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A NEW ITERATIVE METHOD FOR SOLVING LINEAR SYSTEMS OF EQUATIONS

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ABSTRACT. The Jacobi and Gauss-Seidel iterative methods are among iterative methods for solving linear system of equations. In this paper, a new iterative method is introduced, it is based on the linear combination of old and most recent calculated solutions. The new method can be considered as a general method, where the Jacobi and Gauss-Seidel methods as two special cases of it. Some convergence properties are studied, and numerical examples are given to show the effectiveness of the new method. When Jacobi method converges, the new method can be used to accelerate the convergence. In special cases, when one of the two iterative methods, Jacobi or Gauss-Seidel, diverges, the new method can be used to obtain convergence.

1. INTRODUCTION

Direct methods for solving linear systems $Ax = b$, based on the triangularization of the matrix A become prohibitive in terms of computer time and storage if the matrix A is quite large [1]. On the other hand, there are practical situations such as the discretization of partial differential equations, where the matrix size can be as large as several hundred thousand. For such problems, the direct methods become

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impractical. Furthermore, most large problems are sparse, and the sparsity is lost to a considerable extent during the triangularization procedure. Thus, at the end we have to deal with a very large matrix with too many nonzero entries, and storage becomes a crucial issue. For such problems, iterative methods are suitable for this situation.

The Jacobi and Gauss-Seidel iterative methods are classical iterative methods for solving linear systems of equations. They are used primarily to solve a very large and sparse linear system $Ax = b$. Iterative methods are seldom used for solving linear system of equations of small dimension since the time required for sufficient accuracy exceeds that required for direct methods such as Gaussian elimination. They are used for large systems with a high percentage of zero entries, these methods efficient in terms of both computer storage and computations. These large systems arise from engineering applications, as in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations [2]. There are many iterative methods such as the Successive Over-Relaxation method (SOR), where the Gauss-Seidel method is a special case of it. GMRES [3], and BI-CGSTAB [4] algorithms are also used to solve linear systems of equations. The Conjugate Gradient method is used to solve symmetric positive definite systems [1]. The Jacobi and the Gauss-Seidel methods converge for diagonally dominant matrices [1, 5]. The Gauss-Seidel method also converges for symmetric positive definite systems [1].

For special types of matrices, some useful properties for the convergence of the Jacobi, the Gauss-Seidel, and the SOR methods, are presented in [6].

2. Preliminaries

The basic idea behind an iterative method is first to convert the $n \times n$ linear equation

$$Ax = b, \tag{2.1}$$

into an equivalent system of the form

$$x = Tx + c, \tag{2.2}$$

for some fixed matrix T and vector c.

After an initial approximation vector $x^{(0)}$ of the solution vector x is selected, generate a sequence of approximation $\{x^{(k)}\}$ by computing

$$x^{(k)} = Tx^{(k-1)} + c, \qquad \text{for } \; k = 1, 2, \ldots \tag{2.3}$$

with a hope that the sequence $\{x^{(k)}\}$ converges to the solution x of the system $Ax = b$ as $k \to \infty$.

To implement this process, consider the linear system of equations (2.1), where $A \in \mathbb{R}^{n\times n}$, is nonsingular matrix, x and $b \in \mathbb{R}^n$.

The method works under the assumption that $a_{ii} \neq 0$, for each $i = 1, \ldots, n$, to guarantee this, we convert the system (2.1) into an equivalent system

$$PAx = Pb,$$

where P is a permutation matrix. For simplicity, we assume that the matrix A satisfies the assumption.

Thus, write the matrix A in the form

$$A = D - L - U,$$

where D is the diagonal matrix whose diagonal entries are those of A, $-L$ is the strictly lower- triangular part of A, and $-U$ is the strictly upper- triangular part of A. i.e.

