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ON QUASI-EINSTEIN WARPED PRODUCTS

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Abstract. In this paper we investigate when an warped product manifold is a quasi-Einstein manifold and we give the expressions of the Ricci tensors and scalar curvatures for the bases and fibres. In some cases we give some obstructions to the existence of such manifolds.

1. Introduction

Warped products were introduced in ([4]), where it served to give examples of new Riemannian manifolds. In ([5]) it was introduced the notion of quasi-Einstein manifold, notion that was generalize in ([3], [6]).

In this paper we will investigate when an warped product manifold is quasi-Einstein manifold. According to ([5]) we have the following definitions.

Definition 1.1. A non-flat Riemannian manifold (M, g_M) is said to be a *quasi-Einstein* manifold if its Ricci tensor Ric_M satisfies the condition $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y)$ for every $X, Y \in \Gamma(TM)$ where a, b are real scalars and A is a non-zero 1-form on M such that $A(X) = g(X,U)$ for all vector field $X \in \Gamma(TM)$,

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U being an unit vector field which is called the generator of the manifold. If $b = 0$ then the manifold reduces to an Einstein space.

Definition 1.2. A non-flat Riemannian manifold (M^n, g), $n > 2$ is called a *generalized quasi-Einstein* manifold if its Ricci tensor Ric_M of type $(0, 2)$ is non-zero and satisfies the condition $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$ where a, b, c are real scalars and A, B two non-zero 1-forms. The unit vector fields U and V corresponding to the 1-forms A and respectively B are defined by $A(X) = g(X,U)$, $B(X) = g(X,V)$ and are orthogonal, i.e. $g(U,V) = 0$. If $c = 0$ the manifold reduces to a quasi Einstein manifold.

Definition 1.3. A non-flat Riemannian manifold (M^n, g), $n > 2$ is called a *mixed generalized quasi-Einstein* manifold if its Ricci tensor Ric_M of type $(0, 2)$ is non-zero and satisfies the condition $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)]$ for every $X, Y \in \Gamma(TM)$ where a, b, c, d are real scalars and A, B two non-zero 1-forms. The unit vector fields U and V corresponding to the 1-forms A and respectively B are defined by $A(X) = g(X,U)$, $B(X) = g(X,V)$ and are orthogonal, i.e. $g(U,V) = 0$. If $d = 0$ the manifold reduces to a generalized quasi Einstein manifold.

We have the following well-known definition of an warped product ([2]).

Definition 1.4. Let $(N, g_N), (F, g_F)$ be two Riemannian manifolds with $\dim N = m > 0$, $\dim F = k > 0$ and $f : N \longrightarrow (0, \infty)$, $f \in C^\infty(N)$. The *warped product* $M = N \times_f F$ is the Riemannian manifold $N \times F$ furnished with the metric $g_M = g_N + f^2 g_F$. B is called the base of M, F the fibre and the warped product is called a simply Riemannian product if f is a constant function. We denote by Ric_N, Ric_F and H^f the lifts to M of the Ricci curvatures of N and F, and the $Hessian$ of f, respectively.

We give the Ricci curvature of an warped product.

Proposition 1.1. *([17]) The Ricci curvature Ric_M of the warped product $M = N \times_f F$ satisfies:*

(1) $Ric_M(X,Y) = Ric_N(X,Y) - \frac{k}{f} H^f(X,Y)$,

(2) $Ric_M(X,V) = 0$,

(3) $Ric_M(V,W) = Ric_F(V,W) - g(V,W) f^{\#}$, *where* $f^{\#} = -\frac{\triangle f}{f} + \frac{k-1}{f^2} | \bigtriangledown f|^2$ *for any vectors* $X, Y \in \Gamma(TN)$ *and any vectors* $V, W \in \Gamma(TF)$, *where* H^f *and* $\triangle f$ *denote the Hessian of* f *and the Laplacian of* f *given by* $\triangle f = -Tr(H^f)$.

We give the scalar curvature of an warped product.

