space and $B \subseteq X$ be a nonempty compact set. Then, there exists $y_0 \in B$ such that $x\perp^{y_0} B$.*

Proof. Since B is compact, every $x \in X$ has a t-best approximation in B (see [11]). Therefore, there exists $y_0 \in B$ such that $x\perp^{y_0} B$. ■

Now, we consider the following two examples.

Example 2.5. Suppose that $X = \mathbf{R}$ and

$$M(x,y,t) = e^{-\frac{|x-y|}{t}}.$$

Then, $(X, M, \star)$ is a fuzzy metric space, for $a_0 = 0, x, y \in \mathbf{R}$. $x\perp^0 y$ if and only if $|x - y| \leq |x|$.

Example 2.6. Suppose that $X = \{1, 2, 3, \cdots\}$, $a, b \in [0,1]$, $a \star b = ab$ and for $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{\min\{x,y\} + t}{\max\{x,y\} + t}.$$

Then, $(X, M, \star)$ is a fuzzy metric space. If we put $G = \{2, 4, 6, \cdots\}$, for every odd number $k \in X$, $k\perp^{k+1} g$ if $g \geq k$.

Definition 2.7. *Let $(X, M, \star)$ be a fuzzy metric space, $x \in X$ and G be a nonempty subset of X. Then, $y_0 \in G$ is called a t-best coapproximation for x, if $y\perp^{y_0} x$ for $y \in G$.*

The set of best coapproximation x in G is denoted by $R^t{}_G(x)$. It is clear that $y_0 \in R^t{}_G(x)$ if and only if $M(y_0, y, t) \geq M(x, y, t)$, $\forall y \in G$.

Theorem 2.8. *Let (X, d) be a metric space. Let $(X, M_d, .)$ be the standard fuzzy metric space, G be a nonempty subset of X, $y_0 \in G$ and $x \in X$, $y_0 \in R^t{}_G(x)$ if and only if $d(y, y_0) \leq d(x, y)$ for all $y \in G$.*

Proof. We have

$$\begin{aligned} y_0 \in R^t{}_G(x) &\iff y \perp^{y_0} x \ \forall y \in G \\ &\iff M_d(y, y_0, t) \geq M_d(y, x, t) \ \forall y \in G \\ &\iff \frac{t}{t + d(y, y_0)} \geq \frac{t}{t + d(x, y)} \ \forall y \in G \\ &\iff d(y, y_0) \leq d(x, y) \ \forall y \in G. \ \blacksquare \end{aligned}$$

3. The orthogonality in fuzzy normed spaces

We recall that if X is a normed linear space and $x, y \in X$, then x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. If G_1 and G_2 are subsets of X, it is defined $G_1 \perp G_2$ if and only if $g_1 \perp g_2$ for all $g_1 \in G_1$ and $g_2 \in G_2$, see [8, 9].

The ordered triple $(X, N, \star)$ is a fuzzy normed space if X is a (nonempty) vector space, $\star$ is a continuous t-norm and N is a fuzzy set of $X \times (0, \infty)$ satisfying the following conditions, for all $x, y \in X$, $s, t > 0$ and $\alpha \neq 0$,

(i) $N(x, t) > 0$;

(ii) $N(x, t) = 1$ if and only if $x = 0$;

(iii) $N(\alpha x, t) = N(x, \frac{t}{|\alpha|})$;

(iv) $N(x + y, t + s) \geq N(x, t) \star N(y, s)$;

(v) $N(x, -) : (0, \infty) \longrightarrow [0, 1]$ is continuous;

(vi) $\lim_{x \to \infty} N(x, t) = 1$.

Let $(X, N, \star)$ be a fuzzy normed space and define $M(x, y, t) = N(x - y, t)$. Then, $(X, M, \star)$ is fuzzy metric space, which is called the fuzzy metric space induced by the fuzzy normed space $(X, N, \star)$. For $G \subseteq X$, $x \in X$ and $t > 0$, we have

$$M(G, x, t) = \sup\{M(y, x, t) : \ y \in G\} = \sup\{N(x - y, t) : \ y \in G\}$$

Let $(X, M, \star)$ be a fuzzy metric space induced by the fuzzy normed space $(X, N, \star)$. Then, for all $x, y, z \in X$ and every scalar $\alpha \neq 0$, clearly we have

(1) $M(x + z, y + z, t) = M(x, y, t)$;
(2) $M(\alpha x, \alpha y, t) = M(x, y, \frac{t}{|\alpha|})$.

