CHARACTERIZATIONS OF DIMENSION FUNCTIONS OF WAVELET PACKETS

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ABSTRACT. This paper deals with the characterizations of dimension functions of wavelet packets. As a corollary, I prove that if ω_n are orthonormal wavelet packets such that $|\hat{\omega}_n|$ is continuous and $|\hat{\omega}_n(\xi)| = O\left(|\xi|^{-1/2-\alpha}\right)$ at ∞ for some $\alpha > 0$, then ω_n are MRA wavelet packets.

1. INTRODUCTION

A wavelet is a function $\psi \in L^2(\mathbb{R})$ such that $\left\{\psi_{j,k} = 2^{j/2}\psi\left(2^j \cdot -k\right)\right\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. The dimension function of an orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$D_\psi(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\hat{\psi}\left(2^j(\xi+2k\pi)\right)\right|^2.$$

The function D_ψ is well defined and is finite a.e. The importance of the dimension function was discovered by Lemarié, who used it to prove that certain wavelets are associated with a MRA of $L^2(\mathbb{R})$ [16,17]. Auscher [5] proved that if ψ is a wavelet, then the function D_ψ is the dimension of certain closed subspaces of the sequence $\ell^2(\mathbb{Z})$ (hence the name dimension function, a term coined by Guido Weiss). This

result, in particular, proves that D_ψ is integer valued a.e. Gripenberg [13] and Wang [20], independently, characterized all wavelets of $L^2(\mathbb{R})$ associated with an MRA. This well known characterization states that a wavelet ψ of $L^2(\mathbb{R})$ is associated with an MRA if and only if $D_\psi = 1$ a.e. In [6], Bownik studied characterizations of all dimension functions.

Wavelet packet analysis is an important generalization of wavelet analysis, pioneered by Coifman, Meyer, Wickerhauser and other researchers [10, 11, 12, 21]. Discrete Wavelet packets have been thoroughly studied by M.V. Wickerhauser [22] who has also developed computer programmes and implemented them. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Wavelet packets are organized naturally into collections, and each collection is an orthogonal basis for $L^2(\mathbb{R})$.

Wavelet packet functions are generated by scaling and translating a family of basic function shapes, which include father wavelet φ and mother wavelet ψ. In addition to φ and ψ there is a whole range of wavelet packet functions ω_n. These functions are parametrized by an oscillation or frequency index n. A father wavelet corresponds to $n = 0$, so $\varphi = \omega_0$. A mother wavelet corresponds to $n = 1$, so $\psi = \omega_1$. Larger values of n correspond to wavelet packets with more oscillations and higher frequency.

Very recently Ahmad and Kumar have studied band-limited wavelet packets in [1] and pointwise convergence of wavelet packet series in [2]. Jarrah, Kumar and Ahmad have studied certain characterization of wavelet packets in [15]. Fourier transforms of wavelet packets have been studied by Ahmad, Kumar and Debnath in [3]. Ahmad, Kumar and Debnath have also studied existence of unconditional wavelet packet bases in [4]. In the present, paper I study characterizations of dimension functions of

wavelet packets. Results in the paper are generalizations of the results of Gripenberg [13], Hernández and Weiss [14].

2. Preliminaries

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and real numbers, respectively and $\mathbb{T}$ denote the unit circle in the complex plane, which can be identified with the interval $[-\pi, \pi)$. For basic ideas, results on wavelets, wavelet packets and multiresolution analysis, we refer to $[1, 2, 3, 4, 7, 8, 9, 14, 15, 18]$.

Lemma 2.1 [19]. Let $\varphi \in L^2(\mathbb{R})$ be a scaling function. Then $\{\varphi(. - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0 if and only if

$$\sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1 \text{ for a.e. } \xi \in \mathbb{R}.$$

Lemma 2.2 [3]. If $\omega_n \in L^2(\mathbb{R})$ are wavelet packets associated with the scaling function $\varphi = \omega_0$, then

$$(2.1) \quad \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \hat{\omega}_n \left(2^\ell(\xi + 2k\pi) \right) \right|^2 = 1 \text{ for a.e. } \xi \in \mathbb{R},$$

where $\ell = j - u$, $u = 0$ if $j < 0$ and $u = 0, 1, 2, ..., j$ if $j \geq 0$.