Proposition 1.2. *([2]) Let $M = N \times_f F$ be an warped product. Then the scalar curvature of M is given by*

$$\tau_M = \tau_N + \frac{\tau_F}{f^2} + 2k \frac{\triangle f}{f} - k(k-1) \frac{|\nabla f|^2}{f^2}$$

2. Quasi-Einstein warped products

Let $M = N \times_f F$ be an warped product manifold with $f : N \longrightarrow (0, \infty)$, $f \in C^\infty(N)$ and the metric $g_M = g_N + f^2 g_F$ which is also a quasi-Einstein manifold, that means its Ricci tensor satisfies:

$$Ric_M(X,Y) = a g_M(X,Y) + b A(X) A(Y) \qquad (1)$$

Starting from the above formula we want to compute the Ricci tensors of N and F. For that we will consider the following two cases: when U is tangent to N and when U is tangent to F.

Theorem 2.1. *Let $M = N \times_f F$ be an warped product which is also a quasi-Einstein manifold, that is its Ricci tensor satisfies (1).*

a). *When U is tangent to the base N the Ricci tensors of N and F satisfy the following equations:*

$$\begin{cases} Ric_N(X,Y) = ag_N(X,Y) + \frac{k}{f}H^f(X,Y) + bg_N(X,U)g_N(Y,U), \\ Ric_F(X,Y) = g_F(X,Y)[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2]. \end{cases} \quad (2)$$

b). *When U is tangent to the fibre F the Ricci tensors of N and F satisfy the following equations:*

$$\begin{cases} Ric_N(X,Y) = ag_N(X,Y) + \frac{k}{f}H^f(X,Y), \\ Ric_F(X,Y) = g_F(X,Y)[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2] + \\ \qquad bf^4 g_F(X,U)g_F(Y,U). \end{cases} \quad (3)$$

Proof.

a). For $X, Y \in \Gamma(TN)$ we have that $Ric_M(X,Y) = ag_N(X,Y) + bg_N(X,U)g_N(Y,U)$. Hence from proposition 1.1. we have $Ric_M(X,Y) = Ric_N(X,Y) - \frac{k}{f}H^f(X,Y)$. For $X, Y \in \Gamma(TF)$ we have $Ric_M(X,Y) = af^2 g_F(X,Y)$. Thus from proposition 1.1. we have $Ric_M(X,Y) = Ric_F(X,Y) - f^2 g_F(X,Y)[-\frac{\triangle f}{f} + \frac{k-1}{f^2}|\bigtriangledown f|^2]$.

b). For $X, Y \in \Gamma(TN)$ we have that $Ric_M(X,Y) = ag_N(X,Y)$.

Hence from proposition 1.1.we have $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f}H^f(X,Y)$.

For $X, Y \in \Gamma(TF)$ we have $Ric_M(X,Y) = af^2 g_F(X,Y) + bf^4 g_F(X,U)g_F(Y,U)$. Thus from proposition 1.1. we have $Ric_M(X,Y) = Ric_F(X,Y) - f^2 g_F(X,Y)[-\frac{\triangle f}{f} + \frac{k-1}{f^2}|\bigtriangledown f|^2]$.

We can give now the scalar curvatures of M, N and F.

Corollary 2.1. a). *Taking the traces in theorem 2.1., point a) we obtain:*

$$\begin{cases} \tau_M = (m+k)a + b, \\ \tau_N = ma - k\frac{\triangle f}{f} + b, \\ \tau_F = k[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2]. \end{cases} \quad (4)$$

b). *Taking the traces in theorem 2.1., point b) we obtain:*

$$\begin{cases} \tau_M = (m+k)a + b, \\ \tau_N = ma - k\frac{\triangle f}{f}, \\ \tau_F = k[-f\triangle f + (k-1)| \bigtriangledown f|^2 + af^2] + bf^4. \end{cases} \quad (5)$$

3. Obstructions to the existence of quasi-Einstein warped products

In this section we prove some obstructions to the existence of quasi-Einstein warped products. We consider two cases depending on U tangent to the base N or U tangent to the fibre F.

1). When U is tangent to F.

Theorem 3.1. *Let $M = N \times_f F$ be an warped product with N compact and connected,* $\dim N = m \geq 1$, $\dim F = k \geq 1$ *which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ for every $X, Y \in \Gamma(TM)$ with U an unitary vector field tangent to F.* If $m = 1$ or $k = 1$, then M is a simply Riemannian product.

Proof: If $m = 1$, then $\tau_N = 0$ and from the second equation of (5) we get that:

$$0 = ma - k\frac{\triangle f}{f} \Longrightarrow \triangle f = f \cdot \frac{ma}{k}$$

Then the Laplacian has constant sign and hence f is constant.

