SEMI-COMPATIBLE MAPS AND COMMON FIXED POINT THEOREMS IN NON-ARCHIMEDEAN MENGER PM-SPACE

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ABSTRACT. The aim of this paper is to define the concept of semi-compatibility in N. A. Menger PM-space. Our results improve and generalize the results of Amit Singh et al [1], Ume and Kim [10], Rhoades [7] and B. Singh and S. Jain [2, 3, 4, 5].

1. INTRODUCTION

In 1942, K. Menger [12] introduced the notion of probabilistic metric spaces (briefly, PM-space) as a generalization of metric space. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.

In 1975, Istratescu and Crivat [28] first studied the non-Archimedean PM-space. They presented some basic topological preliminaries of N. A. PM-space and later on Istratescu [25], [26], [27] proved some fixed point results on mappings on N. A. Menger PM-space by generalizing the results of Sehgal and Bharucha-Reid [29]. Achari [11] generalized the results of Istratescu and studied some fixed points of qausi-contraction type mappings in non-Archimedean PM-space.

In 2002 Popa [24] obtained a fixed point theorem for d-topological spaces through semi-compatible maps. Since then various mathematicians have extended the concept of semi compatibility in certain spaces like metric spaces, fuzzy metric spaces, 2 metric spaces, PM-spaces etc.

In the present paper we introduce the concept of semi-compatible maps in N. A. Menger PM-space.

2. Preliminaries

Definition 2.1. Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be non-Archimedean probabilistic metric space (briefly N. A. PM-space) if F is a mapping from satisfying the following conditions where the value of F at (is represented by such that

(1) $F(x, y; t) = 1$ for all $t > 0$ if and only if $x = y$;

(2) $F(x, y; t) = F(y, x; t)$;

(3) $F(x, y; 0) = 0$;

(4) If $F(x, y; t_1) = F(y, z; t_2) = 1$, then $F(x, z; max t_1, t_2) = 1$

Definition 2.2. A t-norm is a function $\triangle : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which is associative, commutative, non decreasing in each coordinate and $\triangle(a, 1) = a$ for all $a \in [0, 1]$.

Definition 2.3. A non-Archimedean Menger PM-space is an ordered triplet $(X, F, \triangle)$, where $\triangle$ is a t-norm and (X, F), is a N.A. PM-space satisfying the following condition:

$$F(x, z; \max t_1, t_2) \geq \triangle(F(x, y; t), F(y, z; t)),$$

for all $x, y, z \in X$, $t_1, t_2 \geq 0$.

For details of topological preliminaries on non-Archimedean Menger PM-spaces, we refer to [28].

Definition 2.4. A N. A. Menger PM-space $(X, F, \triangle)$ is said to be of type (C_g) if there exists a $g \in \Omega$ such that $g(F(x, z; t)) \leq g(F(x, y; t)) + g(F(y, z; t))$ for all $x, y, z \in X$, $t \geq 0$ where $\Omega = \{g/g : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing $g(1) = 0$ and $g(0) < \infty\}$.

Definition 2.5. A N. A. Menger PM-space $(X, F, \triangle)$ is said to be of type (D_g) if there exists a $g \in \Omega$ such that $g(\triangle(t_1, t_2)) \leq g(t_1) + g(t_2)$, for all $t_1, t_2 \in [0, 1]$.

Remark 1. (1) If N. A. Menger PM-space is of type (D_g) then $(X, F, \triangle)$ is of type (C_g).

(2) If $(X, F, \triangle)$ is N. A. Menger PM-space is and $\triangle \geq \triangle(r, s) = \max(r + s - 1, 1)$, then $(X, F, \triangle)$ is of type (D_g) for $g \in \Omega$ and $g(t) = 1 - t$.

Through out this paper let $(X, F, \triangle)$ be a complete N.A. Menger PM-space with a continuous strictly increasing t-norm $\triangle$.

Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the condition (Φ); ϕ is semi upper continuous from right and $\phi(t) < t$ for $t > 0$.

