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ON SUM OF G-FRAMES IN HILBERT SPACES

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Abstract. We consider the sum of finite number of g-frames for a Hilbert space H and observed that their sum need not be a g-frame. A necessary and sufficient condition for the sum of two g-frames to be a g-frame has been obtained. Also, we study the sum of finite number of g-Bessel sequences and obtained some results regarding sum of g-Bessel sequences.

1. Introduction

In 1952, Duffin and Schaeffer [6] introduced frames for Hilbert spaces while addressing some difficult problems arising from the theory of non-harmonic Fourier series. In particular, they generalized Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossman and Meyer [2] found a new fundamental application to wavelet and Gabor transforms in which frames played an important role. Today, frames have been widely used in signal processing, data compression, sampling theory and many other fields.

Recently, Sun [9] introduced a g-frame and a g-Riesz bases in a Hilbert space and obtained some results for g-frames and g-Riesz bases. He also observed that frame of subspaces (fusion frames) introduced by Casazza and Kutyniok [5] is a particular case

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of g-frame in a Hilbert space. Also, a system of bounded quasi-projectors introduced by Fornasier [3] is a particular case of g-frame in a Hilbert space.

In the present paper, we consider the sum of finite number of g-frames for a Hilbert space H and observed that their sum need not be a g-frame. Examples have been given in this direction. Also, we give a necessary and sufficient condition for the sum of two g-frames to be a g-frame. Finally, we study sum of finite number of g-Bessel sequences and obtained some results regarding sum of g-Bessel sequences.

2. Preliminaries

Throughout this paper, H and K are two Hilbert spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\{H_i\}_{i\in I}$ is a sequence of Hilbert spaces over $\mathbb{K}$, where I is a subset of integers, $B(H, H_i)$ is the collection of all bounded linear operators from H into H_i and I_H be the identity operator on H.

Definition 2.1. A sequence $\{x_i\}_{i\in I} \subset H$ is called a *frame* for H if there exist constants A, $B > 0$ such that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B\|x\|^2, \quad \text{for all } \ x \in H\,.$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the frame $\{x_i\}_{i\in I}$.

Definition 2.2 ([9])**.** A sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is called a *generalized frame* or simply a *g-frame* for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{i\in I} \|\Lambda_i x\|^2 \le B\|x\|^2, \quad \text{for all } \ x \in H\,. \tag{2.1}$$

The positive constants A and B, respectively, are called the lower and upper frame bounds of the g-frame $\{\Lambda_i\}_{i\in I}$. The g-frame $\{\Lambda_i\}_{i\in I}$ is called *a tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. The sequence $\{\Lambda_i\}_{i\in I}$ is called a

g-Bessel *sequence* for H with respect to $\{H_i\}_{i\in I}$ with bound B if $\{\Lambda_i\}_{i\in I}$ satisfies the right hand side of the inequality (2.1). The sequence $\{\Lambda_i\}_{i\in I}$ is called an *exact g-frame* if it ceases to be a g-frame whenever any one of its elements is removed. The sequence $\{\Lambda_i\}_{i\in I}$ is called *g-complete* if $\{x \in H : \Lambda_i x = 0, \text{ for all } i \in I\} = \{0\}$.

Notation. For each sequence $\{H_i\}_{i\in I}$, define $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ by

$$\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2} = \left\{\{a_i\}_{i\in I} : a_i \in H_i, i \in I \ \text{ and } \sum_{i\in I} \|a_i\|^2 < \infty\right\}$$

with the inner product defined by $\langle\{a_i\}, \{b_i\}\rangle = \sum_{i\in I}\langle a_i, b_i\rangle$.

It is clear that $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ is a Hilbert space with pointwise operations.

The following results which are referred in this paper are listed in the form of lemmas

Lemma 2.3 ([1])**.** *$\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-Bessel sequence for H with bound B if and only if the operator $T : \left(\sum_{i\in I} \oplus H_i\right)_{\ell_2} \to H$ defined by*

$$T(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(x_i)$$

is a well-defined and bounded operator with $\|T\| \le \sqrt{B}$.

