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M^k-TYPE ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL OPERATOR ON SPACE OF HOMOGENEOUS TYPE

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Abstract. In this paper, we prove the M^k-type inequality for multilinear commutator related to singular integral operator on space of homogeneous type. By using the M^k-type inequality, we obtain the weighted L^p-norm inequality and the weighted estimates on the generalized Morrey spaces for the multilinear commutator.

1. Introduction and Preliminaries

As the development of singular integral operators, their commutators have been well studied ([1][11][12][13]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss ([9]) states that commutator $[b,T](f) = T(bf) - bT(f)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In ([11][12][13]), the sharp estimates for some multilinear commutators

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of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the M^k-type inequality for the multilinear commutators related to the singular integral operators on the space of homogeneous type. By using the M^k-type inequality, we obtain the weighted L^p-norm inequality and the weighted estimates on the generalized Morrey spaces for the multilinear commutator.

Given a set X, a function $d : X \times X \to R_0^+$ is called a quasi$-$distance on X if the following conditions are satisfied:

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, for every x and y in X,

(2) $d(x, y) = d(y, x)$, for every x and y in X,

(3) there exists a constant $K \geq 1$ such that

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for every x, y and z in X.

We shall say that two quasi-distance d and d' on X are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y)$ for all $x, y \in X$. In particular, equivalent quasi-distances induce the same topology on X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type. The constants K and A will be called the constants of the space.

Define the singular integral operator T by

$$T(f)(x) = \int_X K(x, y)f(y)d\mu(y),$$

where $K(x, y)$ is a standard calderón-Zygmund kernel.
By the calderón-Zygmund theory ([12]), we know that T is bounded on $L^p(X)$ for any p with $1 < p < \infty$, and is bounded from $L^1(X)$ to weak $L^1(X)$. We can also realize that the kernel function K satisfies the standard Hölder regularity condition,with some $\delta \in (0, 1)$, such that

$$|K(x, y) - K(x_0, y)| \leq C \frac{(d(x, x_0))^\delta}{\mu(B(x, d(y, x)))(d(y, x))^\delta}, \qquad if \;\; d(y, x) \geq 2d(x, x_0).$$

Let b be a $BMO(X)$ function, define the commutator of T and b by

$$T_b(f)(x) = b(x)T(f)(x) - T(bf)(x).$$

Bramanti and Christina proved the boundedness for the commutator of singular integral operator ([1][2][3][4]). And the endpoint estimate are obtained by Chen and Sawyer in [5].

In this paper, we will study the multilinear commutator as following: Suppose b_j are the fixed locally integral functions on X and $j = 1, 2, \cdots, m(m \in N)$. The multilinear commutator related to T is defined by

$$T_{\vec{b}}(f)(x) = \int_X \prod_{j=1}^{m} (b_j(x) - b_j(y))K(x, y)f(y)d\mu(y).$$

Note that when $m = 1$, $T_{\vec{b}}$ is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors ([19][20][21]). Cohen and Gosselin ([6][7][8]) obtained the $L^p(p > 1)$ boundedness of the multilinear singular integral operators. Hu and Yang ([13])proved a variant shape estimate for the multilinear singular integral operators. In[21], the authors prove some shape estimates for the multilinear commutator. As the Morrey space may be considered as an extension of Lebesgue space ([16]), it is natural and important to study the boundedness of such operator on the Morrey space. The purpose of this paper has two-fold,first, we establish a M^k-type estimate for the multilinear commutator related to the singular integral operators, and second, we obtain the weighted L^p-norm inequality and the weighted estimates on the generalized Morrey space for the multilinear commutator by using the sharp inequality.

To state our result,we first give some notions.

In this paper, B will denote a ball in X and $f_B = \mu(B)^{-1}\int_B f(x)d\mu(x)$, we define the sharp function of f as

$$f^{\#}(x) = \sup_{B\ni x}\frac{1}{\mu(B)}\int_B |f(y) - f_B|d\mu(y).$$

It is well-known that ([12])

$$f^{\#}(x) \approx \sup_{B\ni x}\inf_{c\in C}\frac{1}{\mu(B)}\int_B |f(y) - C|d\mu(y).$$

We say that f belongs to $BMO(X)$ if $f^{\#}$ belongs to $L^\infty(X)$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$.

