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FIXED POINTS FOR CONTRACTION MAPPINGS IN GENERALIZED CONE METRIC SPACES

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Abstract. We prove in this paper several fixed point results for mappings that satisfy certain contractive conditions in generalized cone metric spaces. Importantly, our results generalize, extend, and unify some other results in the literature in the sense that they are analogous to those for cone metric spaces, but in a more general setting, where we have here G-cone metric spaces, or in the sense that they are extension or generalization of some other results proved previously in G-metric spaces.

1. INTRODUCTION

Early in sixties, some attempts to generalize the usual notion of metric spaces had been made, see for example [1, 2], in an attempt to obtain analogous results to those for metric spaces, but in a more general setting. Unfortunately, other works conducted by other researchers see for example [3], refuted these generalizations.

In 1992, Dhage [4] introduced a different generalization, but unfortunately, this one has also many fundamental flaws that demonstrated by other workers, see for example [5, 6, 7].

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Later on, generalized metric spaces, or more specifically, G-metric spaces, which are a generalization of the usual notion of metric spaces in an appropriate new structure were introduced by Mustafa and Sims [8]. This new structure was a great alternative to amend the flaws in the concept of D-metric spaces [4]. It is proved in [8] that in this new structure every G-metric space is topologically equivalent to a metric space, which allows transforming directly many concepts and results from metric spaces into the G-metric space setting.

Separately, Huang and Zhang generalized in [9] the notion of metric spaces by replacing the set of real numbers by ordered Banach space, and define the concept of cone metric spaces. Several fixed points theorems were obtained in these new metric spaces for mappings satisfy certain contractive conditions, see for example [10, 11] and references therein.

Recently, Beg *et. al.* [12], benefiting from the new concepts in the previous two metric spaces, introduced a generalization of the G-metric spaces and cone metric spaces in what is called G-cone metric spaces, and proved some convergence properties as well as some fixed point theorems.

In this paper, we introduce new metric spaces, which we call G_K-cone metric spaces obtaining by composing the G-cone metric spaces [12] that are a generalization of the notion of the cone metric spaces, with the norm $\|\cdot\|$. We prove, in G_K-cone metric spaces, analogous results to those for cone metric spaces, but in a more general setting. Also, we prove new fixed point theorems that generalize, extend, and unify other results in the literature, and in addition, we give error bounds for convergence.

2. Preliminaries

We give in this section, preliminaries and basic definitions which will be used throughout the paper.

Throughout the paper, let E be a real Banach space.

Definition 2.1 (See [12]). A subset P of E is called a cone if and only if:

(P1) P is closed, nonempty and $P \neq \{0\}$,

(P2) If $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ then $ax + by \in P$. More generally, if $a, b, c \in \mathbb{R}$, $a, b, c \geq 0$, and $x, y, z \in P$ then $ax + by + cz \in P$,

(P3) $P \cap (-P) = \{0\}$.

A partial ordering $\preceq$ with respect to a given cone $P \subset E$ is defined by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \prec\prec y$ stands for $y - x \in IntP$, i.e., $y - x$ in interior of P. A cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in E$, we have $0 \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P, and it is proved in [13] that there are no normal cones with normal constant $K < 1$.

Definition 2.2 (See [12]). Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to E$ satisfies

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 \prec G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$,

(G3) $G(x, x, y) \preceq G(x, y, z)$ whenever $y \neq z$, for all $x, y, z \in X$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$ (Symmetry in all three variables),

(G5) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$. (Rectangle inequality).

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space, $G-$cone metric space.

For convergence properties of sequences in G-cone metric spaces, one could refer to the paper by Beg *et. al.* [12].

In what follows, we introduce the definition of what we call the G_K-cone metric space which will be used throughout the paper.

Definition 2.3. Let X be a G-cone metric space with normal constant $K \geq 1$. Then the mapping $G_K : X \times X \times X \to [0, \infty)$, which is given by $G_K(x, y, z) = \|G(x, y, z)\|$, is called a generalized K cone metric on X, and X would be called a generalized K cone metric space, G_K-cone metric space.

