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ON A BEST EXTENSION OF A HALF-DISCRETE HILBERT-TYPE INEQUALITY

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Abstract. By using the way of weight functions and the technique of real analysis, a best extension of a half-discrete Hilbert-type inequality with one-pair conjugate exponents and two interval variables is given. The equivalent forms, the operator expressions and the reverses are considered.

1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L^p(0,\infty), g(\geq 0) \in L^q(0,\infty), ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0$, $||g||_q > 0$. Then we have the following famous Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we still have the following discrete Hardy-Hilbert's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{1.2}$$

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Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [2], [3], [4]). In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [5] gave an extension of (1.1) (for $p = q = 2$). Recently, by using the way of weight functions, Yang [6] gave some best extensions of (1.1) and (1.2) as follows: For $r > 1, \frac{1}{r} + \frac{1}{s} = 1$, we have

$$(1.3) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{r}, \frac{\lambda}{s})||f||_{p,\phi}||g||_{q,\psi}(\lambda > 0),$$

$$(1.4) \qquad \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < B(\frac{\lambda}{r}, \frac{\lambda}{s})||a||_{p,\phi}||b||_{q,\psi}(0 < \lambda \leq 2\min\{r, s\}),$$

where, $B(u, v)(= \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt(u, v > 0))$ is the Beta function and $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}, 0 < ||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty, 0 < ||g||_{q,\psi} < \infty$, $0 < ||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^n\}^{\frac{1}{p}} < \infty$ and $0 < ||b||_{q,\psi} < \infty$. Some Hilbert-type inequalities about the other measurable kernels are provided in [7]-[14].

About the case of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided some results in Theorem 351 of [1]. But they did not prove that the the constant factors in the inequalities are the best possible. And Yang [15] gave a result with the kernel $\frac{1}{1+nx}$ similar to $\frac{1}{n+x}$by introducing an interval variable as follows: If $u(t)$ is a differentiable strictly increasing function in $(n_0 - 1, \infty)(n_0 \in \mathbf{N})$, such that $u((n_0 - 1)^+) = 0$ and $u(\infty) = \infty, \lambda > 0$, $(u(t))^{\frac{\lambda-2}{2}} u'(t))(t \in (n_0 - 1, \infty))$ is decreasing, and $f(x), a_n \geq 0, 0 < \int_{n_0-1}^\infty \frac{(u(x))^{1-\lambda}}{u'(x)} f^2(x)dx < \infty$, $0 < \sum_{n=n_0}^\infty \frac{(u(n))^{1-\lambda}}{u'(n)} a_n^2 < \infty$, then

$$\int_{n_0-1}^\infty f(x) \sum_{n=n_0}^\infty \frac{a_n}{(1 + u(n)u(x))^\lambda} dx$$

$$(1.5) \qquad < \quad B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left\{ \int_{n_0-1}^\infty \frac{(u(x))^{1-\lambda}}{u'(x)} f^2(x)dx \sum_{n=n_0}^\infty \frac{(u(n))^{1-\lambda}}{u'(n)} a_n^2 \right\}^{\frac{1}{2}},$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible.

In this paper, by using the way of weight functions and the technique of real analysis, a best extension of (1.5) with one-pair conjugate exponents and two interval variables is given. The equivalent forms, the operator expressions and some reverses are considered.

2. Some Lemmas

Lemma 2.1. *If* $\lambda > 0, u(x)(x \in (b,c)), v(x)(x \in (n_0-1,\infty), n_0 \in \mathbf{N})$ *are strictly increasing differentiable functions and* $[v(x)]^{\frac{\lambda}{2}-1}v'(x)$ *is decreasing with* $u(b^+) = v((n_0-1)^+) = 0, u(c^-) = v(\infty) = \infty$, *define two weight functions as follows*

$$\omega(n) := [v(n)]^{\frac{\lambda}{2}} \int_b^c \frac{u'(x)}{(1+v(n)u(x))^\lambda}[u(x)]^{\frac{\lambda}{2}-1}dx, n \geq n_0 (n \in \mathbf{N}), \tag{2.1}$$

$$\varpi(x) := [u(x)]^{\frac{\lambda}{2}} \sum_{n=n_0}^{\infty} \frac{v'(n)}{(1+v(n)u(x))^\lambda}[v(n)]^{\frac{\lambda}{2}-1}, x \in (b,c). \tag{2.2}$$