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

so we obtain the equivalent system

$$(D - L - U)x = b.$$

The Jacobi iterative method is defined as

$$x^{(k)} = D^{-1}(L + U)x^{(k-1)} + D^{-1}b, \qquad \text{for } \ k = 1, 2, \ldots$$

we call $T_j = D^{-1}(L + U)$ the Jacobi iterative matrix, and $c_j = D^{-1}b$ the Jacobi vector.

The Gauss-Seidel method is a modification of the Jacobi method, which can be defined as

$$x^{(k)} = (D - L)^{-1}Ux^{(k-1)} + (D - L)^{-1}b, \qquad \text{for } \ k = 1, 2, \ldots$$

we call $T_g = (D - L)^{-1}U$ the Gauss-Seidel iterative matrix, and $c_g = (D - L)^{-1}b$ the Gauss-Seidel vector.

3. A New Iterative Method For Solving Linear Systems Of Equations

Consider the linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

where $a_{ii} \neq 0$, for each $i = 1, \ldots, n$. To implement the new method, solve the i^{th} equation of the above system for x_i, and generating each $x_i^{(k)}$ from components of $x^{(k-1)}$, and the most recently calculated values of $x^{(k)}$, for $k \geq 1$. That is,

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} \left(a_{ij} \left(\mu x_j^{(k)} + (1-\mu) x_j^{(k-1)} \right) \right) - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right),$$

for each $i = 1, \ldots, n$. Where $\mu \in [0, 1]$.

To write the method in matrix notations; consider the linear system $Ax = b$, where A is a nonsingular matrix with nonzero diagonal entries, and let $A = D - L - U$ as described earlier. The new iterative method can be written as

$$x^{(k)} = T_\mu x^{(k-1)} + c_\mu \,, \qquad \text{for } \; k = 1, 2, \ldots \tag{3.1}$$

where

$$\begin{aligned} T_\mu &= (D - \mu L)^{-1} \left[(1-\mu)L + U \right] \\ \text{and} \qquad c_\mu &= (D - \mu L)^{-1} b. \end{aligned}$$

Notice that the matrix $(D - \mu L)^{-1}$ exists since $(D - \mu L)$ is a lower triangular matrix with nonzero diagonal entries. Note that if $\mu = 0$, then the iterative method (3.1) results in the Jacobi iterative method, and if $\mu = 1$, then it results in the Gauss-Seidel iterative method.

Definition 3.1. The quantity $\rho(A)$ defined by

$$\rho(A) = \max_i |\lambda_i| \ ,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, is called the spectral radius of A.

Theorem 3.1. *If the spectral radius $\rho(T)$ satisfies $\rho(T) < 1$, then $(I-T)^{-1}$ exists, and*

$$(I-T)^{-1} = I + T + T^2 + \ldots = \sum_{j=0}^{\infty} T^j \ .$$

Theorem 3.2. *For any $x^{(0)} \in \mathbb{R}^n$, the sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ defined by*

$$x^{(k)} = Tx^{(k-1)} + c, \quad \textit{for} \ \ k \geq 1 \ ,$$

converges to the unique solution of $x = Tx + c$ if and only if $\rho(T) < 1$.

Definition 3.2. An $n \times n$ matrix A is said to be strictly row diagonally dominant (SRDD) if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \ , \ \ i = 1, \ldots, n \ .$$

Theorem 3.3. *If A is a strictly row diagonally dominant matrix, then the iterative method (3.1) converges for any arbitrary choice of $x^{(0)}$.*

Proof. Let x be a nonzero vector such that $T_\mu x = \lambda x$, where λ is an eigenvalue of T_μ. This gives

$$((1-\mu)L + U)\, x = (D - \mu L)\, \lambda x,$$

or, equivalently,

$$\sum_{j=1}^{i-1} -(1-\mu)a_{ij}x_j + \sum_{j=i+1}^{n} -a_{ij}x_j = \lambda a_{ii}x_i - \sum_{j=1}^{i-1} -\mu\lambda a_{ij}x_j \ .$$