Lemma 2.3 [3]. For orthonormal wavelet packets $\omega_n \in L^2(\mathbb{R})$ the expression

$$D(\xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \hat{\omega}_n \left(2^\ell(\xi + 2k\pi) \right) \right|^2,$$

where $\ell = j - u$, $u = 0, 1, 2, ..., j$ if $j > 0$, $j \in \mathbb{Z}^+$ is well defined and finite for almost every $\xi \in \mathbb{R}$. Moreover,

$$(2.2) \quad \int_I D(\xi)d\xi = 2\pi$$

for any interval I of length 2π in $\mathbb{R}$.

Lemma 2.4 [1]. Let $\omega_n \in L^2(\mathbb{R})$ are wavelet packets for all $n = 0, 1, 2, ...$ Then

(2.3) $\displaystyle\sum_{k \in \mathbb{Z}} |\hat{\omega}_n(\xi + 2k\pi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$,

and

(2.4) $\displaystyle\sum_{k \in \mathbb{Z}} \hat{\omega}_t \left(2^\ell(\xi + 2k\pi) \right) \overline{\hat{\omega}_n(\xi + 2k\pi)} = 0$ for a. e. $\xi \in \mathbb{R}$, $\ell \geq 1$,

are necessary and sufficient conditions for the orthonormality of the system

$\{\omega_{\ell,n,k} : n = 2^u, 2^u + 1, ..., 2^{u+1} - 1, \ \ell = j - u, \ j, k \in \mathbb{Z}\}$, where $u = 0$ if $j < 0$ and $u = 0, 1, 2, ..., j$ if $j \geq 0$.

Lemma 2.5 [15]. Let $\{\omega_n\}_{n \in \mathbb{N}}$ be a normal sequence of wavelet packets of $L^2(\mathbb{R})$ and $\omega_{\ell,n,k}$ are given by $\{\omega_{l,n,k}(x)\} = 2^{l/2}\omega_n\left(2^l x - k\right)\}$. Then, the function ω_n is an orthonormal wavelet packet if and only if

(2.5) $\displaystyle\sum_{\ell \in \mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left| \hat{\omega}_n(2^\ell \xi) \right|^2 = 1$ for a.e. $\xi \in \mathbb{R}$,

where $\ell = j - u$, $u = 0$ if $j \leq 0$ and $u = 0, 1, 2, ..., j$ if $j > 0$, $j \in \mathbb{Z}$ and

(2.6) $\displaystyle\sum_{\ell=0}^{\infty} \sum_{n=2^u}^{2^{u+1}-1} \hat{\omega}_n(2^\ell \xi) \overline{\hat{\omega}_n\left(2^\ell(\xi + 2m\pi)\right)} = 0$ for a.e. $\xi \in \mathbb{R}$, $m \in 2\mathbb{Z} + 1$.

Lemma 2.6 [14: p. 359]. Let C be a positive integer and let $\{v_j : j \geq 1\}$ be a family of vectors in a Hilbert space $\mathbb{H}$ such that

(1) $\displaystyle\sum_{n=1}^{\infty} \|v_n\|^2 = C$ and

(2) $v_n = \displaystyle\sum_{m=1}^{\infty} \langle v_n, \ v_m \rangle v_m$ for all $n \geq 1$

Let $\mathbb{F} = \overline{\text{span}\{v_j : j \geq 1\}}$. Then, $\dim \mathbb{F} = \sum_{j=1}^{\infty} \|v_j\|^2 = C$ (Number of basis elements of F).

3. Characterizations of dimension functions

The dimension functions of wavelet packets $\omega_n \in L^2(\mathbb{R})$ associated with a dilation by 2 is the function D_{ω_β} given by

$$(3.1) \quad D_{\omega_\beta}(\xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)\right|^2,$$

where $\beta = u + 1$ and $\ell = j - u$, $u = 0, 1, 2, ..., j$ if $j > 0$, $j \in \mathbb{Z}^+$.

Theorem 3.1. The functions $\omega_n \in L^2(\mathbb{R})$ are MRA wavelet packets if and only if $D_{\omega_\beta}(\xi) = 1$ for almost every $\xi \in \mathbb{R}$.