Definition 3.1. *Let the fuzzy metric space induced by the fuzzy normed space $(X, N, \star)$ be a fuzzy normed space and $(X, M, \star)$ be the fuzzy metric space induced by $(X, N, \star)$. If $x, y \in X$, then we write $x \perp y$ if $M(0, x, t) \leq M(< y >, x, t)$, where $< y >$ is the linear subspace generated by y.*

Theorem 3.2. *Let $(X, N, \star)$ be a fuzzy normed space and $x, y \in X$. Then, $x \perp y$ if and only if $N(x + \alpha y, t) \geq N(x, t)$, for every $t > 0$.*

Proof. Suppose that $(X, M, \star)$ is a fuzzy metric space induced by $(X, N, \star)$. Then,

$$\begin{aligned} x \perp y &\iff M(0, x, t) \leq M(< y >, x, t) \\ &\iff M(\alpha y, x, t) \geq M(0, x, t) \ \forall \alpha \in \mathbf{C} \\ &\iff N(x + \alpha y, t) \geq N(x, t) \ \forall \alpha \in \mathbf{C}. \ \blacksquare \end{aligned}$$

Definition 3.3. Let $(X, N, \star)$ be a fuzzy normed space, let G be a subspace of X. We define

(i) $g_0 \in P^t(x)$ if and only if $x - g_0 \perp G$.

(ii) $g_0 \in R^t(x)$ if and only if $G \perp x - g_0$.

Corollary 3.4. *Let $(X, N, \star)$ be a fuzzy normed space and G be a subspace of X. Then,*

(i) $g_0 \in P^t(x)$ *if and only if* $N(x - g_0, t) \leq N(x - y, t)$ *for all* $y \in G$ *and* $t > 0$*;*

(ii) $g_0 \in R^t(x)$ *if and only if* $N(y - g_0, t) \leq N(x - y, t)$ *for all* $y \in G$ *and* $t > 0$.

4. Conclusion

The concept of orthogonality is very important in Mathematics and Physics. For example, in Quantum mechanics one deals with measurable observable which denotes by Hermition operators. These operators satisfy some eigenvalue problems. The eigenvalues are read and the eigenvectors are orthogonal, i.e., $\{<\psi_n|\psi_n>=\delta_{m,n}\}$. A generalization of orthogonality has been given. Indeed, we introduced and discussed the concept of orthogonality in the fuzzy metric spaces and the fuzzy normed spaces, and some results related to orthogonality have been extended.

References

[1] M. Alimohammady, A. Esmaeli, R. Saadati, Completeness results in probabilistic metric spaces, *Chaos Solitons Fractals* **39** (2009), 765-769.

[2] M.A. Erceg, Metric spaces in fuzzy set theory, *J. Math. Anal. Appl.* **69** (1979), 205230.

[3] A.L. Fradkov, R.J. Evans, Control of chaos: methods and applications in engineering, *Chaos Solitons Fractals* **29** (2005), 3356.

[4] A. George, P. Veeramani, Some theorem in fuzzy metric spaces, *J. Fuzzy Math.* **3** (1995), 933-940.

[5] A. George, P. Veeramani, Compact and bounded sets in fuzzy metric spaces, *J. Fuzzy Math.* **8** (2000), 975-980.

[6] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, *Fuzzy sets and Systems* **115** (2000), 485-489.

[7] O. Kaleva, S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems* **212** (1984), 215229.

[8] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, *Kybernetika (Prague)* **11** (1975), 336344.

[9] H. Mazaheri, S.M. Vaezpour, Orthogonality and ϵ-orthogonality in Banach spaces, *Australian J. Math. Anal. Appl.* **2** (2005), 1-5.

[10] J.H. Park, Intuitionistic fuzzy metric spaces, *Chaos Solitons Fractals* **22** (2004), 103946.

[11] L. Rodrgiuez, S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems* **147** (2004), 273283.

[12] R. Saadati, S. Sedghi and N. Shobe, Modified intuitionistic fuzzy metric spaces and some fixed point theorems, *Chaos Solitons Fractals* **38** (2008), 36-47.

[13] R. Saadati, S.M. Vaezpour, Some results on fuzzy Banach spaces, *J. Appl. Math. and Computing* **17** (2005), 475-484.

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