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{B\ni x}\mu(B)^{-1}\int_B |f(y)|d\mu(y).$$

For $0 < p < \infty$, we denote $M_p f(x)$ by

$$M_p(f)(x) = [M(|f|^p)(x)]^{1/p}.$$

For $k \in N$, we denote by M^k the operator M iterated k times,
i.e. $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \qquad when\ \ k \geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \Phi\left(\frac{|f(y)|}{\lambda}\right)d\mu(y) \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in B} ||f||_{\Phi,B}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,B}$, $M_{L(logL)^r}$ and $||\cdot||_{expL^{1/r},B}$, $M_{expL^{1/r}}$. Following [19][20], we know the generalized Hölder's inequality:

$$\frac{1}{\mu(B)}\int_B |f(y)g(y)|d\mu(y) \le ||f||_{\Phi,B}||g||_{\tilde{\Phi},B}.$$

And we can also obtain the following inequalities:

$$||f||_{L(logL)^{1/r},B} \le M_{L(logL)^{1/r}}(f) \le CM_{L(logL)^m}(f) \le CM^{m+1}(f),$$

$$||b - b_B||_{expL^r,B} \le C||b||_{BMO},$$

$$|b_{2^{k+1}B} - b_{2B}| \le Ck||b||_{BMO}.$$

for $r, r_j \ge 1, j = 1, 2, \cdots, m$ with $1/r = 1/r_1 + 1/r_2 \cdots + 1/r_m$, and any $x \in X$, $b \in BMO(X)$.

We say that b belongs to $BMO(X)$ if $M^\#(b)(x)$ belongs to $L^\infty(X)$ and define $||b||_{BMO} = ||b^\#||_{L^\infty}$. It is known that([12])

$$||b - b_{2^kB}||_{BMO} \le Ck||b||_{BMO}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^{j} b_{\sigma(i)}$ and $||\vec{b}_\sigma||_{BMO} = \prod_{i=1}^{j} ||b_{\sigma(i)}||_{BMO}$.

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$([11]), that is

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}$$

and, for $1 < p < \infty$,

$$A_p = \left\{ w : \sup_B \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w(x)^{-1/(p-1)} d\mu(x) \right)^{p-1} < \infty \right\}.$$

Throughout this paper, φ will denote a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \qquad for \;\; t \geq 0.$$

Let w be a weight function on X(that is w is a non-negative locally integrable function on X) and f be a locally integrable function on X. We define the norm as:

$$||f||_{L^{p,\varphi}(w)} = \sup_{x \in X, \; d>0} \left(\frac{1}{\varphi(d)} \int_{B(x,d)} |f(y)|^p w(y) d\mu(y) \right)^{1/p},$$

for $1 \leq p < \infty$, where $B(x,d) = \{y \in X : |x - y| < d\}$.

The generalized weighted Morrey spaces is defined by

$$L^{p,\varphi}(X,w) = \{f \in L^1_{loc}(X) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(X,w) = L^{p,\delta}(X,w)$, which is the classical Morrey spaces ([17][18]).

2. Theorems and Proofs

Now we state our theorems as following.

Theorem 1. Let $b_j \in BMO(X)$ for $j = 1, \cdots, m$. Then for any $0 < r < 1$, $k \geq m+1$, $k \in N$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(X)$ and any $\tilde{x} \in X$,

$$(T_{\vec{b}}(f))^\#_r(\tilde{x}) \leq C||\vec{b}||_{BMO} \left(M^k(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M^k(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}) \right).$$

Theorem 2. Let $b_j \in BMO(X)$ for $j = 1, \cdots, m$. Then $T_{\vec{b}}$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$.

Theorem 3. Let $1 < p < \infty$, $w \in A_p$ and $b_j \in BMO(X)$ for $j = 1, \cdots, m$. Suppose μ satisfies the doubling condition: there exists a constant $n_0 > 0$ such that $\mu(B(x, 2r)) \geq 2^{n_0}\mu(B(x, r))$ for all $x \in X$ and $r > 0$. Then, if $0 < D < 2^{n_0}$,

$$||T_{\vec{b}}(f)||_{L^{p,\varphi}(w)} \leq C||\vec{b}||_{BMO}||f||_{L^{p,\varphi}(w)}.$$

In order to prove the theorems, we need the following lemmas.