Proof. From Lemma 2.2, we have

$$\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)\right|^2 = 1 \Rightarrow D_{\omega_\beta}(\xi) = 1, \text{ a.e.}$$

where ω_n are MRA wavelet packets. Now, to complete the proof of Theorem 3.1 it is sufficient to show that $D_{\omega_\beta}(\xi) = 1$ a.e., so that the wavelet packets are MRA wavelet packets. I shall break up the proof of this into several lemmas.

Lemma 3.2. If ω_n are orthonormal wavelet packets, then

$$(3.2) \quad \hat{\omega}_t(2^m \xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \hat{\omega}_t \left(2^m(\xi + 2k\pi)\right) \overline{\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)} \hat{\omega}_n \left(2^\ell \xi\right)$$

a.e. for all $m \geq 1$ and $t = 2^u, 2^u + 1, ..., 2^{u+1} - 1$, where $\ell = j - u$, $u = 0$ if $j \leq 0$ and $u = 0, 1, 2, ..., j$ if $j > 0$, $j \in \mathbb{Z}$.

Proof. First of all we show that the series in (3.2) is well defined. By using Schwarz's inequality and (2.3), we obtain

$$
\sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \left| \hat{\omega}_t \left(2^m(\xi+2k\pi)\right) \overline{\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)} \right|
$$

$$
\leq \left(\sum_{k\in\mathbb{Z}} \left|\hat{\omega}_t \left(2^m(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}} \sum_{n=2^u}^{2^{u+1}-1} \left(\sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left(\sum_{s\in\mathbb{Z}} \left|\hat{\omega}_t \left(2^m\xi+2s\pi\right)\right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}}
$$

$$
= \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}}.
$$

Taking the sum over all $\ell \geq 1$, using Schwarz's inequality and (2.5), we obtain

$$
\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left| \hat{\omega}_t \left(2^m(\xi+2k\pi)\right) \overline{\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)} \hat{\omega}_n \left(2^\ell\xi\right) \right|
$$

$$
\leq \sum_{\ell=1}^{\infty} \sum_{n=2^u}^{2^{u+1}-1} \left(\sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}} \left(\left|\hat{\omega}_n \left(2^\ell\xi\right)\right| \right)
$$

$$
\leq \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \left|\hat{\omega}_n \left(2^\ell\xi\right)\right|^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\hat{\omega}_n \left(2^\ell(\xi+2k\pi)\right)\right|^2 \right)^{\frac{1}{2}} .1 = \sqrt{D_{\omega_\beta}(\xi)}.
$$

But all the above inequalities are true for a.e. $\xi \in \mathbb{R}$. Therefore, the series in (3.2) is well defined almost everywhere. Let $G_m(\xi)$ be the RHS of (3.2). Then, we show that $G_m(\xi) = \hat{\omega}_t (2^m \xi)$ for a.e. $\xi \in \mathbb{R}$. First of all we show that $G_m(\xi) = G_{m-1}(2\xi)$, and, then, that $G_1(\xi) = \hat{\omega}_t (2\xi)$. Clearly, this gives (3.2).

Replacing ℓ by m in (2.4) and then using it, we get

$$
\begin{aligned}
G_m(\xi) &= \sum_{k \in \mathbb{Z}} \hat{\omega}_t \left(2^m(\xi + 2k\pi)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \overline{\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)} \, \hat{\omega}_n \left(2^\ell \xi\right) \\
&= \sum_{n=2^u}^{2^{u+1}-1} \sum_{k \in \mathbb{Z}} \hat{\omega}_t \left(2^m(\xi + 2k\pi)\right) \overline{\hat{\omega}_n(\xi + 2k\pi)} \, \hat{\omega}_n(\xi) \\
&\quad + \sum_{k \in \mathbb{Z}} \hat{\omega}_t \left(2^m(\xi + 2k\pi)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \overline{\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)} \, \hat{\omega}_n \left(2^\ell \xi\right) \\
&= \sum_{k \in \mathbb{Z}} \hat{\omega}_t \left(2^m(\xi + 2k\pi)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=0}^{\infty} \overline{\hat{\omega}_n \left(2^\ell(\xi + 2k\pi)\right)} \, \hat{\omega}_n \left(2^\ell \xi\right).
\end{aligned}
$$

