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MATRIX TRANSFORMATIONS ON THE SET ces(p, q)

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Abstract. The main purpose of this paper is to characterize the matrices of the classes (ces$(p, q), cs$) and (ces$(p, q), bs$), where cs is the space of convergent series and bs is the space of bounded series.

1. Introduction

Let ω be the space of all (real or complex) sequences, and X, Y are two subsets of ω. If $A = (a_{n,k})_{n,k=1,2,3,\cdots}$ be an infinite matrix of complex numbers we say that A defines a matrix transformation form X into Y and denoted it by $A \in (X, Y)$, if for every sequence $x = (x_k) \in X$ the sequence $A(x) = A_n(x)$ is in Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k \quad \text{for all } n,$$

provided the series on the right is convergent.

In [4] Lim investigated the sequence space ces(p). In [2] Khan and Rahman defined and studied the sequence space ces(p, q). If $p = (p_r)$, then for $\inf p_r > 0$

$$\text{ces}\,(p, q) = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} < \infty \right\},$$

where $q = (q_k)$ is a sequence of positive real numbers, $Q_{2^r} = \sum_r q_k$ and $\sum_r$ denotes a sum over the ranges $2^r \le k < 2^{r+1}$. They showed ces(p, q) is a complete paranorm

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$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |x_k| \right)^{p_r} \right)^{\frac{1}{M}}$$

provided $H = \sup_r p_r < \infty$ and $M = \max\{1, H\}$.

In [5] Stieglitz and Tietz defined convergent and bounded series as

$$cs = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in c \right\},$$

$$c_0 s = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in c_0 \right\},$$

$$bs = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in \ell_\infty \right\},$$

where c, c_0 and ℓ_∞ are the sequence spaces of all convergent, null and bounded sequences respectively.

We state the following inequality (see [3]) which will be used later. For any integer $E > 1$ and any two complex numbers a and b we have

$$|ab| \leq E \left(|a|^t E^{-t} + |b|^p \right), \tag{1.1}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{t} = 1$.

2. MATRIX TRANSFORMATION ON $ces(p, q)$

The following notations are used throughout for all integers $n \geq 1$, we write

$$t_n(Ax) = \sum_{i=1}^{n} A_i(x) = \sum_{k=1}^{\infty} b_{nk} x_k ,$$

where $b_{nk} = \sum_{i=1}^{n} a_{ik}$.

Now we prove:

Theorem 2.1. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p, q), cs)$ if and only if*

(i) there exists an integer $E > 1$ such that

$$T = \sup_n (U_n) < \infty \ ,$$

where

$$U_n = \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |b_{nk}|) \right)^{t_r} E^{-t_r} \ ,$$

$$\frac{1}{p_r} + \frac{1}{t_r} = 1 \qquad r = 0, 1, 2, \cdots$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k$ for all k.

Proof.Necessity. Suppose that $A \in (ces(p, q), cs)$. Then $t_n(Ax)$ exists for each n and $x \in ces(p, q)$. If we put $\sigma_n(x) = t_n(Ax)$, then $(\sigma_n)_n$ is a sequence of continuous real functions on $ces(p, q)$. Also ces(p, q) is complete and further $\sup_n |t_n(Ax)| < \infty$ on $ces(p, q)$. Now arguing with uniform boundedness principle (see Khan and Rahman [2],Th.3) we have condition (i). Condition (ii) is obtained by taking $x = e_k \in$ ces(p, q), where e_k is a sequence with 1 at the k-th place and zeros elsewhere.

Sufficiency. Suppose conditions (i) and (ii) hold. Then the conditions imply

$$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(q_k^{-1} |\alpha_k|\right) \right)^{t_r} E^{-t_r} = \lim_{n\to\infty} (U_n) \le \sup_n (U_n) < \infty.$$

Thus the series $\sum_{k=1}^{\infty} b_{nk} x_k$ and $\sum_{k=1}^{\infty} \alpha_k x_k$ converge for each n and $x \in$ ces(p, q). Put $t_{nk} = b_{nk} - \alpha_k$. Then

$$\sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} t_{nk} x_k + \sum_{k=1}^{\infty} \alpha_k x_k \ .$$

By (ii) for $k_0 \in Z^+$, where Z^+ is the set of positive integers, we have

$$\lim_{n\to\infty} \sum_{k \le 2^{k_0}} t_{nk} x_k = 0 \ .$$

Moreover, for each $x \in$ ces(p, q) and $\epsilon > 0$ we can choose an integer $k_0 \in Z^+$ such that

$$g_{k_0}(x) = \sum_{r=k_0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} < \epsilon^M \ .$$