If we define the function $\theta_\lambda(x)$ *as follows, then we have the following inequality:*

$$0 < B(\frac{\lambda}{2},\frac{\lambda}{2})(1-\theta_\lambda(x)) < \varpi(x) < \omega(n) = B(\frac{\lambda}{2},\frac{\lambda}{2}), \tag{2.3}$$

$$\theta_\lambda(x) = \frac{1}{B(\frac{\lambda}{2},\frac{\lambda}{2})} \int_0^{u(x)v(n_0)} \frac{t^{\frac{\lambda}{2}-1}dt}{(t+1)^\lambda} = O([u(x)]^{\frac{\lambda}{2}}), x \in (b,c). \tag{2.4}$$

Proof. Setting $t = v(n)u(x)$ in (2.1), we find

$$\omega(n) = \int_0^\infty \frac{1}{(t+1)^\lambda} t^{\frac{\lambda}{2}-1}dt = B(\frac{\lambda}{2},\frac{\lambda}{2}).$$

For any fixed $x \in (b,c)$, in view of the fact that the function $\frac{[v(y)]^{\frac{\lambda}{2}-1}v'(y)}{(1+u(x)v(y))^\lambda}$ $(y \in (n_0-1,\infty))$ is strictly decreasing, we find

$$\begin{aligned} \varpi(x) &< [u(x)]^{\frac{\lambda}{2}} \int_{n_0-1}^\infty \frac{1}{(1+u(x)v(y))^\lambda}[v(y)]^{\frac{\lambda}{2}-1}v'(y)dy \\ &\overset{t=u(x)v(y)}{=} \int_0^\infty \frac{1}{(t+1)^\lambda} t^{\frac{\lambda}{2}-1}dt = B(\frac{\lambda}{2},\frac{\lambda}{2}) = \omega(n). \end{aligned}$$

Moreover,

$$\begin{aligned}\varpi(x) &> [u(x)]^{\frac{\lambda}{2}} \int_{n_0}^{\infty} \frac{1}{(1+u(x)v(y))^{\lambda}} [v(y)]^{\frac{\lambda}{2}-1} v'(y)dy \\ &\overset{t=u(x)v(y)}{=} \int_{u(x)v(n_0)}^{\infty} \frac{t^{\frac{\lambda}{2}-1}}{(t+1)^{\lambda}} dt = B(\frac{\lambda}{2},\frac{\lambda}{2})[1-\theta_{\lambda}(x)].\end{aligned}$$

Clearly $\theta_{\lambda}(x) > 0$ and

$$\theta_{\lambda}(x) < \frac{1}{B(\frac{\lambda}{2},\frac{\lambda}{2})} \int_{0}^{u(x)v(n_0)} t^{\frac{\lambda}{2}-1} dt = \frac{2}{\lambda B(\frac{\lambda}{2},\frac{\lambda}{2})} (u(x)v(n_0))^{\frac{\lambda}{2}}.$$

Hence, we have (2.3) and (2.4). □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally, $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \geq n_0 (n \in \mathbf{N}), f(x)$ is a non-negative measurable function in (b, c). Then*

(i) for $p > 1$, we have the following inequalities:

$$\begin{aligned} J_1 &:= \left\{ \sum_{n=n_0}^{\infty} \frac{v'(n)}{[v(n)]^{1-\frac{p\lambda}{2}}} \left[\int_b^c \frac{f(x)}{(1+v(n)u(x))^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} \\ &\leq [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{\frac{1}{q}} \left\{ \int_b^c \varpi(x) \frac{[u(x)]^{p(1-\frac{\lambda}{2})-1}}{[u'(x)]^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} L_1 &:= \left\{ \int_b^c \frac{[\varpi(x)]^{1-q} u'(x)}{[u(x)]^{1-\frac{q\lambda}{2}}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ B(\frac{\lambda}{2},\frac{\lambda}{2}) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}; \end{aligned} \tag{2.6}$$

(ii) for $0 < p < 1$, we have the reverses of (2.5) and (2.6).