By (2.6), the terms in the summation over ℓ where k is odd are zero a.e. Therefore, on replacing k by $2s$, we get

$$
\begin{aligned}
G_m(\xi) \;&=\; \sum_{s\in\mathbb{Z}} \hat{\omega}_t\left(2^m(\xi+4s\pi)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=0}^{\infty} \overline{\hat{\omega}_n\left(2^\ell(\xi+4s\pi)\right)} \,\hat{\omega}_n\left(2^\ell\xi\right) \\[2mm]
&=\; \sum_{s\in\mathbb{Z}} \hat{\omega}_t\left(2^{m+1}\left(\frac{\xi}{2}+2s\pi\right)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=0}^{\infty} \overline{\hat{\omega}_n\left(2^{\ell+1}\left(\frac{\xi}{2}+2s\pi\right)\right)} \\
&\quad \times \hat{\omega}_n\left(2^{\ell+1}\frac{\xi}{2}\right) \\[2mm]
&=\; \sum_{s\in\mathbb{Z}} \hat{\omega}_t\left(2^{m+1}\left(\frac{\xi}{2}+2s\pi\right)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \overline{\hat{\omega}_n\left(2^\ell\left(\frac{\xi}{2}+2s\pi\right)\right)} \\
&\quad \times \hat{\omega}_n\left(2^\ell\frac{\xi}{2}\right) \\[2mm]
&=\; G_{m+1}\left(\frac{\xi}{2}\right).
\end{aligned}
$$

This shows that $G_m(\xi) = G_{m-1}(2\xi)$ almost everywhere.

We, now, calculate $G_1(\xi)$. Changing variables in the sum over ℓ, we obtain

$$
\begin{aligned}
G_1(\xi) \;&=\; \sum_{k\in\mathbb{Z}} \hat{\omega}_t\left(2(\xi+2k\pi)\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \overline{\hat{\omega}_n\left(2^\ell(\xi+2k\pi)\right)} \,\hat{\omega}_n\left(2^\ell\xi\right) \\[2mm]
&=\; \sum_{k\in\mathbb{Z}} \hat{\omega}_t\left(2\xi+4k\pi\right) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=0}^{\infty} \overline{\hat{\omega}_n\left(2^\ell\left(2\xi+4k\pi\right)\right)} \hat{\omega}_n\left(2^\ell 2\xi\right).
\end{aligned}
$$

Further, in the last sum over k we add all the corresponding terms with $2k$ replaced by $2k+1$, which are zero by (2.6). This gives us

$$
G_1(\xi) = \sum_{k\in\mathbb{Z}} \hat{\omega}_t(2\xi+2k\pi) \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=0}^{\infty} \overline{\hat{\omega}_n\left(2^\ell(2\xi+2k\pi)\right)} \,\hat{\omega}_n\left(2^\ell 2\xi\right).
$$

Interchanging the order of summation, using (2.4) when $\ell \geq 1$ and (2.3) when $\ell = 0$ we obtain $G_1(\xi) = \hat{\omega}_t(2\xi)$.

This completes the proof of Lemma 3.2.

Now, consider $\ell^2(\mathbb{Z})$, and denote its (usual) norm by $\|\cdot\|_{\ell^2}$ and the inner product by $\langle\cdot,\cdot\rangle_{\ell^2}$. If ω_n are orthonormal wavelet packets, we define the vector

$$\Psi_{n,\ell}(\xi) = \left\{\hat\omega_n\left(2^\ell(\xi+2k\pi)\right) : n = 2^u, 2^u+1, ..., 2^{u+1}-1; \; \ell = j-u;\right.$$
$$\left. j, k \in \mathbb{Z};\right\}, \ell \geq 1.$$

But (2.3) implies that

$$\|\Psi_{n,\ell}(\xi)\|_{\ell^2} = \left(\sum_{k\in\mathbb{Z}}\left|\hat\omega_n\left(2^\ell(\xi+2k\pi)\right)\right|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{s\in\mathbb{Z}}\left|\hat\omega_n\left(2^\ell\xi+2s\pi\right)\right|^2\right)^{\frac{1}{2}}$$

$$= 1 \text{ for a.e.} \xi \in \mathbb{R}.$$

Therefore, for almost every ξ the vector $\Psi_{n,\ell}(\xi) \in \ell^2(\mathbb{Z})$.