Definition 2.6. A sequence $\{x_n\}$ in N. A. Menger PM-space $(X, F, \triangle)$ converges to x if and only if for each $\epsilon > 0$, $\lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x; \epsilon)) < g(1 - \lambda)$ for all n, $n > M$.

Definition 2.7. A sequence $\{x_n\}$ in N. A. Menger PM-space $(X, F, \triangle)$ is Cauchy sequence if and only if for each $\epsilon > 0$, $\lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x_{n+p}; \epsilon)) < g(1 - \lambda)$ for all n, $n \geq M$ and $p \geq 1$.

Lemma 2.1. *If a function* $\phi : [0, \infty) \to [0, \infty)$ *satisfies the condition* (Φ) *then we get*

(1) *For all* $t > 0, \lim_{n \to \infty} \phi^n(t) = 0$, *where* $\phi^n(t)$ *is the* n^{th} *iteration of* $\phi(t)$.

(2) *If* $\{t_n\}$ *is a non-decreasing sequence of real numbers and* $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$ *then* $\lim_{n \to \infty} t_n = 0$. *In particular, if* $t \leq \phi(t)$, *for all* $t \geq 0$ *then* $t = 0$.

Example 2.1. *Let X be any set with at least two elements. If we define $F(x, x; t) = 1$ for all $x \in X$, $t > 0$ and $F(x, y; t) = \left\{ \begin{array}{ll} 0, & t \leq 1 \\ 1, & t > 1 \end{array} \right\}$ when $x, y \in X$, $x \neq y$, then, $(X, F, \triangle)$ is N. A. Menger PM-space with $\triangle(a, b) = \min(a, b)$ or ab.*

Example 2.2. *Let $X = R$ be the set of real numbers equipped with metric defined as $d(x, y) = \mid x - y \mid$ and set $F(x, y; t) = \frac{t}{t + d(x, y)}$.*

Then $(X, F, \triangle)$ is N. A. Menger PM-space with $\triangle$ as continuous t-norm satisfying $\triangle(r, s) = \min(r, s)$ or rs.

Definition 2.8. Two self maps $A, B : X \to X$ are said to be compatible if $\lim_n g(F(ABx_n, BAx_n, t)) = 0$ for all $t > 0$ and where $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Bx_n = z$ for some z in X.

Definition 2.9. Two self maps $A, B : X \to X$ are said to be weak compatible if they commute at their coincident points, that is $Ax = Bx$ implies that $ABx = BAx$.

Proposition 2.1. *Let $A, S : X \to X$ be mappings with A as continuous maps. If A and S are compatible maps and $Az = Sz$ for some $z \in X$, then $AAz = ASz = SAz = SSz$.*

Proof. Suppose that $\{x_n\}$ be a sequence in X defined by, $x_n = 1, 2, 3, \dots$ and $Sz = Az$ for some $z \in X$. Since A is continuous so, $ASx_n, AAx_n \to Az$. Also A and S are compatible mappings so $\lim_n g(F(ASx_n, AAx_n, t)) = 0$.

$$
\begin{aligned}
g(F(SAz, AAz, t)) &= \lim_n g(F(ASx_n, AAx_n, t)) \\
&\leq \lim_n [g(F(SAx_n, AAx_n, t)) + g(F(SAx_n, ASx_n, t)) \\
&\quad + g(F(ASx_n, AAx_n, t))] \to 0.
\end{aligned}
$$

Therefore, $SAz = AAz$. But $Sz = Az$ implies that $SSz = AAz = ASz = SAz$. $\square$

Definition 2.10. Two self maps $A, B : X \to X$ are said to be semi-compatible if $\lim_n g(F(ABx_n, Bz, t)) = 0$ for all $t > 0$, where $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Bx_n = z$ for some z in X.

It follows that if (A, B) is semi-compatible and $Ay = By$, then $ABy = BAy$. Thus semi-compatibility implies weak compatibility but the converse is not always true.

In the following examples we discuss the relationship of semi compatible, weak compatible and compatible maps.