Now put $B_r(n) = \max_r \left(q_k^{-1}|t_{nk}|\right) = \max_r \left(q_k^{-1}|b_{nk} - \alpha_k|\right)$. Then by ([2], Th.2) and by inequality (1.1) we have

$$\begin{aligned}
\sum_{k=2^{k_0}}^{\infty} |t_{nk}||x_k| / \left[g_{k_o}(x)\right]^{1/M} &\le \sum_{r=k_0}^{\infty} \max_r(q_k^{-1}|t_{nk}|) \left(\sum_r q_k|x_k|\right) \Big/ \left[g_{k_0}(x)\right]^{1/M} \\
&= \sum_{r=k_0}^{\infty} \left(Q_{2^r} B_r(n) \frac{1}{Q_{2^r}} \sum_r q_k|x_k|\right) \Big/ \left[g_{k_0}(x)\right]^{1/M} \\
&\le E\left[\sum_{r=k_0}^{\infty} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} + \sum_{r=k_0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k|x_k|\right)^{p_r} \Big/ \left[g_{k_0}(x)\right]^{p_r/M}\right] \\
&\le E\left[\sum_{r=k_0}^{\infty} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} + 1\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=2^{k_0}}^{\infty} |t_{nk}||x_k| &\le E\left[\sum_{r=k_0}^{\infty} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} + 1\right] \left[g_{k_0}(x)\right]^{1/M} \\
&< E(2U_n + 1)\epsilon \ ,
\end{aligned}$$

where $\sum_{r=k_0}^{\infty} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} \le 2U_n < \infty$ for all n.

It follows immediately that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

and this completes the proof.

Corollary 2.1. [1] *Let* $1 < p_r \le \sup_r p_r < \infty$*. Then* $A \in (ces(p), cs)$ *if and only if*

(i) there exists an integer $E > 1$ such that

$$T = \sup_n (U_n) < \infty \ ,$$

where

$$U_n = \sum_{r=0}^{\infty} \left(2^r \max_r |b_{nk}|\right)^{t_r} E^{-t_r} \ ,$$

$$\frac{1}{p_r} + \frac{1}{t_r} = 1 \quad , \quad r = 0, 1, 2, \cdots$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k$ for all k.

Proof. If we take $q_n = 1$ for all n in the above theorem then we obtain the results.

Corollary 2.2. *Let* $1 < p_r \leq \sup_r p_r < \infty$*. Then* $A \in (ces(p), c_0 s)$ *if and only if*

the condition (i) of Corollary 2.1 holds, and $\lim_{n\to\infty} b_{nk} = 0$.

Theorem 2.2. *Let* $1 < p_r \leq \sup_r p_r < \infty$*. Then* $A \in (ces(p, q), bs)$ *if and only if*

(i) there exists an integer $E > 1$ such that

$$T = \sup_n (U_n) < \infty \ ,$$

where

$$U_n = \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |b_{nk}|)\right)^{t_r} E^{-t_r} \ ,$$

$$\frac{1}{p_r} + \frac{1}{t_r} = 1 \qquad r = 0, 1, 2, \cdots$$

Proof. Necessity follows by using similar argument as in Theorem 2.1. For sufficiency, suppose that the condition (i) holds and that $x \in$ ces(p, q). Then by inequality (1.1) we have

$$
\begin{aligned}
|t_n(Ax)| &= \left|\sum_{k=1}^{\infty} b_{nk}x_k\right| \\
&\leq \sum_{k=1}^{\infty} |b_{nk}x_k| \\
&\leq \sum_{r=0}^{\infty} \sum_{r} |b_{nk}||x_k| \\
&\leq \sum_{r=0}^{\infty} \max_r \left(q_k^{-1}|b_{nk}|\right) \sum_r q_k|x_k| \\
&= \sum_{r=0}^{\infty} Q_{2^r} \max_r \left(q_k^{-1}|b_{nk}|\right) \frac{1}{Q_{2^r}} \sum_r q_k|x_k| \\
&\leq E\left[\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(q_k^{-1}|b_{nk}|\right)\right)^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k|x_k|\right)^{p_r}\right] \\
&\leq E\left(T + \sum_{r=0}^{\infty} \left(\frac{1}{Q^{2^r}} \sum_r q_k|x_k|\right)^{p_r}\right).
\end{aligned}
$$

We conclude $\sup_n |t_n(Ax)| < \infty$.

Therefore, $A \in (\text{ces}(p, q), bs)$ and this completes the proof.

Corollary 2.3. [1] *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (ces(p), bs)$ *if and only if there exists an integer* $E > 1$ *such that*

$$T = \sup_n (U_n) < \infty \ ,$$

where

$$U_n = \sum_{r=0}^{\infty} \left(2^r \max_r |b_{nk}|\right)^{t_r} E^{-t_r} \ ,$$

$$\frac{1}{p_r} + \frac{1}{t_r} = 1 \quad , \quad r = 0, 1, 2, \cdots$$

Proof. When $q_n = 1$ for all n in the above Theorem 2.2 then we get the result.

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