Then,

$$\lambda a_{ii}x_i = -\lambda \sum_{j=1}^{i-1} \mu a_{ij}x_j - \sum_{j=1}^{i-1}(1-\mu)a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j \ .$$

Let x_k be the largest component (having the magnitude 1) of the vector x. Then

$$|\lambda||a_{kk}| \leq |\lambda| \sum_{j=1}^{k-1} \mu |a_{kj}| + (1-\mu) \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}|.$$

Thus

$$|\lambda| \leq \left((1-\mu) \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}| \right) \Big/ \left(|a_{kk}| - \mu \sum_{j=1}^{k-1} |a_{kj}| \right). \tag{3.2}$$

Since A is strictly diagonally dominant, then

$$|a_{kk}| > \sum_{j=1, j\neq k}^{n} |a_{kj}| = (\mu + (1-\mu)) \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}|, \text{ and}$$

$$|a_{kk}| - \mu \sum_{j=1}^{k-1} |a_{kj}| > (1-\mu) \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}| \ .$$

From the inequality (3.2) we conclude $|\lambda| < 1$, then $\rho(T_\mu) < 1$, which implies that T_μ is a convergent matrix. Therefore, the iterative method (3.1) converges by theorem (3.2). □

Theorem 3.4. *If A is a symmetric positive definite matrix. Then the iterative method (3.1), in the case $\mu = 1$, converges for any arbitrary choice of $x^{(0)}$.*

Proof. Since A is symmetric, then $A = D - L - U = D - L - L^T$.
We need to show that $\rho(T_\mu) < 1$, where

$$T_\mu = (D - \mu L)^{-1} [(1-\mu)L + U] = (D - \mu L)^{-1} \left[(1-\mu)L + L^T\right].$$

Let $-\lambda$ be an eigenvalue of T_μ, and u its associated eigenvector. Then

$$(D - \mu L)^{-1} \left[(1-\mu)L + L^T\right] u = -\lambda u, \text{ and}$$

$$\left[(1-\mu)L + L^T\right] u = -\lambda (D - \mu L) u \ .$$

Using the equation $L^T = D - L - A$, the last equation becomes

$$[D - \mu L - A]\, u = -\lambda (D - \mu L) u \ .$$

By multiplying both sides by $u^\star$, we can write

$$u^\star A u = (1+\lambda) u^\star (D - \mu L) u. \tag{3.3}$$

Taking conjugate transpose, we obtain

$$u^\star A u = (1+\bar{\lambda}) u^\star (D - \mu L^T) u. \tag{3.4}$$

From equations (3.3) and (3.4), we obtain

$$\left(\frac{1}{(1+\lambda)} + \frac{1}{\left(1+\bar{\lambda}\right)} \right) u^\star A u = (2-\mu) u^\star D u + \mu u^\star A u.$$

If $\mu = 1$ (Gauss-Seidel method), then

$$\left(\frac{1}{(1+\lambda)} + \frac{1}{\left(1+\bar{\lambda}\right)} \right) u^\star A u > u^\star A u.$$

This implies

$$\left(\frac{1}{(1+\lambda)} + \frac{1}{\left(1+\bar{\lambda}\right)} \right) > 1, \quad \text{or} \quad \frac{2+\lambda+\bar{\lambda}}{(1+\lambda)\left(1+\bar{\lambda}\right)} > 1.$$

Let $\lambda = \alpha + i\beta$, then $\bar{\lambda} = \alpha - i\beta$, so the last inequality becomes

$$\tfrac{2(1+\alpha)}{(1+\alpha)^2+\beta^2} > 1 \ ,$$

from which it follows that $\alpha^2 + \beta^2 < 1$. That is, $\rho(T_g) < 1$, because $|\lambda| = \sqrt{\alpha^2+\beta^2}$. Therefore, the iterative method (3.1), in the case $\mu = 1$, converges for any arbitrary choice of $x^{(0)}$ by theorem (3.2). □

Definition 3.3. A real matrix A of order N is an L-matrix if

$$a_{i,i} > 0, \quad i = 1, 2, \ldots, N,$$

$$\text{and} \qquad a_{i,j} \leq 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, N \ .$$

In the next theorem, we let $|B|$ denote the matrix whose elements are the moduli of the corresponding elements of B.