Lemma 2.4 ([1])**.** *A sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is g-frame for H if and only if the operator T: $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2} \to H$ defined by*

$$T(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(x_i)$$

is a well-defined and bounded operator from $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ onto H.

The operator T is called the synthesis operator of $\{\Lambda_i\}_{i\in I}$ and the adjoint T^* of the synthesis operator T is called the analysis operator of $\{\Lambda_i\}_{i\in I}$.

Lemma 2.5 ([1]). *If $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$ with bounds A and B, then the g-frame operator $S : H \to H$ defined by*

$$Sx = \sum_{i\in I} \Lambda_i^* \Lambda_i x, \quad \textit{for all} \;\; x \in H \tag{2.2}$$

is a positive invertible operator. Further, S satisfies

$$A\|x\|^2 \le \langle Sx, x\rangle \le B\|x\|^2, \quad \textit{for any} \;\; x \in H\,.$$

Furthermore, S is a bounded self-adjoint operator satisfying

$$B^{-1}\|x\|^2 \le \langle S^{-1}x, x\rangle \le A^{-1}\|x\|^2, \quad \textit{for any } x \in H.$$

3. Sum of g-frames

Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ and $\{\theta_i\}_{i\in I} \subseteq B(H, H_i)$ be two g-frame for H, then $\{\Lambda_i + \theta_i\}_{i\in I} \subseteq B(H, H_i)$ may not be a g-frame for H. In this direction, we give the following examples

Example 3.1. Let $\{e_n\}$ be an orthonormal basis for Hilbert space H. Let $\{x_n\}$ be a sequence in H such that

$$x_i = 0, \quad i = 1, 2, \ldots, n \quad \text{and}$$

$$x_{n+i} = e_i, \quad i \in \mathbb{N}.$$

Now, for each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H.$$

Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame for H with respect to $\mathbb{C}$. Further, for each $i \in \mathbb{N}$, define $\theta_i : H \to \mathbb{C}$ as

$$\theta_i(x) = \langle x, e_i\rangle, \quad x \in H.$$

Then $\{\theta_i\}_{i\in\mathbb{N}}$ is also a g-frame for H with respect to $\mathbb{C}$. But $\{\Lambda_i + \theta_i\}_{i\in\mathbb{N}} \subseteq B(H, H_i)$ is not a g-frame for H.

Example 3.2. Let $\{e_n\}$ be an orthonormal basis for Hilbert space H. Let $\{x_n\}$ be a sequence in H such that

$$x_i = e_i, \quad i = 1, 2, \ldots, n \quad \text{and}$$

$$x_{n+i} = e_i, \quad i \in \mathbb{N}.$$

Now, for each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i \rangle, \quad x \in H.$$

Then $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\mathbb{C}$. Further, let $\{y_n\}$ be a sequence in H such that

$$y_i = 0, \quad i = 1, 2, \ldots, n \quad \text{and}$$

$$y_{n+i} = e_i, \quad i \in \mathbb{N}$$

For each $i \in \mathbb{N}$, define $\theta_i : H \to \mathbb{C}$ as

$$\theta_i(x) = \langle x, y_i \rangle, \quad x \in H.$$

Then $\{\theta_i\}_{i\in\mathbb{N}}$ is also a g-frame for H with respect to $\mathbb{C}$. Furthermore, $\{\Lambda_i + \theta_i\}_{i\in\mathbb{N}} \subseteq B(H, \mathbb{C})$ is also a g-frame for H. Indeed, we have

$$\|x\|^2 \leq \sum_{i\in\mathbb{N}} \|(\Lambda_i + \theta_i)(x)\|^2 \leq 3\|x\|^2, \quad x \in H.$$

In view of Examples 3.1 and 3.2, we give the following result

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ and $\{\theta_i\}_{i\in I} \subseteq B(H, H_i)$ be two g-frames for H. Then $\{\Lambda_i + \theta_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H, if $\theta_i = \Lambda_i S^a$, $i \in I$, $a \in \mathbb{R}$.*