Proof. (1) By Hölder's inequality (cf. [16]) and (2.3), we have

$$\begin{aligned}
&\left[\int_b^c \frac{f(x)}{(1+v(n)u(x))^\lambda}dx\right]^p \\
= &\left\{\int_b^c \frac{1}{(1+v(n)u(x))^\lambda}\left[\frac{[u(x)]^{(1-\frac{\lambda}{2})/q}}{[v(n)]^{(1-\frac{\lambda}{2})/p}}\frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}}f(x)\right]\right. \\
&\left.\times\left[\frac{[v(n)]^{(1-\frac{\lambda}{2})/p}}{[u(x)]^{(1-\frac{\lambda}{2})/q}}\frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}}\right]dx\right\}^p \\
\leq &\int_b^c \frac{v'(n)}{(1+v(n)u(x))^\lambda}\frac{[u(x)]^{(1-\frac{\lambda}{2})(p-1)}}{[v(n)]^{1-\frac{\lambda}{2}}[u'(x)]^{p-1}}f^p(x)dx \\
&\times\left\{\int_b^c \frac{u'(x)}{(1+v(n)u(x))^\lambda}\frac{[v(n)]^{(1-\frac{\lambda}{2})(q-1)}}{[u(x)]^{1-\frac{\lambda}{2}}[v'(n)]^{q-1}}dx\right\}^{p-1} \\
= &\frac{[B(\frac{\lambda}{2},\frac{\lambda}{2})]^{p-1}}{[v(n)]^{\frac{p\lambda}{2}-1}v'(n)}\int_b^c \frac{v'(n)f^p(x)}{(1+v(n)u(x))^\lambda}\frac{[u(x)]^{(1-\frac{\lambda}{2})(p-1)}}{[v(n)]^{1-\frac{\lambda}{2}}[u'(x)]^{p-1}}dx.
\end{aligned}$$

Then by Lebesgue term by term integration theorem (cf. [17]), we have

$$\begin{aligned}
J_1 &\leq [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{\frac{1}{q}}\left\{\sum_{n=n_0}^\infty\int_b^c \frac{v'(n)f^p(x)}{(1+v(n)u(x))^\lambda}\frac{[u(x)]^{(1-\frac{\lambda}{2})(p-1)}}{[v(n)]^{1-\frac{\lambda}{2}}[u'(x)]^{p-1}}dx\right\}^{\frac{1}{p}} \\
&= [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{\frac{1}{q}}\left\{\int_b^c\sum_{n=n_0}^\infty \frac{v'(n)f^p(x)}{(1+v(n)u(x))^\lambda}\frac{[u(x)]^{(1-\frac{\lambda}{2})(p-1)}}{[v(n)]^{1-\frac{\lambda}{2}}[u'(x)]^{p-1}}dx\right\}^{\frac{1}{p}} \\
&= [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{\frac{1}{q}}\left\{\int_b^c \varpi(x)\frac{[u(x)]^{p(1-\frac{\lambda}{2})-1}}{[u'(x)]^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}},
\end{aligned}$$

and (2.5) follows. Still by Hölder's inequality, we have

$$\begin{aligned}
&\left[\sum_{n=n_0}^\infty \frac{a_n}{(1+u(x)v(n))^\lambda}\right]^q \\
= &\left\{\sum_{n=n_0}^\infty \frac{1}{(1+u(x)v(n))^\lambda}\left[\frac{[u(x)]^{(1-\frac{\lambda}{2})/q}}{[v(n)]^{(1-\frac{\lambda}{2})/p}}\frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}}\right]\right. \\
&\left.\times\left[\frac{[v(n)]^{(1-\frac{\lambda}{2})/p}}{[u(x)]^{(1-\frac{\lambda}{2})/q}}\frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}}a_n\right]\right\}^q
\end{aligned}$$

$$\leq \left\{ \sum_{n=n_0}^{\infty} \frac{1}{(1+u(x)v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{2})(p-1)}}{[v(n)]^{1-\frac{\lambda}{2}}} \frac{v'(n)}{[u'(x)]^{p-1}} \right\}^{q-1}$$

$$\times \sum_{n=n_0}^{\infty} \frac{1}{(1+u(x)v(n))^{\lambda}} \frac{[v(n)]^{(1-\frac{\lambda}{2})(q-1)}}{[u(x)]^{1-\frac{\lambda}{2}}} \frac{u'(x)}{[v'(n)]^{q-1}} a_n^q$$

$$= \frac{[u(x)]^{1-\frac{q\lambda}{2}}}{[\varpi(x)]^{1-q} u'(x)} \sum_{n=n_0}^{\infty} \frac{[u(x)]^{\frac{\lambda}{2}-1} u'(x) [v(n)]^{\frac{\lambda}{s}}}{(1+u(x)v(n))^{\lambda}} \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} a_n^q.$$

Then we have

$$\begin{aligned} L_1 &\leq \left\{ \int_b^c \{ \sum_{n=n_0}^{\infty} \frac{[u(x)]^{\frac{\lambda}{2}-1} u'(x) [v(n)]^{\frac{\lambda}{2}}}{(1+u(x)v(n))^{\lambda}} \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} a_n^q \} dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=n_0}^{\infty} \left[[v(n)]^{\frac{\lambda}{2}} \int_b^c \frac{[u(x)]^{\frac{\lambda}{2}-1} u'(x)}{(1+u(x)v(n))^{\lambda}} dx \right] \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_{n=n_0}^{\infty} \omega(n) \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and then in view of (2.3), since $\omega(n) = B(\frac{\lambda}{2}, \frac{\lambda}{2})$, inequality (2.6) follows.