Theorem 3.5. *If $A \geq |B|$, then $\rho(A) \geq \rho(B)$.*

Proof of theorem (3.5) is given by Oldenberger [7].

Theorem 3.6. *If $A \geq 0$, then $\rho(A)$ is an eigenvalue of A, and there exists a nonnegative eigenvector of A associated with $\rho(A)$.*

Proof of theorem (3.6) is given by Frobenius [8].

Theorem 3.7. *If A is an L-matrix, then*

a. *If $\rho(T_j) < 1$ and $\rho(T_\mu) < 1$, then*

$$\rho(T_\mu) \leq \rho(T_j) < 1.$$

b. *If $\rho(T_j) \geq 1$ and $\rho(T_\mu) \geq 1$, then*

$$\rho(T_\mu) \geq \rho(T_j) \geq 1.$$

Proof. Let $\tilde{L} = D^{-1}L$, and $\tilde{U} = D^{-1}U$, so we can rewrite the Jacobi, and the new method iterative matrices, respectively as,

$$\begin{aligned} T_j &= \tilde{L} + \tilde{U}, \\ T_\mu &= (I - \mu\tilde{L})^{-1}\left[(1-\mu)\tilde{L} + \tilde{U}\right]. \end{aligned}$$

Since $\tilde{L}$ is a strictly lower triangular matrix, then $\tilde{L}^N = 0$, for some positive integer N, and since A is an L-matrix, and $0 \leq \mu \leq 1$, we have

$$(I - \mu\tilde{L})^{-1} = I + \mu\tilde{L} + \mu^2\tilde{L}^2 + \ldots + \mu^{N-1}\tilde{L}^{N-1} \geq 0, \text{ and}$$
$$T_\mu = (I - \mu\tilde{L})^{-1}\left[(1-\mu)\tilde{L} + \tilde{U}\right] \geq 0.$$

Let $\bar{\lambda} = \rho(T_\mu)$, and $\tilde{\lambda} = \rho(T_j)$.

By theorem (3.6), $\bar{\lambda}$ is an eigenvalue of T_μ, and for some $w \neq 0$, we have $T_\mu w = \bar{\lambda}w$, and

$$\left[(1-\mu+\mu\bar{\lambda})\tilde{L} + \tilde{U}\right]w = \bar{\lambda}w.$$

Since $\bar{\lambda}$ is an eigenvalue of $(1-\mu+\mu\bar{\lambda})\tilde{L}+\tilde{U}$, we have

$$\bar{\lambda} \leq \rho\left((1-\mu+\mu\bar{\lambda})\tilde{L}+\tilde{U}\right).$$

If $\bar{\lambda} \leq 1$, then $\rho\left((1-\mu+\mu\bar{\lambda})\tilde{L}+\tilde{U}\right) \leq \rho(\tilde{L}+\tilde{U}) = \tilde{\lambda}$, by theorem (3.5), and

$$\bar{\lambda} \leq \tilde{\lambda}.$$

On the other hand, if $\bar{\lambda} \geq 1$, then

$$\bar{\lambda} \leq \rho\left((1-\mu+\mu\bar{\lambda})\tilde{L}+\tilde{U}\right) \leq \rho\left((\bar{\lambda}-\mu\bar{\lambda}+\mu\bar{\lambda})\tilde{L}+\bar{\lambda}\tilde{U}\right) = \bar{\lambda}\rho(\tilde{L}+\tilde{U}) = \bar{\lambda}\tilde{\lambda}, \text{ and}$$

$$\tilde{\lambda} \geq 1.$$

We have thus shown

(i) If $\bar{\lambda} \leq 1$, then $\bar{\lambda} \leq \tilde{\lambda}$.