Example 2.3. *Let $X = [0, 1]$ equipped with metric defined as $d(x, y) = \mid x - y \mid$ and set $F(x, y; t) = \frac{t}{t + d(x,y)}$. Then $(X, F, \triangle)$ is N. A. Menger PM-space with $\triangle$ as continuous t-norm satisfying*

$\triangle(r, s) = \min(r, s)$ or (rs). Define $A, B : X \to X$ as

$$Ax = 1 - x, \; Bx = \left\{ \begin{array}{ll} x, & 0 \le x \le \frac{1}{2}; \\ 1, & \frac{1}{2} < x \le 1. \end{array} \right\}.$$

Consider a sequence $x_n = \left(\frac{1}{2} - \frac{1}{n}\right)$ for all n. Then $Ax_n = 1 - \left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{n}$ and $Bx_n = \frac{1}{2} - \frac{1}{n}$. Here $Ax_n, Bx_n \to \frac{1}{2}$ for all n. Also $BAx_n = B\left(\frac{1}{2} + \frac{1}{n}\right) \to 1$ and $ABx_n \to \frac{1}{2}$, $B\left(\frac{1}{2}\right) = \frac{1}{2} = A\left(\frac{1}{2}\right)$. Now, $\lim_n g(F(ABx_n, B\frac{1}{2}, t)) = 0$ for all $t > 0$, which implies that (A, B) is semi-compatible. But $\lim_n g(F(BAx_n, A\frac{1}{2}, t)) \ne 0$, for all $t > 0$, which implies that (B, A) is not semi-compatible.

Thus semi-compatibility of the pair (A, B) does not imply the semi-compatibility of the pair (B, A).

Moreover, weak-compatibility need not imply the semi-compatibility. Here B and A are weak compatible as they commute at their coincident point $\frac{1}{2}$ but they are not semi-compatible.

Also, semi-compatible maps need not be compatible. Here (A, B) is semi-compatible but not compatible as,

$\lim_n g(F(ABx_n, BAx_n, t)) = \lim_n g(F(\frac{1}{2} + \frac{1}{n}, 1, t)) \ne 0$ for all x in X and $t > 0$.

Again, weak compatibility does not imply compatibility as the maps B and A are weak compatible but not compatible.

Example 2.4. *Let $X = [0,1]$ equipped with metric defined as $d(x,y) = \mid x - y \mid$ and set $F(x,y;t) = \frac{t}{t+d(x,y)}$. Then $(X, F, \triangle)$ is N. A. Menger PM-space with $\triangle$ as continuous t-norm satisfying*

$\triangle(r,s) = \min(r,s)$ *or* (rs). *Define $A, B : X \to X$ as*

$$Ax = x, \ Bx = \left\{ \begin{array}{ll} x, & 0 \le x < \frac{1}{2}; \\ 1, & x \ge \frac{1}{2}. \end{array} \right\}.$$

Consider the sequence $x_n = \left(\frac{1}{2} - \frac{1}{n}\right)$. Then $Ax_n, Bx_n \to \frac{1}{2}$, $ABx_n = A\left(\frac{1}{2} - \frac{1}{n}\right) = \left(\frac{1}{2} - \frac{1}{n}\right) \to \frac{1}{2}$ and $BAx_n = B\left(\frac{1}{2} - \frac{1}{n}\right) = \left(\frac{1}{2} - \frac{1}{n}\right) \to \frac{1}{2}$. Thus $\lim_n g(F(ABx_n, BAx_n, t)) = 0$, which implies that A and B are compatible. But $\lim_n g(F(ABx_n, B\frac{1}{2}, t)) \ne 0$, which implies that A and B are not semi-compatible. Hence compatibility does not imply semi-compatibility.