(ii) By the reverse Holder's inequality (cf. [16]) and the same way, for $q < 0$, we have the reverses of (2.5) and (2.6). □

3. Main Results

Setting $\Phi(x) := \frac{[u(x)]^{p(1-\frac{\lambda}{2})-1}}{[u'(x)]^{p-1}}, \widetilde{\Phi}(x) := (1-\theta_{\lambda}(x))\Phi(x)(x \in (b,c))$,

$$\Psi(n) := \frac{[v(n)]^{q(1-\frac{\lambda}{2})-1}}{[v'(n)]^{q-1}} (n \in \mathbf{N}, n \geq n_0),$$

we have $[\Phi(x)]^{1-q} = \frac{u'(x)}{[u(x)]^{1-\frac{q\lambda}{2}}}$, $[\Psi(n)]^{1-p} = \frac{v'(n)}{[v(n)]^{1-\frac{p\lambda}{2}}}$ and

Theorem 3.1. *Let the assumptions of Lemma 2.1 be fulfilled and additionally,* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x) \geq 0 (x \in (b,c)), a_n \geq 0, n \geq n_0 (n \in \mathbf{N})$,

$f \in L_{p,\Phi}(b,c), a = \{a_n\}_{n=n_0}^{\infty} \in l_{q,\Psi},\ 0 < ||f||_{p,\Phi} = \{\int_b^c \Phi(x) f^p(x) dx\}^{\frac{1}{p}} < \infty$ and

$$0 < ||a||_{q,\Psi} = \{\sum_{n=n_0}^{\infty} \Psi(n) a_n^q\}^{\frac{1}{q}} < \infty.$$

Then we have the following equivalent inequalities:

$$I := \sum_{n=n_0}^{\infty} \int_b^c \frac{a_n f(x) dx}{(1+v(n)u(x))^{\lambda}} = \int_b^c \sum_{n=n_0}^{\infty} \frac{a_n f(x) dx}{(1+u(x)v(n))^{\lambda}}$$

$$< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi} ||a||_{q,\Psi}, \tag{3.1}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c \frac{f(x)}{(1+v(n)u(x))^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}}$$

$$< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi}, \tag{3.2}$$

and

$$L := \left\{ \int_b^c [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}}$$

$$< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||a||_{q,\Psi}, \tag{3.3}$$

where the same constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in the above inequalities is the best possible.

Proof. By Lebesgue term by term integration theorem (cf. [17]), there are two expressions for I in (3.1). In view of (2.3) and (2.5), for $\varpi(x) < B(\frac{\lambda}{r}, \frac{\lambda}{s})$, we have (3.2). By Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty} [\Psi^{\frac{-1}{q}}(n) \int_b^c \frac{f(x) dx}{(1+v(n)u(x))^{\lambda}}][\Psi^{\frac{1}{q}}(n) a_n] \le J ||a||_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_b^c \frac{f(x)}{(1+v(n)u(x))^{\lambda}} dx \right]^{p-1}, n \ge n_0,$$

then $J^{p-1} = ||a||_{q,\Psi}$. By (2.5), we find $J < \infty$. If $J = 0$, then (3.2) is naturally valid; if $J > 0$, then by (3.1), we have

$$\begin{aligned} ||a||_{q,\Psi}^q &= J^p = I < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi}, \\ ||a||_{q,\Psi}^{q-1} &= J < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\Phi}, \end{aligned}$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.3) and (2.6), for $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^{1-q}$, we have (3.3). By Hölder's inequality, we find

$$I = \int_b^c [\Phi^{\frac{1}{p}}(x)f(x)][\Phi^{\frac{-1}{p}}(x) \sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda}]dx \le ||f||_{p,\Phi} L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda} \right]^{q-1}, x \in (b, c),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (2.6), we find $L < \infty$. If $L = 0$, then (3.3) is naturally valid; if $L > 0$, then by (3.1), we have

$$\begin{aligned} ||f||_{p,\Phi}^p &= L^q = I < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi}, \\ ||f||_{p,\Phi}^{p-1} &= L < B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\Psi}, \end{aligned}$$

and we have (3.3) which is equivalent to (3.1).