To illustrate the new concepts we give the following examples.

Example 2.1 (See [12]). *Let (X, d) be any cone metric space. Define the mapping $G : X \times X \times X \to E$ by*

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z), \forall x, y, z \in X.$$

Then G is a generalized cone metric on X and X is a $G-$cone metric space. Moreover, $G_K(x, y, z) = \|G(x, y, z)\|$ is a generalized K cone metric on X, and X is a G_K-cone metric space.

Therefore, any cone metric space can define a G-cone metric space and a G_K-cone metric space.

Example 2.2. *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be a cone in E, and $X \subseteq \mathbb{R}$. Define the mapping $G : X \times X \times X \to E$ by*

$$G(x, y, z) = (\frac{1}{3}(|x - y| + |y - z| + |x - z|), \frac{2}{3}(|x - y| + |y - z| + |x - z|)), \forall x, y, z \in X.$$

Then G is a generalized cone metric on X and X is a $G-$cone metric space. Moreover, $G_K(x, y, z) = \|G(x, y, z)\|$ is a generalized K cone metric on X, and X is a G_K-cone metric space. To find G_K, one might take, for instance, the one norm and have

$$G_K(x, y, z) = \|G(x, y, z)\|_1 = |x - y| + |y - z| + |x - z|,$$

or the infinity norm to have

$$G_K(x,y,z) = \|G(x,y,z)\|_\infty = \frac{2}{3}(|x-y| + |y-z| + |x-z|).$$

Definition 2.4. Let X be a G_K-cone metric space, and x_n be a sequence in X. We say that x_n is

- G_K-Cauchy sequence if for every $\varepsilon > 0$, there exists N such that $G_K(x_n, x_m, x_\ell) < \varepsilon$, for all $n, m, \ell > N$.
- convergent sequence if for every $\varepsilon > 0$ there exists N such that for all $n, m > N$, $G_K(x_n, x_m, x) < \varepsilon$ for some fixed $x \in X$. Here, x is called the limit of the sequence x_n, and is denoted by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

A G_K-cone metric space X is said to be complete if every G_K-Cauchy sequence in X is convergent in X.

3. Fixed point theorems for contraction mappings

We prove here, some new fixed point theorems in G_K-cone metric spaces introduced in Section 2.

Theorem 3.1. *Let X be a complete G_K-cone metric space with a normal constant $K \geq 1$, and let $T : X \to X$ be a mapping satisfying, for each $x, y \in X$,*

(3.1)

$$\begin{aligned} G_K(Tx,Ty,Ty) \leq\ & \alpha(x,y,y)G_K(x,y,y) \\ & +\beta(x,y,y)[G_K(x,Tx,Tx) + 2G_K(y,Ty,Ty)] \\ & +\gamma(x,y,y)[G_K(x,Ty,Ty) + G_K(y,Ty,Ty) + G_K(y,Tx,Tx)] \\ & +\delta(x,y,y)[G_K(y,y,Tx) + G_K(y,y,Ty) + G_K(x,x,Ty)] \\ & +\sigma(x,y,y)[G_K(x,x,Tx) + 2G_K(y,y,Ty)], \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0,1)$ such that

$$\begin{aligned} \lambda = \sup\{ & \alpha(x,y,y) + 3\beta(x,y,y) + \gamma(x,y,y) \\ & +(2\gamma(x,y,y) + 9\delta(x,y,y) + 12\sigma(x,y,y))K : x, y \in X\} < 1. \end{aligned}$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$ with an error bound $G_K(T^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} K G_K(x, Tx, Tx)$.