Lemma 1. Let $1 < r < \infty$ and $b_j \in BMO(X)$ with $j = 1, \cdots, k$ and $k \in N$. Then, we have

$$\frac{1}{\mu(B)}\int_B \prod_{j=1}^{k} |b_j(y) - (b_j)_B| d\mu(y) \leq C \prod_{j=1}^{k} ||b_j||_{BMO},$$

$$\left(\frac{1}{\mu(B)}\int_B \prod_{j=1}^{k} |b_j(y) - (b_j)_B|^r d\mu(y)\right)^{1/r} \leq C \prod_{j=1}^{k} ||b_j||_{BMO}.$$

Similarly, for $\sigma \in C_k^m$,when $k \leq m$ and $m \in N$, we have:

$$\frac{1}{\mu(B)}\int_B |(b(y) - (b_j)_B)_\sigma| d\mu(y) \leq C||b_\sigma||_{BMO}$$

and

$$\left(\frac{1}{\mu(B)}\int_B |(b(y) - (b_j)_B)_\sigma|^r d\mu(y)\right)^{1/r} \leq C||b_\sigma||_{BMO}.$$

In fact, we just need to choose $p_j > 1$ and $q_j > 1$, where $1 \leq j \leq k$, such that $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$. After that, using the Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$. respectively, we may get the results.

Lemma 2.([11],p.485) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in X : f(x) > \lambda\}|^{1/q}, \qquad N_{p,q}(f) = \sup_B ||f\chi_B|_{L^p}/||\chi_B||_{L^r},$$

where the sup is taken for all measurable sets B with $0 < \mu(B) < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 3.([11]) Let $0 < p, \eta < \infty$ and $w \in \cup_{1\le r<\infty} A_r$. Then

$$||M_\eta(f)||_{L^p(w)} \le C||f_\eta^\#(f)||_{L^p(w)}.$$

Lemma 4. Let $1 < p < \infty$, $w \in A_1$ and $1 \le q < p$. Suppose μ satisfies the doubling condition: there exists a constant $n_0 > 0$ such that
$\mu(B(x,2r)) \ge 2^{n_0}\mu(B(x,r))$ for all $x \in X$ and $r > 0$. Then, if $0 < D < 2^{n_0}$,

$$||M_q(f)||_{L^{p,\varphi}(\omega)} \le C||f||_{L^{p,\varphi}(w)}.$$

Proof. Let $f \in L^{p,\varphi}(X,w)$. Note that $1 \le q < p$ and for any $u \in A_1$,

$$\int_X |M_q(f)(y)|^p u(y)d\mu(y) \le C\int_X |f(y)|^p u(y)d\mu(y).$$

For a ball $B = B(x,d) \subset X$, we get

$$\begin{aligned}
& \int_B |M_q(f)(y)|^p w(y)d\mu(y) \\
\le\ & \int_X |M_q(f)(y)|^p M(w\chi_B)(y)d\mu(y) \\
\le\ & C\int_X |f(y)|^p M(w\chi_B)(y)d\mu(y) \\
=\ & C\left[\int_B |f(y)|^p M(w\chi_B)(y)d\mu(y) + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB} |f(y)|^p M(w\chi_B)(y)d\mu(y)\right] \\
\le\ & C\left[\int_B |f(y)|^p w(y)d\mu(y) + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB} |f(y)|^p \frac{w(y)}{\mu(2^{k+1}B)}d\mu(y)\right] \\
\le\ & C\left[\int_B |f(y)|^p w(y)d\mu(y) + \sum_{k=0}^{\infty}\int_{2^{k+1}B} |f(y)|^p \frac{M(w)(y)}{2^{n_0(k+1)}}d\mu(y)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq C\left[\int_B |f(y)|^p w(y)d\mu(y) + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f(y)|^p\frac{w(y)}{2^{n_0k}}d\mu(y)\right]\\
&\leq C||f||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}2^{-n_0k}\varphi(2^{k+1}d)\\
&\leq C||f||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}(2^{-n_0}D)^k\varphi(d)\\
&\leq C||f||^p_{L^{p,\varphi}(w)}\varphi(d).
\end{aligned}$$

thus,

$$||M_q(f)||_{L^{p,\varphi}(\omega)} \leq C||f||_{L^{p,\varphi}(w)}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$. Then, for $f \in L^{p,\varphi}(X, w)$,

$$||M(f)||_{L^{p,\varphi}(w)} \leq C||f^{\#}||_{L^{p,\varphi}(w)}.$$

The proof of the Lemma is similar to that of Lemma 4 by Lemma 3, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^{\infty}(X)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |T_{\vec{b}}(f)(x) - C_0|^r d\mu(x)\right)^{1/r}\\
&\leq C||\vec{b}||_{BMO}\left(M^k(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M^k(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).
\end{aligned}$$