If $k = 1$, then $\tau_F = 0$ and from the third equation of (5) we have that:

$$0 = -f\triangle f + af^2 + bf^2 \Longrightarrow \triangle f = f(a+b)$$

Thus the Laplacian has constant sign and hence f is constant.

Theorem 3.2. *Let $M = N \times_f F$ be an warped product with* $\dim N = m \geq 2$, $\dim F = k \geq 2$ *which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ for every $X, Y \in \Gamma(TM)$ with U an unitary vector field tangent to F. If $b \neq 0$ then M reduces to a simply Riemannian product.*

Proof: Consider in the second equation of (3) that X, Y are orthogonal vectod fields tangent to F such that $g_M(X,U) \neq 0$ and $g_M(Y,U) \neq 0$. Then taking in consideration the different domains of definition of the functions that appear in the second equation of (3), we obtain that f is constant.

Remark 3.1. Since for $b = 0$ we obtain that M is an Einstein space, we conlude that there does not exist any warped product which is quasi-Einstein in the case when U is tangent to F.

2). When U is tangent to N.

Remark 3.2. From now on we consider in the second equation of (2) that:

$$-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = c \qquad (6)$$

Remark 3.3. So the Ricci tensors from (2) become:

$$\begin{cases} Ric_N(X,Y) = ag_N(X,Y) + \frac{k}{f}H^f(X,Y) + bg_N(X,U)g_N(Y,U), \\ Ric_F(X,Y) = cg_F(X,Y) \\ -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = c \end{cases} \qquad (7)$$

Theorem 3.3. *Let $M = N \times_f F$ be an warped product with N compact and connected,* $\dim N = m \geq 1$, $\dim F = k \geq 1$ *which is also a quasi-Einstein manifold* *with* $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ *for every* $X, Y \in \Gamma(TM)$ *with* U *an unitary vector field tangent to* N. *If* $m = 1$ *or* $k = 1$ *then* M *is a simply Riemannian product.*

Proof: If $m = 1$, then we have that $\tau_N = 0$ and from the second equation of (4) it follows that:

$$0 = ma - k\frac{\triangle f}{f} + b \Longrightarrow \triangle f = f \cdot \frac{ma+b}{k}$$

Hence the Laplacian has constant sign on a compact manifold, thus f is constant. If $k = 1$, then $\triangle f = af$ and thus the Laplacian has constant sign. So f is constant.

Theorem 3.4. *Let* $M = N \times_f F$ *be an warped product with* N *compact and connected,* $\dim N = m \geq 2$, $\dim F = k \geq 2$ *which is also a quasi-Einstein manifold with* $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ *for every* $X, Y \in \Gamma(TM)$ *with* U *an unitary vector field tangent to* N. *If* $a \leq 0$ *then* M *reduces to a simply Riemannian product.*

Proof: Let $z \in N$ such that $f(z)$ is the maximum of f on N. Then $\bigtriangledown f(z) = 0$ and $\triangle f(z) \geq 0$. Writing the equation (6) in the point z we obtain:

$$-f(z)\triangle f(z) + af^2(z) = c \qquad (8)$$

Now, from (6) and (8) we obtain that:

$$-f(z)\triangle f(z) + af^2(z) = -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 \Longrightarrow$$
$$f\triangle f = (k-1)|\bigtriangledown f|^2 + af^2 + f(z)\triangle f(z) - af^2(z) \Longrightarrow$$
$$f\triangle f = (k-1)|\bigtriangledown f|^2 + f(z)\triangle f(z) + a[f^2 - f^2(z)] \geq 0 \Longrightarrow \triangle f \geq 0$$

Thus f is constant.

From now on we will consider that $a > 0$.

Theorem 3.5. *Let $M = N \times_f F$ be an warped product with N compact and connected, $\dim N = m \geq 2$, $\dim F = k \geq 2$ which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ for every $X, Y \in \Gamma(TM)$ with U an unitary vector field tangent to N. If F has negative scalar curvature, then M reduces to a simply Riemannian product.*

Proof: From the third equation of (4) and (6) we get that $\tau_F = kc$. Since $\tau_F \leq 0$ it follows that $c \leq 0$. Then (6) becomes:

$$-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = c \Longrightarrow$$
$$-f\triangle f + af^2 = c - (k-1)|\bigtriangledown f|^2 \leq 0 \Longrightarrow$$
$$f\triangle f \geq af^2 > 0 \Longrightarrow \triangle f > 0$$

Thus f is constant.