Proof. Let $x_0 \in X$ be an arbitrary initial guess, and let the sequence $\{x_n\}$ be defined by $x_n = T^n x_0$, or equivalently, $x_n = T x_{n-1}$, $n \geq 1$. Then, from (3.1), using (G1) from Definition 2.2, we get

$$
\begin{aligned}
G_K(x_n, x_{n+1}, x_{n+1}) &= G_K(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq \alpha G_K(x_{n-1}, x_n, x_n) \\
&\quad +\beta[G_K(x_{n-1}, x_n, x_n) + 2G_K(x_n, x_{n+1}, x_{n+1})] \\
&\quad +\gamma[G_K(x_{n-1}, x_{n+1}, x_{n+1}) + G_K(x_n, x_{n+1}, x_{n+1})] \\
&\quad +\delta[G_K(x_n, x_n, x_{n+1}) + G_K(x_{n-1}, x_{n-1}, x_{n+1})] \\
&\quad +\sigma[G_K(x_{n-1}, x_{n-1}, x_n) + 2G_K(x_n, x_n, x_{n+1})],
\end{aligned}
\tag{3.2}
$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are evaluated at (x_{n-1}, x_n, x_n). By rectangle inequality, (G5) in Definition 2.2, we have

$$
\begin{cases}
G(x_{n-1}, x_{n+1}, x_{n+1}) \preceq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), \\
G(x_{n-1}, x_{n-1}, x_{n+1}) \preceq 2G(x_{n-1}, x_{n+1}, x_{n+1}), \\
G(x_n, x_n, x_{n+1}) \preceq G(x_{n-1}, x_{n-1}, x_{n+1}) + G(x_{n-1}, x_n, x_n), \\
G(x_{n-1}, x_{n-1}, x_n) \preceq 2G(x_{n-1}, x_n, x_n),
\end{cases}
\tag{3.3}
$$

and hence, using the normality of the cone and that $\|.\|$ satisfies the triangle inequality, we get

$$\begin{cases} G_K(x_{n-1}, x_{n+1}, x_{n+1}) & \le K\|G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\| \\ & \le K(G_K(x_{n-1}, x_n, x_n) + G_K(x_n, x_{n+1}, x_{n+1})), \\ G_K(x_{n-1}, x_{n-1}, x_{n+1}) & \le 2K\|G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\| \\ & \le 2K(G_K(x_{n-1}, x_n, x_n) + G_K(x_n, x_{n+1}, x_{n+1})), \\ G_K(x_n, x_n, x_{n+1}) & \le K\|3G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\| \\ & \le K(3G_K(x_{n-1}, x_n, x_n) + 2G_K(x_n, x_{n+1}, x_{n+1})), \\ G_K(x_{n-1}, x_{n-1}, x_n) \le 2K\|G(x_{n-1}, x_n, x_n)\| = 2KG_K(x_{n-1}, x_n, x_n). \end{cases}$$

Together with equation (3.2), we have

$$\begin{aligned} G_K(x_n, x_{n+1}, x_{n+1}) \le (\alpha + 3\beta) \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\} \\ +(1 + 2K)\gamma \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\} \\ +9K\delta \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\} \\ +12K\sigma \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\}, \end{aligned}$$

or

$$\begin{aligned} & G_K(x_n, x_{n+1}, x_{n+1}) \\ & \quad \le (\alpha + 3\beta + \gamma + (2\gamma + 9\delta + 12\sigma)K) \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\}, \end{aligned}$$

which could be written as

$$G_K(x_n, x_{n+1}, x_{n+1}) \le \lambda \max\{G_K(x_{n-1}, x_n, x_n), G_K(x_n, x_{n+1}, x_{n+1})\}.$$

Since $\lambda < 1$, we have

$$G_K(x_n, x_{n+1}, x_{n+1}) \le \lambda G_K(x_{n-1}, x_n, x_n),$$

and by induction, we get

$$G_K(x_n, x_{n+1}, x_{n+1}) \le \lambda^n G_K(x_0, x_1, x_1). \tag{3.4}$$

Now, for all $n, m \in \mathbb{N}$, with $n < m$, we have, using (G5) from Definition 2.2,

$$G(x_n, x_m, x_m) \preceq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m).$$