Fix a ball $B = B(x_0, d)$ and $\tilde{x} \in B$, we write $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$. Following [P–P], we will consider the cases $m = 1$ and $m > 1$, and choose $C_0 = T(((b_1)_{2B} - b_1)f_2)(x_0)$ and $C_0 = T(\prod_{j=1}^m (b_j - (b_j)_{2B})f_2)(x_0)$, respectively.

We first consider the **Case** $m = 1$**.** For $C_0 = T(((b_1)_{2B} - b_1)f_2)(x_0)$, we write

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2B})T(f)(x) - T((b_1 - (b_1)_{2B})f)(x).$$

Then

$$
\begin{aligned}
& |T_{b_1}(f)(x)-C_0| \\
= & |(b_1(x)-(b_1)_{2B})T(f)(x)+T(((b_1)_{2B}-1f65b_1)f)(x)-T(((b_1)_{2B}-b_1)f_2)(x_0)| \\
\leq & |(b_1(x)-(b_1)_{2B})T(f)(x)|+|T(((b_1)_{2B}-b_1)f_1)(x)| \\
& +|T(((b_1)_{2B}-b_1)f_2)(x)-T(((b_1)_{2B}-b_1)f_2)(x_0)| \\
= & A(x)+B(x)+C(x).
\end{aligned}
$$

For $A(x)$, we get

$$
\begin{aligned}
& \left(\frac{1}{\mu(B)}\int_B |A(x)|^r d\mu(x)\right)^{1/r} \\
\leq & \frac{1}{\mu(B)}\int_B |A(x)|d\mu(x) \\
\leq & \frac{1}{\mu(B)}\int_B |(b_1(x)-(b_1)_{2B})T(f)(x)|d\mu(x) \\
\leq & ||b_1-(b_1)_{2B}||_{\exp L,2B}||T(f)||_{L(\log L),2B} \\
\leq & C||b_1||_{BMO}M^2(T(f))(\tilde{x}).
\end{aligned}
$$

For $B(x)$, by the weak type (1,1) of T and Lemma 2 , we obtain

$$
\begin{aligned}
& \left(\frac{1}{\mu(B)}\int_B |B(x)|^r d\mu(x)\right)^{1/r} \\
\leq & \frac{1}{\mu(B)}\int_B |B(x)|d\mu(x) \\
= & \frac{1}{\mu(B)}\int_B |T(((b_1)_{2B}-b_1)f_1)(x)|d\mu(x) \\
\leq & \left(\frac{1}{\mu(B)}\int_{2B} |T((b_1-(b_1)_{2B})f\chi_{2B})(x)|^p d\mu(x)\right)^{1/p} \\
= & \frac{1}{\mu(B)}\frac{1}{\mu(B)^{\frac{1}{p}-1}}||T(((b_1)_{2B}-b_1)(f\chi_{2B})(x))||_{L^p}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\frac{1}{\mu(B)}||T((b_1)_{2B}-b_1)(f\chi_{2B})(x)||_{WL^1}\\
&\le C\frac{1}{\mu(B)}||((b_1)_{2B}-b_1)(f\chi_{2B})(x)||_{L^1}\\
&\le C\frac{1}{\mu(B)}\int_{2B}|((b_1)_{2B}-b_1(x))||f(x)|d\mu(x)\\
&\le C||(b_1)_{2B}-b_1(x)||_{expL,2B}||f||_{L(\log L),2B}\\
&\le C||b_1||_{BMO}M^2(f)(\tilde{x}).
\end{aligned}
$$