3. Results and Discussions

Now, we prove our main results

Theorem 3.1. *Let A, B, S and T be self maps of a complete N. A. Menger PM-space $(X, F, \triangle)$ satisfying*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$;*

(2) *The pair (A, S) is semi-compatible and (B, T) is weak compatible;*

(3) *One of A or S is continuous;*

(4)

$$(3.1) \qquad g(F(Ax, By, t)) \le \phi \left[\max \left\{ \begin{array}{l} g(F(Sx, Ax, t)), g(F(Ty, By, t)), \\ g(F(Sx, Ty, t)), g(F(Ty, Ax, t)) \end{array} \right\} \right]$$

for all $x, y \in X$ and $t > 0$, where $\phi \in \Phi$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. As $A(X) \subset T(X)$ and $B(X) \subset S(X)$ there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1 = y_0$, $Bx_1 = Sx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$(3.2) \qquad y_{2n} = Ax_{2n} = Tx_{2n}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},$$

for $n = 0, 1, \ldots$.

Now using (3.1) with $x = x_{2n}$, $y = x_{2n+1}$, we get

$$g(F(y_{2n}, y_{2n+1}, t)) = g(F(Ax_{2n}, Bx_{2n+1}, t))$$

$$(3.3) \qquad \leq \phi \left[\max \left\{ \begin{array}{l} g(F(Sx_{2n}, Ax_{2n}, t)), g(F(Tx_{2n+1}, Bx_{2n+1}, t)), \\ g(F(Sx_{2n}, Tx_{2n+1}, t)), g(F(Tx_{2n+1}, Ax_{2n}, t)) \end{array} \right\} \right]$$

$$(3.4) \qquad = \phi \left[\max \left\{ \begin{array}{l} g(F(y_{2n-1}, y_{2n}, t)), g(F(y_{2n}, y_{2n+1}, t)), \\ g(F(y_{2n-1}, y_{2n}, t)), g(F(y_{2n}, y_{2n}, t)) \end{array} \right\} \right]$$

If $g(F(y_{2n}, y_{2n+1}, t)) \geq g(F(y_{2n-1}, y_{2n}, t))$, then (3.4) gives

$g(F(y_{2n}, y_{2n+1}, t)) \leq \phi[g(F(y_{2n}, y_{2n+1}, t))] < g(F(y_{2n}, y_{2n+1}, t))$, a contradiction.

If $g(F(y_{2n-1}, y_{2n}, t)) \geq g(F(y_{2n}, y_{2n+1}, t))$, then (3.4) gives

$g(F(y_{2n}, y_{2n+1}, t)) \leq \phi[g(F(y_{2n-1}, y_{2n}, t))].$

Similarly , we can have

$g(F(y_{2n+1}, y_{2n+2}, t)) \leq \phi[g(F(y_{2n}, y_{2n+1}, t))].$

Therefore, for all n even or odd , we have

$g(F(y_n, y_{n+1}, t)) \leq \phi[g(F(y_{n-1}, y_n, t))].$

Then by Lemma (2.1), we get

$\lim_n g(F(y_n, y_{n+1}, t)) = 0$ and therefore , $\{y_n\}$ is a Cauchy sequence. Since $(X, F, \triangle)$ is complete , the sequence $\{y_n\}$ converges to a point $z \in X$ and so the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to a point $z \in X$.

Case I: Let S is continuous $SAx_{2n}, SSx_{2n} \to Sz$. The semi-compatibility of the pair (A, S) gives $\lim_n ASx_{2n} = Sz$.

Step 1. By putting $x = Sx_{2n}, y = x_{2n+1}$ in (3.1), we get

$$g(F(ASx_{2n}, Bx_{2n+1}, t)) \le \phi \left[\max \left\{ \begin{array}{c} g(F(SSx_{2n}, ASx_{2n}, t)), g(F(Tx_{2n+1}, Bx_{2n+1}, t)), \\ g(F(SSx_{2n}, Tx_{2n+1}, t)), g(F(Tx_{2n+1}, ASx_{2n}, t)) \end{array} \right\} \right].$$

Letting $n \to \infty$, we have

$$g(F(Sz, z, t)) \le \phi \left[\max \left\{ \begin{array}{c} g(F(Sz, Sz, t)), g(F(z, z, t)), \\ g(F(Sz, z, t)), g(F(z, Sz, t)) \end{array} \right\} \right],$$

which implies that $z = Sz$.