Using equation (3.4), and using the normality of the cone and that $\|.\|$ satisfies the triangle inequality, implies

(3.5)

$$\begin{aligned} G_K(x_n, x_m, x_m) & \\ &\leq K(G_K(x_n, x_{n+1}, x_{n+1}) + G_K(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G_K(x_{m-1}, x_m, x_m)) \\ &\leq K(\lambda^n G_K(x_0, x_1, x_1) + \lambda^{n+1} G_K(x_0, x_1, x_1) + \cdots + \lambda^{m-1} G_K(x_0, x_1, x_1)) \\ &\leq \tfrac{\lambda^n}{1-\lambda} K G_K(x_0, x_1, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, equation (3.5) implies that

$$\lim_{n,m\to\infty} G_K(x_n, x_m, x_m) = 0,$$

and hence, x_n is a G_K-Cauchy sequence. By completeness, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.6}$$

To show that u is indeed a fixed point for T, we again use (3.1) to get

$$\begin{aligned} G_K(x_n, Tu, Tu) & \\ \leq\ & \alpha G_K(x_{n-1}, x_n, x_n) \\ & +\beta[G_K(x_{n-1}, x_n, x_n) + 2G_K(u, Tu, Tu)] \\ & +\gamma[G_K(x_{n-1}, Tu, Tu) + G_K(u, Tu, Tu) + G_K(u, x_n, x_n)] \\ & +\delta[G_K(u, u, x_n) + G_K(u, u, Tu) + G_K(x_{n-1}, x_{n-1}, Tu)] \\ & +\sigma[G_K(x_{n-1}, x_{n-1}, x_n) + 2G_K(u, u, Tu)], \end{aligned}$$

where α, β, γ, δ, and σ are now evaluated at (x_{n-1}, u, u). Taking the limit as $n \to \infty$, leads to

$$G_K(u, Tu, Tu) \leq 2(\beta + \gamma) G_K(u, Tu, Tu) + 2(\delta + \sigma) G_K(u, u, Tu),$$

and using (G5) from Definition 2.2 to derive the fact that

$$G(u,u,Tu) \preceq 2G(Tu,Tu,u) \Longrightarrow G_K(u,u,Tu) \leq 2KG_K(u,Tu,Tu),$$

we get

$$\begin{aligned} G_K(u,Tu,Tu) &\leq 2(\beta+\gamma+2K(\delta+\sigma))G_K(u,Tu,Tu) \\ &\leq \lambda G_K(u,Tu,Tu). \end{aligned} \tag{3.7}$$

Since $\lambda < 1$, we have $G_K(u,Tu,Tu) = 0$. Therefore, $Tu = u$.

For uniqueness, suppose that $u \neq v = Tv$. Then, from (3.1) after simplifying, we get

$$G_K(u,v,v) \leq \alpha G_K(u,v,v)+\gamma[G_K(u,v,v)+G_K(v,u,u)]+\delta[G_K(v,v,u)+G_K(u,u,v)],$$

where α, γ, δ are evaluated at (u,v,v). Using again (G5) from Definition 2.2, we have

$$G(v,u,u) \preceq 2G(u,v,v),$$

and hence,

$$G_K(v,u,u) \leq 2KG_K(u,v,v).$$

Therefore, after simplifying, we get

$$\begin{aligned} G_K(u,v,v) &\leq (\alpha+\gamma+\delta+2(\gamma+\delta)K)G_K(u,v,v) \\ &\leq \lambda G_K(u,v,v). \end{aligned}$$

Since $\lambda < 1$, we have $G_K(u,v,v) = 0$, which implies $u = v$.