For $C(x)$, we have, for $x\in B$,

$$
\begin{aligned}
&|T(b_1-(b_1)_{2B})(f_2)(x)-T(b_1-(b_1)_{2B})(f_2)(x_0)|\\
=&\left|\int_{(2B)^c}(b_1(y)-(b_1)_{2B})f_2(y)(K(x,y)-K(x_0,y))d\mu(y)\right|\\
\le&\int_{(2B)^c}|(b_1(y)-(b_1)_{2B})||f(y)||(K(x,y)-K(x_0,y))|d\mu(y)\\
\le& C\sum_{k=1}^{\infty}\int_{2^{k+1}B/2^kB}\frac{(d(x,x_0))^\delta}{\mu(B(x,d(x,y)))(d(x,y))^\delta}|f(y)||b_1(y)-(b_1)_{2B}|d\mu(y)\\
\le& C\sum_{k=1}^{\infty}\frac{{r_0}^\delta}{\mu(2^kB)}\frac{1}{(2^kr_0)^\delta}\int_{2^{k+1}B}|f(y)||b_1(y)-(b_1)_{2B}|d\mu(y)\\
\le& C\sum_{k=1}^{\infty}2^{-k\delta}\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}|f(y)||b_1(y)-(b_1)_{2B}|d\mu(y)\\
\le& C\sum_{k=1}^{\infty}2^{-k\delta}||b_1(y)-(b_1)_{2B}||_{expL,2^{k+1}B}||f||_{L(\log L),2^{k+1}B}\\
\le& C||b_1||_{BMO}M^2(f)(\tilde{x}).
\end{aligned}
$$

thus,we can obtain:

$$
\left(\frac{1}{\mu(B)}\int_B|C(x)|^r dx\right)^{1/r}\le C||b_1||_{BMO}M^2(f)(\tilde{x}).
$$

Now, we consider the **Case** $m \geq 2$. we have, for $b = (b_1, \cdots, b_m)$,

$$
\begin{aligned}
& T_{\vec{b}}(f)(x) = \int_X \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x,y) f(y) d\mu(y) \\
= & \int_X \prod_{j=1}^{m} [(b_j(x) - (b_j)_{2B}) - (b_j(y) - (b_j)_{2B})] K(x,y) f(y) d\mu(y) \\
= & \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2B})_\sigma \int_X (b(y) - (b)_{2B})_{\sigma^c} K(x,y) f(y) d\mu(y) \\
= & \prod_{j=1}^{m} (b_j(x) - (b_j)_{2B}) \int_X K(x,y) f(y) d\mu(y) \\
& + (-1)^m \int_X \prod_{j=1}^{m} (b_j(y) - (b_j)_{2B}) K(x,y) f(y) d\mu(y) \\
& + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b_j(x) - (b_j)_{2B})_\sigma \int_X (b_j(y) - (b_j)_{2B})_{\sigma^c} K(x,y) f(y) d\mu(y) \\
= & \prod_{j=1}^{m} (b_j(x) - (b_j)_{2B}) T(f)(x) + (-1)^m T(\prod_{j=1}^{m} (b_j(y) - (b_j)_{2B}) f)(x) \\
& + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} ((b_j(x) - (b_j)_{2B})_\sigma T(b_j - (b_j)_{2B})_{\sigma^c}(f)(x)
\end{aligned}
$$

thus, recall that $C_0 = T(\prod_{j=1}^{m} (b_j(y) - (b_j)_{2B}) f_2)(x_0)$,

$$
\begin{aligned}
& |T_{\vec{b}}(f)(x) - T(\prod_{j=1}^{m} (b_j(y) - (b_j)_{2B}) f_2)(x_0)| \\
\leq & |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2B}) T(f)(x)| \\
& + |T(\prod_{j=1}^{m} (b_j(y) - (b_j)_{2B}) f_1)(x)| \\
& + |\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ((b_j(x) - (b_j)_{2B})_\sigma T(b_j - (b_j)_{2B})_{\sigma^c}(f)(x)|
\end{aligned}
$$

$$
\begin{aligned}
&+|T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x)-T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)| \\
=\ & I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, we get,

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |I_1(x)|^r d\mu(x)\right)^{1/r} \le \frac{1}{\mu(x)}\int_B |I_1(x)|d\mu(x) \\
\le\ & \frac{1}{\mu(B)}\int_B |\prod_{j=1}^{m}(b_j(x)-(b_j)_{2B})||T(f)(x)|d\mu(x) \\
\le\ & C||\prod_{j=1}^{m}(b_j(x)-(b_j)_{2B})||_{\exp L^{1/r_j},2B}||T(f)||_{L(\log L)^r,2B} \\
\le\ & C\prod_{j=1}^{m}||b_j||_{BMO}M^{m+1}(T(f))(\tilde{x}) \\
\le\ & C||\vec{b}||_{BMO}M^k(T(f))(\tilde{x}).
\end{aligned}
$$