Hence, inequalities (3.1), (3.2) and (3.3) are equivalent.

There exists an unified constant $d \in (b, c)$, satisfying $u(d) = 1$. For $0 < \varepsilon < \frac{q\lambda}{2}$, setting $\widetilde{f}(x) = [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}u'(x), x \in (b, d)$; $\widetilde{f}(x) = 0, x \in [d, c)$, $\widetilde{a}_n = [v(n)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(n)$, $n \ge n_0$, if there exists a positive number $k(\le B(\frac{\lambda}{2}, \frac{\lambda}{2}))$, such that (3.1) is still valid as we replace $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ by k, then in particular, we have

$$\begin{aligned} \widetilde{I} \quad : \quad &= \int_b^c \sum_{n=n_0}^{\infty} \frac{\widetilde{a}_n \widetilde{f}(x) dx}{(1+u(x)v(n))^{\lambda}} < k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\ &= k(\frac{1}{\varepsilon})^{\frac{1}{p}} \{v(n_0)v'(n_0) + \sum_{n=n_0+1}^{\infty} [v(n)]^{-\varepsilon-1} v'(n)\}^{\frac{1}{q}} \\ &\leq k(\frac{1}{\varepsilon})^{\frac{1}{p}} \{v(n_0)v'(n_0) + \int_{n_0}^{\infty} [v(y)]^{-\varepsilon-1} v'(y) dy\}^{\frac{1}{q}} \\ &= \frac{k}{\varepsilon} \{\varepsilon v(n_0)v'(n_0) + [v(n_0)]^{-\varepsilon}\}^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

In view of the decreasing property of $\frac{1}{(1+u(x)v(y))^{\lambda}}[v(y)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(y)$, we find

$$\begin{aligned} \widetilde{I} &= \int_b^d [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1} u'(x) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1} v'(n)}{(1+u(x)v(n))^{\lambda}} dx \\ &\geq \int_b^d [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1} u'(x) \left[\int_{n_0}^{\infty} \frac{[v(y)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1} v'(y)}{(1+u(x)v(y))^{\lambda}} dy \right] dx \end{aligned}$$

$$\overset{t=u(x)v(y)}{=} \int_b^d [u(x)]^{-\varepsilon-1} u'(x) \left[\int_{u(x)v(n_0)}^{\infty} \frac{t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}} dt \right] dx$$

$$\begin{aligned} &= \int_b^d [u(x)]^{\varepsilon-1} u'(x) \left[\int_0^{\infty} \frac{t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}} dt - \int_0^{u(x)v(n_0)} \frac{t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}} dt \right] dx \\ &= \frac{1}{\varepsilon} B(\frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q}) - A(x), \end{aligned}$$

where

$$A(x) := \int_b^d [u(x)]^{\varepsilon-1} u'(x) \left[\int_0^{u(x)v(n_0)} \frac{t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}} dt \right] dx. \tag{3.7}$$

Since we find

$$\begin{aligned} 0 \quad < \quad & A(x) < \int_b^d [u(x)]^{\varepsilon-1} u'(x) [\int_0^{u(x)v(n_0)} t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1} dt] dx \\ = \quad & \frac{[v(n_0)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}}}{(\frac{\lambda}{2}-\frac{\varepsilon}{q})(\frac{\lambda}{2}+\frac{\varepsilon}{p})}, \end{aligned}$$

then it follows $A(x) = O(1)(\varepsilon \to 0^+)$. By (3.6) and (3.7), we have

$$B(\frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q}) - \varepsilon O(1) < k\{\varepsilon v(n_0)v'(n_0) + [v(n_0)]^{-\varepsilon}\}^{\frac{1}{q}},$$

and then $B(\frac{\lambda}{2}, \frac{\lambda}{2}) \leq k(\varepsilon \to 0^+)$. Hence $k = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best value of (3.1).

We conform that the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in (3.2) ((3.3)) is the best possible, otherwise we can came to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. □

Remark 1. Set two weight normal spaces as follows:
$L_{p,\Phi}(b,c) = \{f|||f||_{p,\Phi} < \infty\}, l_{q,\Psi} = \{a|||a||_{q,\Psi} < \infty\}$.(i) Define a half-discrete Hilbert's operator $T : L_{p,\Phi}(b,c) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi}(b,c)$, there exists an unified representation $Tf \in l_{p,\Psi^{1-p}}$, satisfying $Tf(n) = \int_b^c \frac{f(x)}{(1+v(n)u(x))^\lambda}dx$, $n \geq n_0$. Then by (3.1), it follows $||Tf||_{p.\Psi^{1-p}} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\Phi}$ and T is bounded with $||T|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Since the constant factor in (3.2) is the best possible, we have $||T|| = B(\frac{\lambda}{2}, \frac{\lambda}{2})$.