(ii) If $\bar{\lambda} \geq 1$, then $\tilde{\lambda} \geq 1$.

which implies

(iii) If $\tilde{\lambda} < 1$, then $\bar{\lambda} < 1$.

Since $T_j = (\tilde{L}+\tilde{U}) \geq 0$, it follows by theorem (3.6) that $\tilde{\lambda}$ is an eigenvalue of T_j. Therefore, for some $x \neq 0$, we have $(\tilde{L}+\tilde{U})x = \tilde{\lambda}x$, and

$$\left(I-(\mu/\tilde{\lambda})\tilde{L}\right)^{-1}\left[(1-\mu)\tilde{L}+\tilde{U}\right]x = \tilde{\lambda}x,$$

then

$$\tilde{\lambda} \leq \rho\left(\left(I-(\mu/\tilde{\lambda})\tilde{L}\right)^{-1}\left[(1-\mu)\tilde{L}+\tilde{U}\right]\right).$$

If $\tilde{\lambda} \geq 1$, and since $(\mu/\tilde{\lambda}) \leq 1$, then we have

$$\begin{aligned}\left(I-(\mu/\tilde{\lambda})\tilde{L}\right)^{-1} &= I+(\mu/\tilde{\lambda})\tilde{L}+\ldots+(\mu/\tilde{\lambda})^{N-1}\tilde{L}^{N-1} \\ &\leq I+\mu\tilde{L}+\ldots+\mu^{N-1}\tilde{L}^{N-1} = \left(I-\mu\tilde{L}\right)^{-1}, \text{and}\end{aligned}$$

$$\left(I-(\mu/\tilde{\lambda})\tilde{L}\right)^{-1}\left[(1-\mu)\tilde{L}+\tilde{U}\right] \leq \left(I-\mu\tilde{L}\right)^{-1}\left[(1-\mu)\tilde{L}+\tilde{U}\right].$$

Hence

$$\tilde{\lambda} \leq \bar{\lambda}.$$

Thus we have shown,

(iv) If $\tilde{\lambda} \geq 1$, then $\tilde{\lambda} \leq \bar{\lambda}$.

By (i) and (iii) we have (a). By (ii) and (iv) we have (b). □

Theorem 3.8. *If A is an L-matrix, then*

a. *If $\rho(T_g) < 1$ and $\rho(T_\mu) < 1$, then*

$$\rho(T_g) \leq \rho(T_\mu) < 1.$$

b. *If $\rho(T_g) \geq 1$ and $\rho(T_\mu) \geq 1$, then*

$$\rho(T_g) \geq \rho(T_\mu) \geq 1.$$

Proof. Let $\tilde{L} = D^{-1}L$, and $\tilde{U} = D^{-1}U$, so we can rewrite the Gauss-Seidel, and the new method iterative matrices, respectively as,

$$\begin{aligned} T_g &= (I - \tilde{L})^{-1}\tilde{U}, \\ T_\mu &= (I - \mu\tilde{L})^{-1}\left[(1-\mu)\tilde{L} + \tilde{U}\right]. \end{aligned}$$

And, as in the proof of theorem (3.7), we have

$$T_\mu = (I - \mu\tilde{L})^{-1}\left[(1-\mu)\tilde{L} + \tilde{U}\right] \geq 0.$$

Let $\bar{\lambda} = \rho(T_\mu)$, and $\hat{\lambda} = \rho(T_g)$.