Step 2. By putting $x = z, y = x_{2n+1}$ in (3.1), we have

$$g(F(Az, Bx_{2n+1}, t)) \le \phi \left[\max \left\{ \begin{array}{c} g(F(Sz, Az, t)), g(F(Tx_{2n+1}, Bx_{2n+1}, t)), \\ g(F(Sz, Tx_{2n+1}, t)), g(F(Tx_{2n+1}, Az, t)) \end{array} \right\} \right].$$

Taking $n \to \infty$, we get $Az = z$. Hence $Sz = Az = z$.

Step 3. As $A(X) \subset T(X)$, there exists $w \in X$ such that $Az = Sz = z = Tw$.

By putting $x = x_{2n}, y = w$ in (3.1), we get

$$g(F(Ax_{2n}, Bw, t)) \le \phi \left[\max \left\{ \begin{array}{c} g(F(Sx_{2n}, Ax_{2n}, t)), g(F(Tw, Bw, t)), \\ g(F(Sx_{2n}, Tw, t)), g(F(Tw, Ax_{2n}, t)) \end{array} \right\} \right].$$

Letting $n \to \infty$, we get $z = Bw$. Hence $Bw = Tw = z$.

Since (B, T) is weak compatible, we get $TBw = BTw$. That is $Bz = Tz$.

Step 4. By putting $x = y = z$ in (3.1) and assuming $Az \ne Bz$, we get

$$g(F(Az, Bz, t)) \le \phi \left[\max \left\{ \begin{array}{c} g(F(Sz, Az, t)), g(F(Tz, Bz, t)), \\ g(F(Sz, Tz, t)), g(F(Tz, Az, t)) \end{array} \right\} \right].$$

Which implies that $Az = Bz = z$.

Combining all the result, we get $Sz = Az = Bz = Tz = z$, which implies that z is a common fixed point of A, B, S and T.

Case II: Let A is continuous $ASx_{2n} \to Az$. The semi-compatibility of the pair (A, S) gives $ASx_{2n} \to Sz$. By uniqueness of limit we get $Sz = Az$.

Step 1. By putting $x = z, y = x_{2n+1}$ in (3.1), we get

$$g(F(Az, Bx_{2n+1}, t)) \leq \phi \left[\max \left\{ \begin{array}{l} g(F(Sz, Az, t)), g(F(Tx_{2n+1}, Bx_{2n+1}, t)), \\ g(F(Sz, Tx_{2n+1}, t)), g(F(Tx_{2n+1}, Az, t)) \end{array} \right\} \right].$$

Letting $n \to \infty$, we get $Az = z$ and rest of the proof follows from Step 3 onwards of the previous case.

Uniqueness of the fixed point can be easily proved by using (3.1) and hence the theorem. $\qquad\square$

Example 3.1. *Let $X = R$ equipped with metric defined as $d(x, y) = | x - y |$ and set $F(x, y; t) = \frac{t}{t + d(x, y)}$. Then $(X, F, \triangle)$ is N. A. Menger PM-space with $\triangle$ as continuous t-norm satisfying $\triangle(r, s) = \min(r, s)$ or (rs). Define $A, B, S, T : X \to X$ as*

$$A(x) = \left\{ \begin{array}{ll} 1, & if \ x < 1; \\ x, & if \ x \geq 1. \end{array} \right. \qquad B(x) = \left\{ \begin{array}{ll} 1, & if \ x \leq 1; \\ \frac{3}{2}, & if \ x > 1. \end{array} \right.$$

$$S(x) = \left\{ \begin{array}{ll} 3 - 2x, & if \ x \leq 1; \\ x, & if \ x > 1. \end{array} \right. \qquad T(x) = \left\{ \begin{array}{ll} 2 - x, & if \ x \leq 1; \\ x + 1, & if \ x > 1. \end{array} \right.$$

Consider a sequence $x_n = \left(1 - \frac{1}{n}\right)$ for all n.