(ii) Define a half-discrete Hilbert's operator $\widetilde{T} : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(b,c)$ as follows: For $a \in l_{q,\Psi}$, there exists an unified representation $\widetilde{T}a \in L_{q,\Phi^{1-q}}(b,c)$, satisfying $(\widetilde{T}a)(x) = \sum_{n=n_0}^\infty \frac{a_n}{(1+u(x)v(n))^\lambda}, x \in (b,c)$. Then by (3.2), it follows $||\widetilde{T}a||_{q.\Phi^{1-q}} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\Psi}$ and $\widetilde{T}$ is bounded with $||\widetilde{T}|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$.Since the constant factor in (3.3) is the best possible, we have $||\widetilde{T}|| = B(\frac{\lambda}{2}, \frac{\lambda}{2}) = ||T||$.

In the following theorem, for $0 < p < 1$, we still use the formal symbols of $||f||_{p,\widetilde{\Phi}}$ and $||a||_{q,\Psi}$ et al.

Theorem 3.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally,* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, f(x) \geq 0 (x \in (b,c)), a_n \geq 0 (n \geq n_0, n \in \mathbf{N})$, $0 < ||f||_{p,\widetilde{\Phi}} = \{\int_b^c (1 - \theta_\lambda(x))\Phi(x)f^p(x)dx\}^{\frac{1}{p}} < \infty$ *and*

$0 < ||a||_{q,\Psi} = \{\sum_{n=n_0}^{\infty} \Psi(n) a_n^q\}^{\frac{1}{q}} < \infty$. *Then we have the following equivalent inequalities:*

$$I = \sum_{n=n_0}^{\infty} \int_b^c \frac{a_n f(x)dx}{(1+v(n)u(x))^\lambda} = \int_b^c \sum_{n=n_0}^{\infty} \frac{a_n f(x)dx}{(1+u(x)v(n))^\lambda}$$

$$> B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \tag{3.8}$$

$$J = \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c \frac{f(x)}{(1+v(n)u(x))^\lambda} dx \right]^p \right\}^{\frac{1}{p}}$$

$$> B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\widetilde{\Phi}}, \tag{3.9}$$

and

$$\widetilde{L} := \left\{ \int_b^c [\widetilde{\Phi}(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda} \right]^q dx \right\}^{\frac{1}{q}}$$

$$> B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\Psi}. \tag{3.10}$$

Moreover, if there exists a constant $\delta_0 > 0$, such that for any $\delta \in [0, \delta_0)$, $[v(y)]^{\frac{\lambda}{2}+\delta-1} v'(y)$ is decreasing in $(n_0 - 1, \infty)$, then the same constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in the above inequalities is the best possible.

Proof. In view of (2.3) and the reverse of (2.5), for $\varpi(x) > B(\frac{\lambda}{2}, \frac{\lambda}{2})(1 - \theta_\lambda(x))$,we have (3.9). By the reverse Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_b^c \frac{f(x)dx}{(1+v(n)u(x))^\lambda} \right] [\Psi^{\frac{1}{q}}(n) a_n] \geq J||a||_{q,\Psi}. \tag{3.11}$$

Then by (3.9), we have (3.8). On the other-hand, assuming that (3.8) is valid, setting a_n as Theorem 1, then $J^{p-1} = ||a||_{q,\Psi}$. By the reverse of (2.5), we find $J > 0$. If $J = \infty$, then (3.9) is naturally valid; if $J < \infty$, then by (3.8), we have

$$\begin{aligned} ||a||_{q,\Psi}^{q} &= J^p = I > B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \\ ||a||_{q,\Psi}^{q-1} &= J > B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\widetilde{\Phi}}, \end{aligned}$$

and we have (3.9) which is equivalent to (3.8).