From now on we will also consider that $c > 0$.

Theorem 3.6. *Let $M = N \times_f F$ be an warped product with N compact and F an Einstein space of constant c, $\dim N = m \geq 2$, $\dim F = k \geq 2$ which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in \mathbb{R}$, $A(X) = g_M(X,U)$ for every $X, Y \in \Gamma(TM)$, with U an unitary vector field tangent to N. Then M reduces to a simply Riemannian product if at least one of the following condition is true:*

a). $|\bigtriangledown f|^2 \geq \frac{c}{k-1}$,

b). $\tau_M \leq \tau_N$,

c). $\tau_M \geq \tau_N + \tau_F$ *and* $c \geq a$,

d). $\tau_N \leq 0$ *and* $b \geq 0$,

e). $\tau_N \geq \frac{mc}{f^2} + b$ *and* $m \geq k$,

f). $\tau_M \geq \tau_N + \frac{\tau_F}{f^2}$.

Proof: a) From (6) we have that:

$$-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = c \Longrightarrow$$
$$-f\triangle f + af^2 = c - (k-1)|\bigtriangledown f|^2 \leq 0 \Longrightarrow$$
$$f\triangle f \geq af^2 \Longrightarrow \triangle f \geq 0$$

Thus f is constant.

b) From the second equation of (4) we have that:

$$\tau_N = ma - k\tfrac{\triangle f}{f} + b \Longrightarrow \tau_N + ak = a(m+k) - k\tfrac{\triangle f}{f} + b = \tau_M - k\tfrac{\triangle f}{f} \Longrightarrow$$
$$\tau_M - \tau_N = ak + k\tfrac{\triangle f}{f} = \tfrac{k}{f}[af + \triangle f] \leq 0 \Longrightarrow \triangle f \leq -af < 0$$

Thus f is constant.

c) From the third equation of (4) and (6) we get that:

$$\tau_F = kc \geq ka \Longrightarrow \tau_F + \tau_N \geq ka + ma - k\tfrac{\triangle f}{f} + b = \tau_M - k\tfrac{\triangle f}{f} \Longrightarrow$$
$$k\tfrac{\triangle f}{f} \geq \tau_M - (\tau_F + \tau_N) \geq 0 \Longrightarrow \triangle f \geq 0$$

Thus f is constant.

d) From the second equation of (4) we have that:

$$\tau_N = ma - k\tfrac{\triangle f}{f} + b \Longrightarrow k\tfrac{\triangle f}{f} = ma - \tau_N + b \geq 0 \Longrightarrow \triangle f \geq 0$$

Thus f is constant.

e) From the second equation of (4) and (6) we have that:

$$\tau_N = ma - k\tfrac{\triangle f}{f} + b \Longrightarrow$$
$$\tau_N \cdot f^2 = maf^2 - kf\triangle f + bf^2 = m(c + f\triangle f - (k-1)|\bigtriangledown f|^2) - kf\triangle f + bf^2 =$$
$$(m-k)f\triangle f + mc - m(k-1)|\bigtriangledown f|^2 + bf^2 \Longrightarrow$$
$$\tau_N \cdot f^2 - mc - bf^2 = (m-k)f\triangle f - m(k-1)|\bigtriangledown f|^2 \geq 0$$

i) $m = k$ implies $-m(m-1)|\bigtriangledown f|^2 \geq 0 \Longrightarrow m(m-1)|\bigtriangledown f|^2 \leq 0$
$\Longrightarrow |\bigtriangledown f|^2 = 0 \Longrightarrow \bigtriangledown f = 0$.

ii). $m > k$ implies $(m-k)f \bigtriangleup f \geq m(k-1)|\bigtriangledown f|^2 \geq 0 \Longrightarrow \bigtriangleup f \geq 0$.

Thus, f is constant.

f) From proposition 1.2. it follows directly that:

$$\tau_M - (\tau_N + \frac{\tau_F}{f^2}) = 2k\frac{\bigtriangleup f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2} \geq 0 \Longrightarrow \bigtriangleup f \geq \frac{k-1}{2} \cdot \frac{|\nabla f|^2}{f} \geq 0$$

Thus f is constant.

Remark 3.4. Similar obstructions can be obtained when the warped product is a generalized quasi Einstein space or a mixed generalized quasi Einstein space.

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