Proof. Note that

$$\sum_{i\in I} \|(\Lambda_i + \theta_i)(x)\|^2 = \sum_{i\in I} \|\Lambda_i x\|^2 + \sum_{i\in I} \|\theta_i x\|^2 + 2\langle S^{a+1}x, x \rangle, \quad x \in H.$$

For $a \geq 0$, we have

$$A(1 + A^a)^2\|x\|^2 \leq \sum_{i\in I} \|(\Lambda_i + \theta_i)(x)\|^2 \leq B(1 + B^a)^2\|x\|^2, \quad x \in H$$

and for $a < 0$, we have

$$\begin{aligned}(A + B^{2a+1} + 2B^{a+1})\|x\|^2 &\leq \sum_{i\in I}\|(\Lambda_i + \theta_i)(x)\|^2 \\ &\leq (B + A^{2a+1} + 2A^{a+1})\|x\|^2, \quad x \in H\end{aligned}$$

where, A and B are frame bounds for g-frames $\{\Lambda_i\}_{i\in I}$. Hence, $\{\Lambda_i + \theta_i\}_{i\in I}$ is a g-frames for H with respect to $\{H_i\}_{i\in I}$.

In view of Theorem 3.3, the following observations arise naturally.

(I) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and $\{\theta_i\}_{i\in I}$ be canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i + \theta_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H.

(II) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and $\{\theta_i\}_{i\in I}$ be canonical parseval g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i + \theta_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H.

(III) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H with g-frame operator S and $\{\theta_i\}_{i\in I}$ be an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{(\Lambda_i + \theta_i)S^a\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H. □

Next, If $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H and $L : H \to H$ is a bounded linear operator on H, then $\{\Lambda_i L\}_{i\in I} \subseteq B(H, H_i)$ may not be g-frame for H. In this direction, we give the following examples.

Example 3.4. Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and $I : H \to H$ be an identity operator on H. Then, $\{\Lambda_i I\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H.

Example 3.5. Let $\{e_n\}$ be an orthonormal basis for Hilbert space H. Let $\{x_n\}$ be a sequence in H such that

$$x_i = e_i, \quad i \in \mathbb{N}.$$

Now, for each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H.$$

Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame for H with respect to $\mathbb{C}$.

Further, define $L : H \to H$ as

$$L(x) = \langle x, e_1\rangle e_1 , \quad x \in H.$$

Then, L is bounded linear operator on H. But $\{\Lambda_i L\}_{i\in I}$ is not a g-frame for H.

In the view of Examples 3.4 and 3.5, we give the following theorem.

Theorem 3.6. *Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H with g-frame operator S and bounds A and B. Let L be a bounded linear operator on H. Then $\{\Lambda_i L\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H if and only if L is invertible on H. Moreover, in this case, the g-frame operator for $\{\Lambda_i L\}_{i\in I}$ is L^*SL and new bounds are $A\|L^{-1}\|^2$ and $B\|L\|^2$.*

Proof. Let L be invertible on H. Then, for each $x \in H$,

$$\sum_{i\in I} \|(\Lambda_i L)(x)\|^2 \geq A\|Lx\|^2 \geq A\|L^{-1}\|^2\|x\|^2$$

and

$$\sum_{i\in I} \|(\Lambda_i L)(x)\|^2 \leq B\|(Lx)\|^2 \leq B\|L\|^2\|x\|^2$$

Therefore, $\{\Lambda_i L\}_{i\in I}$ is a g-frame for H with bounds $A\|L^{-1}\|^2$ and $B\|L\|^2$.

Conversely, let $\{\Lambda_i L\}_{i\in I}$ be a g-frame for H. Therefore, its g-frame operator is invertible on H. But the g-frame operator of $\{\Lambda_i L\}_{i\in I}$ is

$$\sum_{i\in I}(\Lambda_i L)^*(\Lambda_i L)(x) = L^*SL(x), \quad x \in H.$$

So, L is invertible on H. □

Theorem 3.7. *Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H with g-frame operator S and L be a bounded linear operator on H. Then $\{\Lambda_i + \Lambda_i L\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H if and only if $(I + L)$ is invertible on H. Moreover, in the case, the*

g-frame operator for $\{\Lambda_i + \Lambda_i L\}_{i\in I}$ is $S(I+L^*)(I+L)$ and bounds are $A\|(I+L)^{-1}\|^2$ and $B\|I+L\|^2$.