Here, $Ax_n, Sx_n \to 1$ for all n. Also $ASx_n = A\left(1 + \frac{2}{n}\right) \to 1$ and $S(1) = 1$.

Thus $\lim_n g(F(ASx_n, S(1), t)) = 0$ for all $t > 0$, which implies that (A, S) is semi-compatible. Also, $B(1) = T(1) = 1 = BT(1) = TB(1) \Rightarrow (B, T)$ is weakly compatible. Hence all the conditions of our theorem are satisfied and 1 is common fixed point of A, B, S and T.

Theorem 3.2. *Let A, B, S and T be self maps of a complete N. A. Menger PM-space $(X, F, \triangle)$ satisfying*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$;*

(2) *One of A or S is continuous;*

(3)

$$(3.5) \qquad g(F(Ax, By, t)) \le \phi \left[\max \left\{ \begin{array}{l} g(F(Sx, Ax, t)), g(F(Ty, By, t)), \\ g(F(Sx, Ty, t)), g(F(Ty, Ax, t)) \end{array} \right\} \right]$$

for all $x, y \in X$ and $t > 0$, where $\phi \in \Phi$;

(4) The pairs (A, S) and (B, T) are semi-compatible.

Then A, B, S and T have a unique common fixed point in X.

Proof. As semi-compatibility implies weak compatibility, so the proof follows from Theorem 3.1. □

On taking $A = B$ in Theorem 3.1, we have the following theorem.

Theorem 3.3. *Let A, S and T be self maps of a complete N. A. Menger PM-space $(X, F, \triangle)$ satisfying*

(1) $A(X) \subset S(X) \cap T(X)$;

(2) One of A or S is continuous;

(3)

$$(3.6) \qquad g(F(Ax, Ay, t)) \le \phi \left[\max \left\{ \begin{array}{l} g(F(Sx, Ax, t)), g(F(Ty, Ay, t)), \\ g(F(Sx, Ty, t)), g(F(Ty, Ax, t)) \end{array} \right\} \right]$$

for all $x, y \in X$ and $t > 0$, where $\phi \in \Phi$;

(4) The pairs (A, S) and (A, T) are semi-compatible.

Then A, S and T have a unique common fixed point in X.

Theorem 3.4. *Let A, B, S and T be self maps of a complete N. A. Menger PM-space $(X, F, \triangle)$ satisfying*

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$;

(2) One of A or S is continuous;

(3)

$$(3.7) \qquad g(F(Ax, By, t)) \leq \phi \left[\max \left\{ \begin{array}{l} g(F(Sx, Ax, t)), g(F(Ty, By, t)), \\ g(F(Sx, Ty, t)), g(F(Ty, Ax, t)) \end{array} \right\} \right]$$

   for all $x, y \in X$ and $t > 0$, where $\phi \in \Phi$;

   (4) *The pair (A, S) is compatible and (B, T) is weak compatible.*

Then A, B, S and T have a unique common fixed point in X.

Proof. It is sufficient to prove the result when A is continuous. By the proof of Theorem 3.1, the sequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ and $\{Sx_{2n}\}$ converges to $z \in X$.

As A is continuous, therefore, $ASx_n, AAx_n \to Az$.

Since (A, S) is compatible, so $\lim_n SAx_n = \lim_n ASx_n = Az$.

Step 1. By putting $x = Ax_{2n}, y = x_{2n+1}$ in (3.7), we get

$$g(F(AAx_{2n}, Bx_{2n+1}, t)) \leq \phi \left[\max \left\{ \begin{array}{l} g(F(SAx_{2n}, AAx_{2n}, t)), g(F(Tx_{2n+1}, Bx_{2n+1}, t)), \\ g(F(Tx_{2n+1}, AAx_{2n}, t)), g(F(SAx_{2n}, Tx_{2n+1}, t)) \end{array} \right\} \right].$$

Letting $n \to \infty$, we get

$$g(F(Az, z, t)) \leq \phi \left[\max \left\{ \begin{array}{l} g(F(Az, Az, t)), g(F(z, z, t)), \\ g(F(z, Az, t)), g(F(Az, z, t)) \end{array} \right\} \right].$$

Which gives $Az = z$.