Now, since $x_0 \in X$ was arbitrary, then from (3.6) we conclude that $T^n x \to u$ as $n \to \infty$ for all $x \in X$. Finally, taking the limit in (3.5), with $x_0 = x \in X$, as $m \to \infty$, we get the error bound $G_K(T^n x,u,u) \leq \frac{\lambda^n}{1-\lambda} KG_K(x,Tx,Tx)$. □

Corollary 3.1. *Let X be a complete G_K-cone metric space, and let $T : X \to X$ be a mapping satisfying, for each $x,y \in X$,*

$$G_K(Tx,Ty,Ty) \leq \alpha(x,y,y)G_K(x,y,y), \tag{3.8}$$

where α is a function from $X \times X \times X$ into $[0,1)$, with $\lambda = \sup\{\alpha(x,y,y) : x,y \in X\} < 1$. Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$ with an error bound $G_K(T^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} K G_K(x, Tx, Tx)$.

Proof. Take in Theorem 3.1, $\beta = \gamma = \delta = \sigma = 0$, then $\lambda = \sup\{\alpha(x,y,y) : x,y \in X\} < 1$, and the proof follows straightforwardly from proof of Theorem 3.1. □

Remark 3.1. As one can see from Corollary 3.1, other selections or variations from the condition in (3.1) are valid similarly, i.e., one could take $\alpha = \gamma = \delta = \sigma = 0$, $\alpha = \beta = \gamma = 0$, $\alpha = \beta = \delta = 0$, or $\alpha = \beta = 0$, ..., and so on, and a similar proof would follows, because any map satisfies the new condition using one of the above would definitely, satisfies the original condition (3.1) in Theorem 3.1.

Corollary 3.2. *Let X be a complete G_K-cone metric space with a normal constant $K \geq 1$, and let $T : X \to X$ be a mapping satisfying, for each $x, y \in X$,*

(3.9)

$$\begin{aligned} G_K(Tx,Ty,Ty) \leq \max\{&\alpha(x,y,y)G_K(x,y,y), \\ &\beta(x,y,y)[G_K(x,Tx,Tx) + 2G_K(y,Ty,Ty)], \\ &\gamma(x,y,y)[G_K(x,Ty,Ty) + G_K(y,Ty,Ty) + G_K(y,Tx,Tx)], \\ &\delta(x,y,y)[G_K(y,y,Tx) + G_K(y,y,Ty) + G_K(x,x,Ty)], \\ &\sigma(x,y,y)[G_K(x,x,Tx) + 2G_K(y,y,Ty)]\}, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0,1)$ such that

$$\begin{aligned} \lambda = \sup\{&\alpha(x,y,y) + 3\beta(x,y,y) + \gamma(x,y,y) \\ &+(2\gamma(x,y,y) + 9\delta(x,y,y) + 12\sigma(x,y,y))K : x,y \in X\} < 1. \end{aligned}$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$ with an error bound $G_K(T^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} K G_K(x, Tx, Tx)$.

Proof. Assume we have the case

$$\begin{aligned}&\max\{\alpha(x,y,y)G_K(x,y,y),\beta(x,y,y)[G_K(x,Tx,Tx)+2G_K(y,Ty,Ty)],\\&\qquad\gamma(x,y,y)[G_K(x,Ty,Ty)+G_K(y,Ty,Ty)+G_K(y,Tx,Tx)],\\&\qquad\delta(x,y,y)[G_K(y,y,Tx)+G_K(y,y,Ty)+G_K(x,x,Ty)],\\&\qquad\sigma(x,y,y)[G_K(x,x,Tx)+2G_K(y,y,Ty)]\}\\&\quad=\alpha(x,y,y)G_K(x,y,y).\end{aligned}$$

Then we have $G_K(Tx,Ty,Ty)\leq\alpha(x,y,y)G_K(x,y,y)$, and by taking $\beta=\gamma=\delta=\sigma=0$, we get $\lambda=\sup\{\alpha(x,y,y):x,y\in X\}<1$, and the proof follows from proof of Corollary 3.1. Similar proofs hold for other cases, e.g., if the maximum is $\beta(x,y,y)[G_K(x,Tx,Tx)+2G_K(y,Ty,Ty)]$, then the same proof of Theorem 3.1 holds with $\alpha=\gamma=\delta=\sigma=0$, and $\lambda=\sup\{3\beta(x,y,y):x,y\in X\}<1$, i.e., $\sup\{\beta(x,y,y):x,y\in X\}<\frac{1}{3}$. See also Remark 3.1. □