In view of (2.3) and the reverse of (2.6), for $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{r},\frac{\lambda}{s})(1-\theta_\lambda(x))]^{1-q}(q < 0)$, we have (3.10). By the reverse Hölder's inequality, we have

$$I = \int_b^c [\widetilde{\Phi}^{\frac{1}{p}}(x)f(x)]\left[\widetilde{\Phi}^{\frac{-1}{p}}(x)\sum_{n=n_0}^{\infty}\frac{a_n}{(1+u(x)v(n))^\lambda}\right]dx \geq ||f||_{p,\widetilde{\Phi}}\widetilde{L}. \tag{3.12}$$

Then by (3.10), we have (3.8). On the other-hand, assuming that (3.8) is valid, setting

$$f(x) := [\widetilde{\Phi}(x)]^{1-q}\left[\sum_{n=n_0}^{\infty}\frac{a_n}{(1+u(x)v(n))^\lambda}\right]^{q-1}, x \in (b,c),$$

then $\widetilde{L}^{q-1} = ||f||_{p,\widetilde{\Phi}}$. By the reverse of (2.6), we find $\widetilde{L} > 0$. If $\widetilde{L} = \infty$, then (3.10) is naturally valid; if $\widetilde{L} < \infty$, then by (3.8), we have

$$\begin{aligned} ||f||_{p,\widetilde{\Phi}}^{p} &= \widetilde{L}^q = I > B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \\ ||f||_{p,\widetilde{\Phi}}^{p-1} &= \widetilde{L} > B(\frac{\lambda}{2},\frac{\lambda}{2})||a||_{q,\Psi}, \end{aligned}$$

and we have (3.10) which is equivalent to (3.8).

Hence inequalities (3.8), (3.9) and (3.10) are equivalent.

For $0 < \varepsilon < \min\{\frac{|q|\lambda}{2}, |q|\delta_0\}$, setting $\widetilde{f}(x) = [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}u'(x), x \in (b,d)$; $\widetilde{f}(x) = 0, x \in [d,c), \widetilde{a}_n = [v(n)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(n), n \geq n_0$, if there exists a positive number $k(\geq B(\frac{\lambda}{2},\frac{\lambda}{2}))$, such that (3.8) is still valid as we replace $B(\frac{\lambda}{2},\frac{\lambda}{2})$ by k, then in particular, for $q < 0$, we have

$$\widetilde{I} := \int_b^c \sum_{n=n_0}^{\infty}\frac{\widetilde{a}_n\widetilde{f}(x)dx}{(1+u(x)v(n))^\lambda} > k||\widetilde{f}||_{p,\widetilde{\Phi}}||\widetilde{a}||_{q,\Psi}$$

$$
\begin{aligned}
&= k\{\int_b^d (1 - O([u(x)]^{\frac{\lambda}{2}}))[u(x)]^{-\varepsilon-1}u'(x)dx\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty}[v(n)]^{-\varepsilon-1}v'(n)\right\}^{\frac{1}{q}} \\
&= k\{\frac{1}{\varepsilon} - O(1)\}^{\frac{1}{p}} \left\{[v(n_0)]^{-\varepsilon-1}v'(n_0) + \sum_{n=n_0+1}^{\infty}[v(n)]^{-\varepsilon-1}v'(n)\right\}^{\frac{1}{q}} \\
&\geq k\{\frac{1}{\varepsilon} - O(1))^{\frac{1}{p}}\{[v(n_0)]^{-\varepsilon-1}v'(n_0) + \int_{n_0}^{\infty}[v(y)]^{-\varepsilon-1}v'(y)dy\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\{1 - \varepsilon O(1)\}^{\frac{1}{p}}\{\varepsilon[v(n_0)]^{-\varepsilon-1}v'(n_0) + [v(n_0)]^{-\varepsilon}\}^{\frac{1}{q}}. \qquad (3.13)
\end{aligned}
$$

In view of the decreasing property of $\frac{[v(y)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(y)}{(1+u(x)v(y))^{\lambda}}$, setting $t = u(x)v(y)$, we find

$$
\begin{aligned}
\widetilde{I} &= \int_b^d [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}u'(x)\sum_{n=n_0}^{\infty}\frac{[v(n)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(n)}{(1+u(x)v(n))^{\lambda}}dx \\
&\leq \int_b^d [u(x)]^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}u'(x)\left[\int_{n_0-1}^{\infty}\frac{[v(y)]^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}v'(y)}{(1+u(x)v(y))^{\lambda}}dy\right]dx \\
&= \int_b^d [u(x)]^{\varepsilon-1}u'(x)dx\int_0^{\infty}\frac{t^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}}dt \\
&= \frac{1}{\varepsilon}B(\frac{\lambda}{2}-\frac{\varepsilon}{q}, \frac{\lambda}{2}+\frac{\varepsilon}{q}). \qquad (3.14)
\end{aligned}
$$