By theorem (3.6), $\bar{\lambda}$ is an eigenvalue of T_μ, and for some $v \neq 0$, we have $T_\mu v = \bar{\lambda}v$, and

$$\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1}\tilde{U}v = \bar{\lambda}v,$$

therefore

$$\bar{\lambda} \leq \rho\left(\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1}\tilde{U}\right).$$

if $\bar{\lambda} < 1$, and since $\left(\mu + (1-\mu)/\bar{\lambda}\right) \geq 1$, then we have

$$\begin{aligned}\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1} &= I + \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L} + \ldots + \left(\mu + (1-\mu)/\bar{\lambda}\right)^{N-1}\tilde{L}^{N-1} \\ &\geq I + \tilde{L} + \ldots + \tilde{L}^{N-1} = \left(I - \tilde{L}\right)^{-1}, \text{and}\end{aligned}$$

$$\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1}\tilde{U} \geq \left(I - \tilde{L}\right)^{-1}\tilde{U} \ ,$$

and hence,

$$\bar{\lambda} \geq \hat{\lambda}.$$

Thus, we have shown

(i) If $\bar{\lambda} < 1$, then $\hat{\lambda} \leq \bar{\lambda} < 1$.

But, if $\bar{\lambda} \geq 1$, and since $\left(\mu + (1-\mu)/\bar{\lambda}\right) \leq 1$, then we have

$$\begin{aligned}\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1} &= I + \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L} + \ldots + \left(\mu + (1-\mu)/\bar{\lambda}\right)^{N-1}\tilde{L}^{N-1} \\ &\leq I + \tilde{L} + \ldots + \tilde{L}^{N-1} = \left(I - \tilde{L}\right)^{-1}, \text{ and}\end{aligned}$$

$$\left[I - \left(\mu + (1-\mu)/\bar{\lambda}\right)\tilde{L}\right]^{-1}\tilde{U} \leq \left(I - \tilde{L}\right)^{-1}\tilde{U} \ ,$$

and hence,

$$\bar{\lambda} \leq \hat{\lambda}.$$

We have thus shown,

(ii) If $\bar{\lambda} \geq 1$, then $\hat{\lambda} \geq \bar{\lambda} \geq 1$.

By (i) we have (a). By (ii) we have (b). □

Corollary 3.1. *If A is an L-matrix, then*

a. *If $\rho(T_j) < 1$, $\rho(T_\mu) < 1$, and $\rho(T_g) < 1$, then*

$$\rho(T_g) \leq \rho(T_\mu) \leq \rho(T_j) < 1.$$

b. *If $\rho(T_j) \geq 1$, $\rho(T_\mu) \geq 1$, and $\rho(T_g) \geq 1$, then*

$$\rho(T_g) \geq \rho(T_\mu) \geq \rho(T_j) \geq 1.$$

Proof. The proof follows from theorems (3.7), and (3.8). □

4. Numerical examples

Example 4.1. *Consider the following linear system:*

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7 \\ x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

The exact solution is $(1, 2, -1)^t$. *The Gauss-Seidel method does not converge in this case, since* $\rho(T_g) = 2 > 1$. *On the contrary, the method (3.1) with* $\mu = 0.15$ *converges, where* $\rho(T_\mu) = 0.9378$, *we can choose any value for* μ *close to zero. It is clear that the convergence is slow, but it avoids the problem with the Gauss-Seidel method. See Table 1. The stopping criterion* $\|x - x^{(k)}\|_\infty \leq 10^{-5}$ *was used, and the initial solution was taken to be the zero vector, the approximation is rounded and the norm criterion is met at iteration* 204.

x	Iteration i					
	1	2	3	$\cdots$	203	204
$x_1^{(i)}$	7.00000	10.33000	4.86970	$\cdots$	1.00000	1.00000
$x_2^{(i)}$	0.95000	-8.11450	1.66870	$\cdots$	2.00001	1.99999
$x_3^{(i)}$	2.61500	-9.17965	-0.72787	$\cdots$	-0.99999	-1.00001

TABLE 1. Solution of Example 4.1 using the method (3.1) with $\mu = 0.15$.