Corollary 3.3. *Let X be a complete G_K-cone metric space with a normal constant $K\geq 1$, and let $T:X\to X$ be a mapping satisfying, for each $x,y\in X$,*

(3.10)

$$\begin{aligned}G_K(Tx,Ty,Ty)\ &\leq\alpha(x,y,y)G_K(x,y,y)\\&\quad+\beta(x,y,y)[G_K(x,Tx,Tx)+G_K(y,Ty,Ty)]\\&\quad+\gamma(x,y,y)[G_K(x,Ty,Ty)+G_K(y,Ty,Ty)+G_K(y,Tx,Tx)]\\&\quad+\delta(x,y,y)[G_K(y,y,Tx)+G_K(y,y,Ty)+G_K(x,x,Ty)]\\&\quad+\sigma(x,y,y)[G_K(x,x,Tx)+2G_K(y,y,Ty)],\end{aligned}$$

where $\alpha,\beta,\gamma,\delta,\sigma$ are some functions from $X\times X\times X$ into $[0,1)$ such that

$$\begin{aligned}\lambda=\sup\{&\alpha(x,y,y)+3\beta(x,y,y)+\gamma(x,y,y)\\&+(2\gamma(x,y,y)+9\delta(x,y,y)+12\sigma(x,y,y))K:x,y\in X\}<1.\end{aligned}$$

Then T has a unique fixed point, say $u\in X$. Moreover, $T^nx\to u$ as $n\to\infty$ for all $x\in X$ with an error bound $G_K(T^nx,u,u)\leq\frac{\lambda^n}{1-\lambda}KG_K(x,Tx,Tx)$.

Proof. From Theorem 3.1, one can see that every map satisfies (3.10) would definitely, satisfy (3.1), and hence, the proof is straightforward from proof of Theorem 3.1. □

Remark 3.2. In condition (3.10), we remove the coefficient 2 of $G_K(y,Ty,Ty)$ which was in condition (3.1). Therefore, following the proof of Theorem 3.1, one would realize that λ in Corollary 3.3, could be replaced by

$$\lambda = \sup\{\alpha(x,y,y) + 2\beta(x,y,y) + \gamma(x,y,y) \\ +(2\gamma(x,y,y) + 9\delta(x,y,y) + 12\sigma(x,y,y))K : x,y \in X\} < 1.$$

where we have 2β instead of 3β, and the proof follows straightforwardly. Similarly, one could also remove the coefficient 2 of $G_K(y,y,Ty)$ in condition (3.1) and similar Corollary as Corollary 3.3, and similar arguments as those for Corollary 3.3 follow straightforwardly.

Corollary 3.4. *Let X be a complete G_K-cone metric space with a normal constant $K \geq 1$, and let $T : X \to X$ be a mapping satisfying for some $m \in \mathbb{N}$, for each $x, y \in X$,*

(3.11)

$$\begin{aligned} G_K(T^m x, T^m y, T^m y) \leq\ & \alpha(x,y,y)G_K(x,y,y) \\ & +\beta(x,y,y)[G_K(x,T^m x,T^m x) + 2G_K(y,T^m y,T^m y)] \\ & +\gamma(x,y,y)[G_K(x,T^m y,T^m y) + G_K(y,T^m y,T^m y) + G_K(y,T^m x,T^m x)] \\ & +\delta(x,y,y)[G_K(y,y,T^m x) + G_K(y,y,T^m y) + G_K(x,x,T^m y)] \\ & +\sigma(x,y,y)[G_K(x,x,T^m x) + 2G_K(y,y,T^m y)], \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0,1)$ such that

$$\lambda = \sup\{\alpha(x,y,y) + 3\beta(x,y,y) + \gamma(x,y,y) \\ +(2\gamma(x,y,y) + 9\delta(x,y,y) + 12\sigma(x,y,y))K : x,y \in X\} < 1.$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$ with an error bound $G_K(T^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} K G_K(x, Tx, Tx)$.