For $I_2(x)$, by the boundness of T on $L^p(X)$ and similar to the proof of $B(x)$, using Lemma 2 , we get

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |I_2(x)|^r d\mu(x)\right)^{1/r} \le \frac{1}{\mu(B)}\int_B |I_2(x)|d\mu(x) \\
=\ & \frac{1}{\mu(B)}\int_B |T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_1)(x)|d\mu(x) \\
\le\ & \left(\frac{1}{\mu(B)}\int_B |T(\prod_{j=1}^{m}(b_j-(b_j)_{2B})f_1)(x)|^p d\mu(x)\right)^{1/p} \\
=\ & \frac{1}{\mu(B)}\frac{1}{\mu(B)^{\frac{1}{p}-1}}||T(\prod_{j=1}^{m}(b_j-(b_j)_{2B})f_1)(x))||_{L^p}
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{\mu(B)}||T(\prod_{j=1}^{m}(b_j-(b_j)_{2B})f_1)(x))||_{WL^1}\\
&\le \frac{1}{\mu(B)}||(\prod_{j=1}^{m}(b_j-(b_j)_{2B})f_1)(x))||_{L^1}\\
&\le \frac{1}{\mu(B)}\int_B|\prod_{j=1}^{m}(b_j-(b_j)_{2B})||f_1(x)|d\mu(x)\\
&\le C\prod_{j=1}^{m}||(b_j-(b_j)_{2B})||_{\exp L^{1/r_j},2B}||f||_{L(\log L)^r,2B}\\
&\le C||\vec{b}||_{BMO}M^{m+1}(f)(\tilde{x})\\
&\le C||\vec{b}||_{BMO}M^{k}(f)(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, by Lemma 2 ,

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B|I_3(x)|^r d\mu(x)\right)^{1/r} \le \frac{1}{\mu(B)}\int_B|I_3(x)|d\mu(x)\\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\mu(B)}\int_B|(b_j(x)-(b_j)_{2B})_\sigma||T(b_j-(b_j)_{2B})_{\sigma^c}(f)(x)|d\mu(x)\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b_j(x)-(b_j)_{2B})_\sigma||_{\exp L^{1/r_j},2B}||T(b_j-(b_j)_{2B})_{\sigma^c}(f)||_{L(\log L)^r,2B}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||b_\sigma||_{BMO}M^{m+1}(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}||_{BMO}M^{k}(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, similar to the proof of $C(x)$ in the **Case** $m=1$. We have :

$$\begin{aligned}
&|T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x)-T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)|\\
&= |\int_{(2B)^c}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f(y)(K(x,y)-K(x_0,y))d\mu(y)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{(2B)^c} |\prod_{j=1}^m (b_j(y)-(b_j)_{2B})||f(y)||(K(x,y)-K(x_0,y))|d\mu(y) \\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}B/2^kB} \frac{(d(x,x_0))^\delta}{\mu(B(x,d(x,y)))(d(x,y))^\delta}|f(y)||\prod_{j=1}^m (b_j(y)-(b_j)_{2B})|d\mu(y) \\
&\leq C\sum_{k=1}^\infty \frac{{r_0}^\delta}{\mu(2^kB)}\frac{1}{(2^kr_0)^\delta}\int_{2^{k+1}B}|f(y)||\prod_{j=1}^m (b_j(y)-(b_j)_{2B})|d\mu(y) \\
&\leq C\sum_{k=1}^\infty 2^{-k\delta}\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}|f(y)||\prod_{j=1}^m (b_j(y)-(b_j)_{2B})|d\mu(y) \\
&\leq C\sum_{k=1}^\infty 2^{-k\delta}||\prod_{j=1}^m (b_j(y)-(b_j)_{2B})||_{\exp L^{1/r_j},2B}||f||_{L(\log L)^r,2B} \\
&\leq C||\vec{b}||_{BMO}M^{m+1}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$
\left(\frac{1}{\mu(B)}\int_B |I_4(x)|^r d\mu(x)\right)^{1/r} \leq ||\vec{b}||_{BMO}M^k(f)(\tilde{x}).
$$

This completes the proof of the theorem.

Proof of Theorem 2. By Theorem 1 and the $L^p(w)$-boundedness of M^k, we may obtain the conclusion of Theorem 2 by induction.