Let $\mathbb{F}_{\omega_\beta}(\xi)$ be the closure of the span of the set of vectors $\{\Psi_{n,\ell}(\xi) : \ell \geq 1;$ $n = 2^u, 2^u+1, ..., 2^{u+1}-1\}$. Then, $\mathbb{F}_{\omega_\beta}(\xi)$ is a well defined subspace of $\ell^2(\mathbb{Z})$ for almost every $\xi \in \mathbb{R}$. We can rewrite (3.2) in terms of the above notation as

$$\hat\omega_t(2^m\xi) = \sum_{n=2^u}^{2^{u+1}-1}\sum_{\ell=1}^{\infty}\langle\Psi_{t,m}(\xi), \Psi_{n,\ell}(\xi)\rangle_{\ell^2}\hat\omega_n(2^\ell\xi) \text{ for a.e. } \xi \in \mathbb{R},$$

for all $t = 2^u, 2^u+1, ..., 2^{u+1}-1$. In the above replacing ξ by $\xi+2s\pi$ we obtain, for $n \geq 1$,

$$\hat\omega_t(2^m(\xi+2s\pi)) = \sum_{n=2^u}^{2^{u+1}-1}\sum_{\ell=1}^{\infty}\langle\Psi_{t,m}(\xi), \Psi_{n,\ell}(\xi)\rangle_{\ell^2}\hat\omega_n(2^\ell(\xi+2s\pi)) \text{ a.e.}$$

But $\langle \Psi_{t,m}(\xi), \Psi_{n,\ell}(\xi) \rangle_{\ell^2}$ is 2π-periodic. Therefore, we can write this equality vectorially as

$$(3.3) \ \Psi_{t,m}(\xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \langle \Psi_{t,m}(\xi), \Psi_{n,\ell}(\xi) \rangle_{\ell^2} \Psi_{n,\ell}(\xi).$$

Further, since $D_{\omega_\beta}(\xi)$ is finite a.e., simple calculation shows that

$$(3.4) \ D_{\omega_\beta}(\xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \|\Psi_{n,\ell}(\xi)\|_{\ell^2}^2 \,.$$

Let S be the subset of $\mathbb{T}$ on which $D_{\omega_\beta}(\xi) < \infty$. Then, the vectors $\Psi_{n,\ell}(\xi)$, $l \geq 1$, are well defined on S (observe that $|S| = 2\pi$). For $\xi \in S$, let $\mathbb{F}_{\omega_\beta}(\xi)$ be closure, in $\ell^2(\mathbb{Z})$, of the span of $\{\Psi_{n,\ell}(\xi) : \ell \geq 1; n = 2^u, 2^u + 1, ..., 2^{u+1} - 1\}$. Then the hypothesis of Lemma 2.6 are satisfied if $v_\ell = \Psi_{n,\ell}(\xi)$. This gives us

$$(3.5) \ \dim \mathbb{F}_{\omega_\beta}(\xi) = D_{\omega_\beta}(\xi) \text{ on } S.$$

Now, we are now ready to prove the sufficient part of Theorem 3.1. Let $\omega_n \in L^2(\mathbb{R})$ be wavelet packets for which $D_{\omega_\beta}(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$. Then, by (3.5), $\dim \mathbb{F}_{\omega_\beta}(\xi) = 1$ for a.e. $\xi \in \mathbb{T}$. This shows that, for each $\xi \in S$, $\mathbb{F}_{\omega_\beta}(\xi)$ is generated by a single unit vector $U(\xi)$. We now choose a particular one. For $\ell \geq 1$, let

$$E_\ell = \left\{ \xi \in S : \Psi_{n,\ell}(\xi) \neq \vec{0} \text{ and } \Psi_{n,r}(\xi) = \vec{0} \text{ for all } r < \ell \right\}.$$