By (3.13) and (3.14), we have

$$
B(\frac{\lambda}{2}-\frac{\varepsilon}{q}, \frac{\lambda}{2}+\frac{\varepsilon}{q}) > k\{1-\varepsilon O(1)\}^{\frac{1}{p}}\{\varepsilon[v(n_0)]^{-\varepsilon-1}v'(n_0) + [v(n_0)]^{-\varepsilon}\}^{\frac{1}{q}},
$$

and then $B(\frac{\lambda}{2}, \frac{\lambda}{2}) \geq k(\varepsilon \to 0^+)$. Hence $k = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best value of (3.8).

We confirm that the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in (3.9) ((3.10)) is the best possible, otherwise we can came to a contradiction by (3.11) ((3.12)) that the constant factor in (3.8) is not the best possible. □

Remark 2. (i) If $\alpha > 0, u(x) = x^{\alpha}, b = 0, c = \infty, v(n) = n^{\alpha}, n_0 = 1$, then for $0 < \alpha\lambda \leq 2$, $[v(x)]^{\frac{\lambda}{2}-1}v'(x) = \alpha x^{\frac{\alpha\lambda}{2}-1}$ is decreasing. In particular, for $\alpha = 1, 0 < \lambda \leq 2, u(x) = x(x \in (0,\infty)), v(n) = n(n \in \mathbf{N})$ in (3.1), (3.2) and (3.3), we have the following equivalent inequalities:

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{a_n f(x)dx}{(1+nx)^{\lambda}} = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)dx}{(1+nx)^{\lambda}}$$

$$< B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\phi}||a||_{q,\psi}, \tag{3.15}$$

$$\left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\int_0^{\infty} \frac{f(x)}{(1+nx)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\phi}, \tag{3.16}$$

and

$$\left\{ \int_0^{\infty} x^{\frac{q\lambda}{2}-1} \left[\sum_{n=1}^{\infty} \frac{a_n}{(1+nx)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\psi}. \tag{3.17}$$

(ii) For $p = q = 2, b = n_0 - 1 = 0, c = \infty, v(x) = u(x)$ in (3.1), we have (1.5). Hence, (3.1) is a best extension of (1.5).

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References

[1] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934

[2] D. S. Mitrinović, J.E. Pečarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer Acaremic Publishers, Boston, 1991

[3] B. Yang, *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd., 2009

[4] B. Yang, *Discrete Hilbert-type inequalities*, Bentham Science Publishers Ltd., 2011

[5] B. Yang, On Hilbert's integral inequality, *Journal of Mathematical Analysis and Applications*, **220**(1998), 778–785

[6] B. Yang, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijin, 2009 (China)

[7] B. Yang, I. Brnetić, M. Krnić, J. Pečarić, Generalization of Hilbert and Hardy-Hilbert integral inequalities, *Math. Ineq. and Appl.*, **8(2)**(2005), 259–272

[8] Y. Hong, All-side generalization about Hardy-Hilbert integral inequalities, *Acta mathematica Sinica*, **44(4)**(2001), 619–625

[9] J. Jin, L. Debnath, On a Hilbert-type linear series operator and its applications, *Journal of Mathematical Analysis and Applications*, **371**(2010), 691–704

[10] L. Azar, On some extensions of Hardy-Hilbert's inequality and Applications, *Journal of Inequalities and Applications*, 2009, no. 546829

[11] B. Yang, T. Rassias, On the way of weight coefficient and research for Hilbert-type inequalities, *Math. Ineq. Appl.*, 2003,**6(4)**(2003), 625–658

[12] B. Arpad, O. choonghong, Best constant for certain multilinear integral operator, *Journal of Inequalities and Applications*, 2006, no. 28582

[13] W. Zhong, The Hilbert-type integral inequality with a homogeneous kernel of Lambda-degree, *Journal of Inequalities and Applications*, 2008, no. 917392

[14] Y. Li, B. He, On inequalities of Hilbert's type, *Bulletin of the Australian Mathematical Society*, **76(1)**(2007), 1–13

[15] B. Yang, A mixed Hilbert-type inequality with a best constant factor, *International Journal of Pure and Applied Mathematcs*, **20(3)**(2005), 319–328

[16] J. Kuang, *Introduction to real analysis*, Hunan Education Press, Chansha, 1996 (China)

[17] J. Kuang, *Applied inequalities*, Shangdong Science Technic Press, Jinan, 2004 (China)

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