Step 2. As, $A(X) \subset T(X)$, there exists $w \in X$ such that $Az = z = Tw$.

By putting $x = x_{2n}, y = w$ in (3.7), we get

$$g(F(Ax_{2n}, Bw, t)) \leq \phi \left[\max \left\{ \begin{array}{l} g(F(Sx_{2n}, Ax_{2n}, t)), g(F(Tw, Bw, t)), \\ g(F(Tw, Ax_{2n}, t)), g(F(Sx_{2n}, Tw, t)) \end{array} \right\} \right].$$

Taking $n \to \infty$, we get $Bw = z$. Hence $Tw = Bw$. Since (B, T) is weak compatible, we get $BTw = TBw$, i.e., $Bz = Tz$.

Step 3. By putting $x = x_{2n}, y = z$ in (3.7), we get

$$g(F(Ax_{2n}, Bz, t)) \leq \phi \left[\max \left\{ \begin{array}{c} g(F(Sx_{2n}, Ax_{2n}, t)), g(F(Tz, Bz, t)), \\ g(F(Tz, Ax_{2n}, t)), g(F(Sx_{2n}, Tz, t)) \end{array} \right\} \right].$$

Taking $n \to \infty$, we get $Bz = z = Tz$.

Step 4. As, $B(X) \subset S(X)$, there exists $v \in X$ such that $Bz = z = Sv$.

By putting $x = v, y = z$ in (3.7), we get

$$g(F(Av, Bz, t)) \leq \phi \left[\max \left\{ \begin{array}{c} g(F(Sv, Av, t)), g(F(Tz, Bz, t)), \\ g(F(Tz, Av, t)), g(F(Sv, Tz, t)) \end{array} \right\} \right]$$

$$g(F(Av, z, t)) \leq \phi \left[\max \left\{ \begin{array}{c} g(F(z, Av, t)), g(F(z, z, t)), \\ g(F(z, Av, t)), g(F(z, z, t)) \end{array} \right\} \right].$$

Which gives $Av = z = Sv$. By proposition 2.1, we get $AAv = ASv = SSv = SAv$. Which implies that $Az = Sz$.

Step 5. By putting $x = z, y = z$ in (3.7), we get

$$g(F(Az, Bz, t)) \leq \phi \left[\max \left\{ \begin{array}{c} g(F(Sz, Az, t)), g(F(Tz, Bz, t)), \\ g(F(Tz, Az, t)), g(F(Sz, Tz, t)) \end{array} \right\} \right].$$

Which gives $Az = Bz$. Thus $Sz = Az = Bz = Tz = z$. That is z is a common fixed point of A, B, S and T.

Uniqueness of the fixed point can be proved by using condition (3.7). □

Theorem 3.5. *Let A, B, S and T be self maps of a complete N. A. Menger PM-space $(X, F, \triangle)$ satisfying*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$;*

(2) *One of A or S is continuous;*

(3)

$$(3.8) \qquad g(F(Ax, By, t)) \leq \phi \left[\max \left\{ \begin{array}{c} g(F(Sx, Ax, t)), g(F(Ty, By, t)), \\ g(F(Sx, Ty, t)), g(F(Ty, Ax, t)) \end{array} \right\} \right]$$

for all $x, y \in X$ *and* $t > 0$ *and* $\phi \in \Phi$ *where* $\phi : [0,1] \to [0,1]$ *is some continuous function such that* $\phi(t) < t$ *and* $\phi(1) = 1$;

(4) *The pairs* (A, S) *and* (B, T) *are compatible.*

Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. As compatibility implies weak compatibility, so Theorem 3.5 follows directly from Theorem 3.4. □

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