The sets E_ℓ, for $\ell \geq 1$, are mutually disjoint and together with

$$E_0 = \left\{ \xi \in \mathbb{T} : D_{\omega_\beta}(\xi) = 0 \right\},$$

form a partition of S. Hence, for $\xi \in S \backslash E_0$, there exists a unique $\ell \geq 1$ such that $\xi \in E_\ell$. But E_0 has measure 0. Therefore

$$U(\xi) = \frac{1}{\|\Psi_{t,\ell}(\xi)\|_{\ell^2}} \Psi_{t,\ell}(\xi), \ \xi \in E_\ell \text{ for some } \ell \geq 1,$$

is well defined and $\|U(\xi)\|_{\ell^2} = 1$ for almost every $\xi \in \mathbb{T}$. Write

$$U(\xi) = \{u_k : k \in \mathbb{Z}\}.$$

If we find the scaling function $\varphi = \omega_0$, in view of Lemma 2.1, we hope that $u_k(\xi) = \hat{\omega}_0(\xi + 2k\pi)$. Thus, we let

$$\hat{\omega}_0(\xi) = u_k(\xi - 2k\pi) \text{ if } \xi \in \mathbb{T} + 2k\pi \text{ for some } k \in \mathbb{Z}.$$

This defines $\hat{\omega}_0$ on $\mathbb{R}$. We claim $\hat{\omega}_0 \in L^2(\mathbb{R})$:

$$\|\hat{\omega}_0\|_2^2 = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} |\hat{\omega}_0(\xi + 2k\pi)|^2 d\xi = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} |u_k(\xi)|^2 d\xi = \int_{\mathbb{T}} \|U(\xi)\|_{\ell^2}^2 d\xi = 2\pi$$

since $U(\xi)$ is a unit vector. We also have

(3.6) $\quad \displaystyle\sum_{k\in\mathbb{Z}} |\hat{\omega}_0(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |u_k(\xi)|^2 = \|U(\xi)\|_{\ell^2}^2 = 1 \text{ for a.e. } \xi \in \mathbb{R},$

which, is equivalent to the fact that $\{\omega_0(. - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. Define $V_0^{\#}$ as the closed subspace of $L^2(\mathbb{R})$ generated by $\{\varphi(. - k) : k \in \mathbb{Z}\}$. We claim that

(3.7) $V_0^{\#} = V_0 = \displaystyle\bigoplus_{j<0} W_j.$

From this it follows that $\{V_j : j \in \mathbb{Z}\}$ is the desired MRA.

For each $\ell \geq 1$, there exists a measurable function ν_ℓ, defined on $\mathbb{T}$, such that

$$\Psi_{t,\ell}(\xi) = \nu_{t,\ell}(\xi) U(\xi) \text{ for a.e. } \xi \in \mathbb{T}.$$

Componentwise,

$$\hat{\omega}_t \left(2^\ell(\xi + 2k\pi) \right) = \nu_{t,\ell}(\xi)\, \hat{\omega}_0(\xi + 2k\pi) \text{ for a.e. } \xi \in \mathbb{T},\ k \in \mathbb{Z}.$$

Hence, by (3.6), for a.e. $\xi \in \mathbb{T}$,

$$(3.8) \quad \sum_{k \in \mathbb{Z}} \left| \hat{\omega}_t \left(2^\ell(\xi + 2k\pi) \right) \right|^2 = \sum_{k \in \mathbb{Z}} |\nu_{t,\ell}(\xi)|^2\, |\hat{\omega}_0(\xi + 2k\pi)|^2 = |\nu_{t,\ell}(\xi)|^2,$$

which shows that $\nu_{t,\ell} \in L^2(\mathbb{T})$ with $\|\nu_{t,\ell}\|_{L^2(\mathbb{T})}^2 = 2^{-\ell}(2\pi)$. Write the Fourier series of $\nu_{t,\ell},\ \ell \geq 1$, as

$$\nu_{t,\ell}(\xi) = \sum_{k \in \mathbb{Z}} a_k^{t,\ell}\, e^{-ik\xi} \text{ for a.e. } \xi \in \mathbb{T},$$

with convergence in the $L^2(\mathbb{T})$-norm, and $\{a_k^{t,\ell}\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$. Extending $\nu_{t,\ell},\ 2\pi$-periodically, we obtain

$$(3.9) \quad \hat{\omega}_t \left(2^\ell \xi \right) = \nu_{t,\ell}(\xi)\hat{\omega}_0(\xi) \text{ for a.e. } \xi \in \mathbb{R},\ \ell \geq 1.$$