Example 4.2. *Consider the following linear system:*

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5 \end{aligned}$$

The exact solution is $(1, 2, -1)^t$. *The Jacobi method does not converge in this case, since* $\rho(T_j) = \sqrt{5}/2 > 1$. *On the contrary, the method (3.1) with* $\mu = 0.5$ *converges, where* $\rho(T_\mu) = 0.7588$, *we can choose any value for* μ *close to one. We obtained an excellent approximation in* 45 *iterations. See Table 2. The stopping criterion* $\|x - x^{(k)}\|_\infty \leq 10^{-5}$ *was used, and the initial solution was taken to be the zero vector, the approximation is rounded and the norm criterion is met at iteration* 45.

x	Iteration i					
	1	2	3	$\cdots$	44	45
$x_1^{(i)}$	-0.50000	1.65625	1.63086	$\cdots$	0.99999	1.00000
$x_2^{(i)}$	2.25000	3.48438	1.13379	$\cdots$	1.99999	2.00000
$x_3^{(i)}$	-2.06250	-0.77734	-0.52368	$\cdots$	-1.00000	-1.00000

TABLE 2. Solution of Example 4.2 using the method (3.1) with $\mu = 0.5$.

Example 4.3. *Consider the following linear system:*

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 5 \\ -x_1 + 4x_2 - x_4 &= -3 \\ -x_1 + 4x_3 - x_4 &= -7 \\ - x_2 - x_3 + 4x_4 &= 9 \end{aligned}$$

The exact solution is $(1, 0, -1, 2)^t$. *The Jacobi method converges in* 18 *iterations, where* $\rho(T_j) = 0.5$, *while the method (3.1), with* $\mu = 0.7$, *converges in* 12 *iterations,* *where* $\rho(T_\mu) = 0.375$, *we can choose any value for* μ *in* $(0, 1)$. *See Table 3, and Table 4. The stopping criterion* $\|x - x^{(k)}\|_\infty \leq 10^{-5}$ *was used, and the initial solution was taken to be the zero vector. Notice that the coefficient matrix* A *is a strictly row diagonally dominant, symmetric positive definite, and it is an L-matrix.*

x	Iteration i					
	1	2	3	$\cdots$	11	12
$x_1^{(i)}$	1.25000	0.73438	0.97227	$\cdots$	0.99998	0.99999
$x_2^{(i)}$	-0.53125	-0.05547	-0.04955	$\cdots$	-0.00002	-0.00001
$x_3^{(i)}$	-1.53125	-1.05547	-1.04955	$\cdots$	-1.00002	-1.00001
$x_4^{(i)}$	1.88906	1.90090	1.97434	$\cdots$	1.99999	2.00000

TABLE 3. Solution of Example 4.3 using the method (3.1) with $\mu = 0.7$.

x	Iteration i					
	1	2	3	$\cdots$	17	18
$x_1^{(i)}$	1.25000	0.62500	1.06250	$\cdots$	1.00000	0.99999
$x_2^{(i)}$	-0.75000	0.12500	-0.18750	$\cdots$	-0.00001	0.00000
$x_3^{(i)}$	-1.75000	-0.87500	-1.18750	$\cdots$	-1.00001	-1.00000
$x_4^{(i)}$	2.25000	1.62500	2.06250	$\cdots$	2.00000	1.99999

TABLE 4. Solution of Example 4.3 using the Jacobi method.

5. CONCLUSIONS

A new general iterative method, by linear combination of old and most recent calculated solutions, was introduced. It works as a general method where the Jacobi and Gauss-Seidel methods are special cases of it. We have proved some convergence properties. some numerical examples where presented to show the effectiveness of the new method. In special cases, where the matrix A is an L-matrix, we have shown theoretically and numerically that the new method converged faster than the Jacobi method when the spectral radius of both iteration matrices is less than one. And when it compared with the Gauss-Seidel method, in the case of divergence, we have seen that the divergence was more pronounced for the Gauss-Seidel method. When Jacobi method converges, the new method can be used to accelerate the convergence. In special cases, when one of the two iterative methods, Jacobi or Gauss-Seidel, diverges, the new method can be used to obtain convergence.

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