Proof. Let $(I+L)$ is invertible on H. Therefore, we have

$$\begin{aligned} A\|(I+L)^{-1}\|^2\|x\|^2 &\leq \sum_{i\in I}\|(\Lambda_i+\Lambda_i L)(x)\|^2 \\ &\leq B\|(I+L)\|^2\|x\|^2, \quad x\in H. \end{aligned}$$

Hence, $\{\Lambda_i + \Lambda_i L\}_{i\in I}$ is a g-frame for H with bounds $A\|(I+L)^{-1}\|^2$ and $B\|(I+L)\|^2$.

Conversely, let $\{\Lambda_i + \Lambda_i L\}_{i\in I}$ be a g-frame for H. Therefore, its g-frame operator is invertible on H. Since

$$\begin{aligned} \sum_{i\in I}(\Lambda_i+\Lambda_i L)^*(\Lambda_i+\Lambda_i L)(x) &= Sx + SLx + L^*Sx + L^*SLx \\ &= S(I+L^*)(I+L)(x), \quad x\in H. \end{aligned}$$

So, $(I+L)$ is invertible on H. □

Corollary 3.8. *If $\{\Lambda_i\}_{i\in I}\subseteq B(H,H_i)$ is a g-frame for H, then for all $a\neq -1$, $\{\Lambda_i + a\Lambda_i P\}_{i\in I}$ is a g-frame for H, where P is an orthogonal projection on H.*

Theorem 3.9. *Let $\{\Lambda_i\}_{i\in I}\subseteq B(H,H_i)$ and $\{\theta_i\}_{i\in I}\subseteq B(H,H_i)$ be two g-Bessel sequences for H with operators T_1 and T_2, respectively, defined as in* Lemma 2.3. *Also, Let L_1, L_2 be two bounded linear operators on H. Then the following statements are equivalent*

(1) $\{\Lambda_i L_1 + \theta_i L_2\}_{i\in I}\subseteq B(H,H_i)$ is a g-frame for H.

(2) $L_1^*T_1 + L_2^*T_2$ is an invertible operator on H.

Proof. (1)⇔(2): Note that $\{\Lambda_i L_1 + \theta_i L_2\}_{i\in I}$ is a g-frame for H if and only if its synthesis operator is invertible on H, where

$$\begin{aligned} T(\{x_i\}_{i\in I}) &= \sum_{i\in I}(\Lambda_i L_1 + \theta_i L_2)^*(x_i) \\ &= L_1^* T_1(\{x_i\}_{i\in I}) + L_2^* T_2(\{x_i\}_{i\in I}) \\ &= (L_1^* T_1 + L_2^* T_2)(\{x_i\}_{i\in I}), \quad \{x_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}. \quad \square \end{aligned}$$

Corollary 3.10. *If $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ and $\{\theta_i\}_{i\in I} \subseteq B(H, H_i)$ are two g-frames for H with g-frame operators S_1 and S_2, respectively. Then, $S = L_1^* S_1 L_1 + L_2^* S_2 L_2 + L_1^* T_1 T_2^* L_2 + L_2^* T_2 T_1^* L_1$ is a g-frame operator of g-frame $\{\Lambda_i L_1 + \theta_i L_2\}_{i\in I}$.*

Proof. The g-frame operator of $\{\Lambda_i L_1 + \theta_i L_2\}_{i\in I}$ is given by

$$\begin{aligned} S &= TT^* \\ &= (L_1^* T_1 + L_2^* T_2)(L_1^* T_1 + L_2^* T_2)^* \\ &= L_1^* S_1 L_1 + L_2^* S_2 L_2 + L_1^* T_1 T_2^* L_2 + L_2^* T_2 T_1^* L_1. \quad \square \end{aligned}$$

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