Taking inverse Fourier transform on both sides, we obtain

$$\omega_{-\ell,t,0}(x) = 2^{-\ell/2}\omega_t(2^{-\ell}x) = 2^{\ell/2} \sum_{k \in \mathbb{Z}} a_k^{t,\ell}\omega_0(x - k),\ \ell \geq 1.$$

Hence, $\omega_{-\ell,t,0} \in V_0^{\#}$ for $\ell \geq 1$. Since $V_0^{\#}$ is invariant under integral translations and $\omega_{-\ell,t,k}(x) = 2^{-\ell/2}\omega_t \left(2^{-\ell} \left(x - 2^\ell k \right) \right)$, we have $\omega_{-\ell,t,k} \in V_0^{\#}$ for all $t = 2^u, 2^u + 1, ..., 2^{u+1} - 1,\ k \in \mathbb{Z}$ and $\ell \geq 1$. Thus, $W_{-\ell} \subseteq V_0^{\#}$ for all $\ell \geq 1$ and, hence $V_0 \subseteq V_0^{\#}$.

Now, we need to show that $V_0^{\#} \subseteq V_0$. We do this by showing ω_0 is perpendicular to W_j for all $j \geq 0$. For $j \geq 0$ and $s \in \mathbb{Z}$, the Plancherel theorem, a change of variables and a periodization argument allow us to write

$$2\pi \sum_{n=2^u}^{2^{u+1}-1} \langle \omega_0 \; \omega_{\ell,n,s} \rangle \;=\; \sum_{n=2^u}^{2^{u+1}-1} \langle \hat{\omega}_0 \; (\omega_{\ell,n,s})^\wedge \rangle$$

$$= \sum_{n=2^u}^{2^{u+1}-1} 2^{-\frac{\ell}{2}} \int_{\mathbb{R}} \hat{\omega}_0(\xi) \overline{\hat{\omega}_n(2^{-\ell}\xi)} e^{i2^{-\ell}s\xi} \, d\xi$$

$$= \sum_{n=2^u}^{2^{u+1}-1} 2^{\frac{\ell}{2}} \int_{\mathbb{R}} \hat{\omega}_0\left(2^\ell\xi\right) \overline{\hat{\omega}_n(\xi)} e^{is\xi} \, d\xi$$

$$(3.10) \qquad = \; 2^{\frac{\ell}{2}} \int_{\mathbb{T}} \left(\sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \hat{\omega}_0\left(2^\ell(\xi+2k\pi)\right) \overline{\hat{\omega}_n(\xi+2k\pi)} \right)$$
$$\times e^{is\xi} \, d\xi.$$

The convergence of the last series in $L^2(\mathbb{T})$ is guaranteed by the fact that $\omega_n \in L^2(\mathbb{R})$, $n = 0,1,2,...$ From (3.8) and our assumption $D_{\omega_\beta}(\xi) = 1$ a.e. we obtain

$$\sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} |\nu_{n,\ell}(\xi)|^2 \;=\; \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \sum_{k\in\mathbb{Z}} \left| \hat{\omega}_n\left(2^\ell(\xi+2k\pi)\right) \right|^2$$

$$= \; 1 \text{ for a.e. } \xi \in \mathbb{R}.$$

Hence, for such ξ and for each $\ell \geq 0$, there exists $\ell_0 \equiv \ell_0(2^\ell\xi) \geq 1$ such that $\nu_{t,\ell_0}(2^\ell\xi) \neq 0$. This and (3.9) imply, for such ξ,

$$\hat{\omega}_0\left(2^\ell(\xi+2k\pi)\right) = \frac{1}{\nu_{t,\ell_0}(2^\ell\xi)} \hat{\omega}_t\left(2^{\ell+\ell_0}(\xi+2k\pi)\right), \; k \in \mathbb{Z}.$$

We now use (2.4) to obtain (observe that $\ell + \ell_0 \geq 1$)

$$\sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \hat{\omega}_0\left(2^\ell(\xi+2k\pi)\right) \overline{\hat{\omega}_n(\xi+2k\pi)}$$

$$= \frac{1}{\nu_{t,\ell_0}(2^\ell\xi)} \sum_{n=2^u}^{2^{u+1}-1} \sum_{k\in\mathbb{Z}} \hat{\omega}_t\left(2^{\ell+\ell_0}(\xi+2k\pi)\right) \overline{\hat{\omega}_n(\xi+2k\pi)}$$

$$= 0,$$

for a.e. $\xi \in \mathbb{T}$ and for all $t = 2^u, 2^u + 1, ..., 2^{u+1} - 1, \ell \geq 0$. Therefore, from this result and (3.10), we obtain

$$\langle \omega_0, \ \omega_{\ell,n,s}\rangle = 0 \text{ for all } n = 2^u, 2^u+1, ..., 2^{u+1}-1, \ s \in \mathbb{Z} \text{ and } \ell \geq 0.$$