Proof. From Theorem 3.1, T^m has a unique fixed point, say u, i.e., $T^m u = u$, and since $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$, we have Tu as another fixed point for T^m, and by uniqueness, $Tu = u$. The rest of the proof follows similarly. □

Theorem 3.2. *Let X be a complete G_K-cone metric space with a normal constant $K \geq 1$, and let $T : X \to X$ be a mapping satisfying, for each $x, y, z \in X$,*

$$\begin{aligned} G_K(Tx,Ty,Tz) \ &\leq \alpha(x,y,z)G_K(x,y,z) \\ &+\beta(x,y,z)[G_K(x,Tx,Tx)+G_K(y,Ty,Ty)+G_K(z,Tz,Tz)] \\ &+\gamma(x,y,z)[G_K(x,Ty,Ty)+G_K(y,Tz,Tz)+G_K(z,Tx,Tx)] \\ &+\delta(x,y,z)[G_K(y,y,Tx)+G_K(z,z,Ty)+G_K(x,x,Tz)] \\ &+\sigma(x,y,z)[G_K(x,x,Tx)+G_K(y,y,Ty)+G_K(z,z,Tz)], \end{aligned} \tag{3.12}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0, 1)$ such that

$$\begin{aligned} \lambda = \sup\{&\alpha(x,y,z)+3\beta(x,y,z)+\gamma(x,y,z) \\ &+(2\gamma(x,y,z)+9\delta(x,y,z)+12\sigma(x,y,z))K : x,y,z \in X\} < 1. \end{aligned}$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$ with an error bound $G_K(T^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} K G_K(x, Tx, Tx)$.

Proof. Taking $z = y$ in (3.12) leads to the condition (3.1) in Theorem 3.1, and hence, the proof follows directly from proof of Theorem 3.1. □

Note that, previous corollaries following Theorem 3.1 also follow Theorem 3.2 straightforwardly.

We give in what follows some examples to validate our results.

Example 3.1. *Let $X = [-1, 1] \subset \mathbb{R}$, $E = \mathbb{R}$, and $P = \{x \in \mathbb{R} : x \geq 0\}$ be a cone in E. Define the mapping $G : X \times X \times X \to E$ by*

$$G(x,y,z) = d(x,y) + d(y,z) + d(x,z), \forall x,y,z \in X,$$

with $d(x,y) = |x-y|$. *Hence,* $G_K(x,y,z) = |x-y| + |y-z| + |x-z|$. *Let* $T : X \to X$ *be given by*

$$T(x) = \begin{cases} -\frac{1}{6}xe^{-\frac{1}{|x|}}, & x \in [-1,0) \cup (0,1], \\ 0, & x = 0. \end{cases}$$

Then, for all $x, y \in X$, *we have*

$$\begin{aligned} G_K(Tx,Ty,Ty) &= |Tx-Ty| + |Tx-Ty| + |Ty-Ty| \\ &= |Tx-Ty| + |Tx-Ty| \\ &= \left| -\frac{1}{6}xe^{-\frac{1}{|x|}} + \frac{1}{6}ye^{-\frac{1}{|y|}} \right| + \left| -\frac{1}{6}xe^{-\frac{1}{|x|}} + \frac{1}{6}ye^{-\frac{1}{|y|}} \right| \\ &\leq \frac{1}{6}|x| + \frac{1}{6}|y| + \frac{1}{6}|x| + \frac{1}{6}|y| \\ &\leq \frac{1}{6}\left| x + \frac{1}{6}xe^{-\frac{1}{|x|}} \right| + \frac{1}{6}\left| y + \frac{1}{6}ye^{-\frac{1}{|y|}} \right| \\ &\qquad + \frac{1}{6}\left| x + \frac{1}{6}xe^{-\frac{1}{|x|}} \right| + \frac{1}{6}\left| y + \frac{1}{6}ye^{-\frac{1}{|y|}} \right| \\ &= \frac{1}{6}|Tx-x| + \frac{1}{6}|Ty-y| + \frac{1}{6}|Tx-x| + \frac{1}{6}|Ty-y| \\ &= \frac{1}{6}(|Tx-x| + |Tx-x|) + \frac{1}{6}(|Ty-y| + |Ty-y|) \\ &= \frac{1}{6}(G_K(x,Tx,Tx) + G_K(y,Ty,Ty)) \\ &\leq \frac{1}{6}(G_K(x,Tx,Tx) + 2G_K(y,Ty,Ty)). \end{aligned}$$