Proof of Theorem 3. We first consider the case m=1. Taking $0 < r < 1$ in Theorem 1, by Lemma 4 and Lemma 5, we obtain

$$
\begin{aligned}
&||T_{\vec{b}}(f)||_{L^{p,\varphi}(w)} \leq ||M(T_{\vec{b}}(f))||_{L^{p,\varphi}(w)} \leq C||(T_{\vec{b}})_r^{\#}(f)||_{L^{p,\varphi}(w)} \\
&\leq C||\vec{b}||_{BMO}\left(||M^k(f)||_{L^{p,\varphi}(w)} + ||M^k(T(f))||_{L^{p,\varphi}(w)}\right) \\
&\leq C||\vec{b}||_{BMO}\left(||f||_{L^{p,\varphi}(w)} + ||T(f))||_{L^{p,\varphi}(w)}\right) \\
&\leq C||\vec{b}||_{BMO}\left(||f||_{L^{p,\varphi}(w)} + ||f||_{L^{p,\varphi}(w)}\right) \\
&\leq C||\vec{b}||_{BMO}||f||_{L^{p,\varphi}(w)}.
\end{aligned}
$$

When $m \geq 2$, we may get the conclusion of Theorem 3 by induction.

This completes the proof of Theorem 3.

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References

[1] A. Bernardis, S. Hartzstein and G. Pradolini, Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type, *J. Math. Anal. Appl.* **322**(2006), 825–846

[2] J. J. Betancor, A commutator theorem for fractional integrals in spaces of homogeneous type, *Inter. J. Math. & Math. Sci.* **6**(2000), 419–421

[3] M. Bramanti and M. C. Cerutti, Commutators of singular integrals and fractional integrals on homogeneous spaces, *Harmonic Analysis and Operator Theory, Contemp. Math., Amer. Math. Soc.* **189**(1995), 81–94

[4] M. Bramanti and M. C. Cerutti, Commutators of singular integrals on homogeneous spaces, *Boll. U. M. I.* **10(B)**(1996), 843–883

[5] W. Chen and E. Sawyer, Endpoint estimates for commutators of fractional integrals on spaces of homogeneous type, *J. Math. Anal. Appl.* **282**(2003), 553–566

[6] J. Cohen, A sharp estimate for a multilinear singular integral on R^n, *Indiana Univ. Math. J.* **30**(1981), 693–702

[7] J. Cohen and J. Gosselin, On multilinear singular integral operators on R^n, *Studia Math.* **72**(1982), 199–223.

[8] J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integral operators, *Illinois J. Math.* **30**(1986), 445-465

[9] R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.* **103**(1976), 611–635

[10] Y. Ding and S. Z. Lu, Weighted boundedness for a class rough multilinear operators, *Acta Math. Sinica* **17**(2001), 517–526

[11] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math., **116**, Amsterdam, 1985

[12] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weighted Theory for Integral Transforms on Spaces of Homogeneous Type*, Piman Monogr. and Surveys in Pure and Appl. Math., **92**, Addison-Wesley/Longman, 1998

[13] G. E. Hu and D. C. Yang, A variant sharp estimate for multilinear singular integral operators, *Studia Math.* **141**(2000), 25–42

[14] L. Z. Liu, Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators, *Acta Math. Scientia* **25(B)(1)**(2005), 89–94

[15] R. Macias and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. Math.* **33**(1979), 257–270

[16] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, *Harmonic Analysis*, Proceedings of a conference held in Sendai, Japan, 1990, 183–189

[17] J. Peetre, On convolution operators leaving $L^{p,\lambda}$-spaces invariant, *Ann. Mat. Pura. Appl.* **72**(1966), 295–304

[18] J. Peetre, On the theory of $L^{p,\lambda}$-spaces, *J. Func. Anal.* **4**(1969), 71–87

[19] C. Pérez, Endpoint estimate for commutators of singular integral operators, *J. Func. Anal.* **128**(1995), 163–185

[20] C. Pérez and G. Pradolini, Sharp weighted endpoint estimates for commutators of singular integral operators, *Michigan Math. J.* **49**(2001), 23–37

[21] C. Pérez and R. Trujillo-Gonzalez, Sharp Weighted estimates for multilinear commutators, *J. London Math. Soc.* **65**(2002), 672–692

[22] G. Pradolini, O. Salinas, Commutators of singular integrals on space of homogeneous type, preprint, *available at http://math.unl.edu.ar/preprints*

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