This shows that ω_0 is orthogonal to W_j is invariant under integral translation, we deduce that $V_0^\# \perp W_j$ for all $j \geq 0$. Hence $V_0^\# \subseteq V_0$, and the proof of the Theorem 3.1 is finished.

Theorem 3.3. For orthonormal wavelet packets $\omega_n \in L^2(\mathbb{R})$, the following statements are equivalent:

(1). ω_n are MRA wavelet packets;

(2). $D_{\omega_\beta}(\xi) = 1$ for a.e. $\xi \in \mathbb{T}$;

(3). $D_{\omega_\beta}(\xi) > 0$ for a.e. $\xi \in \mathbb{T}$;

(4). $\dim \mathbb{F}_{\omega_\beta}(\xi) = 1$ for a.e. $\xi \in \mathbb{T}$,

where $\mathbb{F}_{\omega_\beta}(\xi)$ is the closure, in $\ell^2(\mathbb{Z})$, of the span of $\{\Psi_{n,\ell}(\xi) : \ \ell \geq 1\}$ and $\Psi_{n,\ell}(\xi)$ is the vector $\left\{\hat{\omega}_n\left(2^\ell(\xi+2k\pi)\right) : \ k \in \mathbb{Z}; n = 2^u, 2^u+1, ..., 2^{u+1}-1; \ \ell = j - u\right\}$.

Proof. The equivalence between (1), (2) and (4) has already been proved (see Theorem 3.1 and Equality (3.5)). But (2) implies (3), it is Sufficient to prove that (3) implies (2). If $D_{\omega_\beta}(\xi) > 0$ for a.e. $\xi \in \mathbb{T}$, the fact that $D_{\omega_\beta}(\xi)$ is an integer a.e. implies that $D_{\omega_\beta}(\xi) \geq 1$ almost everywhere. But this and the equality

$$\int_{\mathbb{T}} D_{\omega_\beta}(\xi)\, d\xi = 2\pi.$$

Corollary 3.4. If ω_n are orthonormal wavelet packets such that $|\hat{\omega}_n|$ is continuous and $|\hat{\omega}_n(\xi)| = O\left(|\xi|^{-\frac{1}{2}-\alpha}\right)$ at ∞ for some $\alpha > 0$, then ω_n are MRA wavelet packets.

Proof. The behavior at infinity of $|\hat{\omega}_n|$ tells us that the series

$$s(\xi) = \sum_{n=2^u}^{2^{u+1}-1} \sum_{\ell=1}^{\infty} \left|\hat{\omega}_n\left(2^\ell \xi\right)\right|^2,$$

where $\ell = j - u$, $u = 0, 1, 2, ..., j$ if $j > 0$, $j \in \mathbb{Z}^+$, converges uniformly on compact subsets of $\mathbb{R}\backslash\{0\}$. Moreover, an easy calculation shows that $s(\xi) = O\left(|\xi|^{-1-2\alpha}\right)$ at ∞. It follows that

$$\sum_{k \in \mathbb{Z}} s(\xi + 2k\pi) = D_{\omega_\beta}(\xi)$$

converges uniformly on compact subsets of $\mathbb{T}$. Thus, $D_{\omega_\beta}(\xi)$ is continuous on $(0, 2\pi)$. Since $D_{\omega_\beta}(\xi)$ is integer-valued and $\int_{\mathbb{T}} D_{\omega_\beta}(\xi)\, d\xi = 2\pi$ (see Lemma 2.3), we must have $D_{\omega_\beta}(\xi) = 1$ a.e. on $\mathbb{T}$.

Corollary 3.5. If ω_n are band-limited wavelet packets such that $|\hat{\omega}_n|$ are continuous, then ω_n are MRA wavelet packets

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