If we take $\beta = \frac{1}{6} \in [0,1)$, *and* $\alpha = \gamma = \delta = \sigma = 0$ *in Theorem 3.1, then the contraction condition is satisfied with* $\lambda = 3\beta < 1$, *and* T *has a unique fixed point in* $[-1,1]$, *namely* $u = 0$.

Similarly, one can see that for all $x, y, z \in X$, *we have*

$$\begin{aligned} G_K(Tx,Ty,Tz) &= |Tx-Ty| + |Tx-Tz| + |Ty-Tz| \\ &\leq \frac{1}{6}(G_K(x,Tx,Tx) + G_K(y,Ty,Ty) + G_K(z,Tz,Tz)), \end{aligned}$$

and if we take $\beta = \frac{1}{6} \in [0,1)$*, and* $\alpha = \gamma = \delta = \sigma = 0$ *in Theorem 3.2, then the contraction condition is satisfied with* $\lambda = 3\beta < 1$*, and* T *has a unique fixed point in* $[-1,1]$*, namely* $u = 0$*.*

Example 3.2. *Consider the* $G-$*cone metric space introduced in Example 2.2, where* $E = \mathbb{R}^2$*,* $P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$ *a cone in* E*,* $X \subseteq \mathbb{R}$*, and* $G : X \times X \times X \to E$ *be defined by*

$$G(x,y,z) = (\frac{1}{3}(|x-y|+|y-z|+|x-z|), \frac{2}{3}(|x-y|+|y-z|+|x-z|)), \forall x,y,z \in X.$$

Let $X = [-1,1]$ *and* $T : X \to X$ *be given by*

$$T(x) = \begin{cases} -\frac{1}{6}xe^{-\frac{1}{|x|}}, & x \in [-1,0) \cup (0,1], \\ 0, & x = 0. \end{cases}$$

Then, considering the one norm implies

$$G_K(x,y,z) = \|G(x,y,z)\|_1 = |x-y|+|y-z|+|x-z|,$$

and considering the infinity norm gives

$$G_K(x,y,z) = \|G(x,y,z)\|_\infty = \frac{2}{3}(|x-y|+|y-z|+|x-z|).$$

In both cases, using similar arguments as those in Example 3.1, one can see that T *has a unique fixed point in* $[-1,1]$*, namely* $u = 0$*.*

4. CONCLUSIONS

We propose in this paper new fixed point results, with error bounds for convergence. Our results generalize, extend, and unify some other results in the literature in the sense that they are analogous to those for cone metric spaces, but in a more general setting, where we have here G-cone metric spaces, or in the sense that they are extension or generalization of some other results proved in G-metric spaces. To demonstrate this, one could find for example that our proved theorem, Theorem 3.2, is in a more general setting than the proved theorem, namely, Theorem 1, in [11] which is proved in cone metric space. Another example is the existence of a unique fixed point results in [10], namely, Theorem 3.1 and 3.2, which are direct consequences from our results, i.e., special cases from our results that are more general where we have more terms, as well as coefficients which are functions but not only constants. As a last example, one could verify easily that our proved theorem, Theorem 3.2, unifies and extends the proved existence of a unique fixed point result, namely, Theorem 2.1, in [14], and therefore, this result is a consequence from our result, i.e., our result is more general. Hence, many results in the literature could be added here as corollaries to our results.

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