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ON AN INFINITE DIMENSIONAL LESLIE MATRIX IN THE BANACH SPACE c_0

OMER FARAJ MUKHERIJ

ABSTRACT: In this paper, we introduce Leslie matrix with an infinite dimensional components. The top row $\{\alpha_n\}_1^\infty$ and sub diagonal $\{\omega_n\}_1^\infty$ of such matrix are assumed to be elements of the Banach space c_0.We prove that an infinite dimensional Leslie matrices have a positive real eigenvalue. In addition such matrix defines a compact linear operator from c_0 into c_0 .

1. INTRODUCTION

The use of matrices in population mathematics has been discussed by Leslie [6] and arised his common model "Leslie model" which described the population growth [2,11]. This model is represented by the finite square matrix with constant non-negative elements in the first row and elements between 0 and 1 in the subdiagonal immediately below the principle and all other elements are zero.

An eigenvalue of Leslie matrix has been evaluated and it has become an important value for describing the limiting behavior of the population [6,11].

It is well known that the Leslie matrix has only one real positive eigenvalue which dominates the others in modulus and other eigenvalues are non-positive real or complex [11].

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In this paper we consider that the Leslie matrix has infinite dimension with its top row and subdiagonal are elements of the Banach space c_0, the space of all scalar sequences converging to zero (null-sequences) with norm defined by $\|x\|_0 = \sup_i |x_i|$.

We prove that such matrix has a positive real eigenvalue and corresponding eigenvector has positive entries, we also prove that the infinite dimensional Leslie matrix defines a compact linear operator from c_0 into c_0.

Definition 1: An infinite matrix L defined by

$$L = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \omega_1 & 0 & 0 & \cdots \\ 0 & \omega_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $0 < \omega_j < 1$ and $\alpha_j \in \mathbf{R}+0 \ \forall j = 1,2,3,\ldots$ is called an infinite-dimensional Leslie matrix.

We will consider there is an infinite population, and as Leslie assumed in his model in case of finite matrix, we consider a population divided into an infinite number of age groups, the top row α_j is consider to denoted the number of individuals produced by each individual in an age group at time t ,and the subdiagonal ω_j denotes the probability of survival from age j to age group $j+1$.

Let X and Y are sequence spaces such that X contains c_0. Consider an infinite matrix of scalars $L = (a_{ij})_{i,j=1}^{\infty}$. We say that L defines a linear map from X into Y if for every $x = \{x_j\}_{j=1}^{\infty} \in X$ and each $i = 1,2,\ldots$ the series $\sum_{j=1}^{\infty} a_{ij} x_j$

is convergent. If we let

$$Lx_i = y_i = \sum_{j=1}^{\infty} a_{ij} x_j \quad , \quad i = 1,2,\ldots .$$

Then $\qquad Lx \in Y.\ (cf\,[10])$.

The norm of L is given by

$$\|L\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|.$$

Lemma 1: An infinite matrix $A = (a_{ij})_{i,j=1}^{\infty}$ defines a linear map from c_0 into c_0 if and only if

(a) $\|A\|_0 = \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$ and

(b) $\lim_{i\to\infty} a_{ij} = 0$ for every $j = 1,2,3,\ldots$.

Proof: Assume conditions (a) and (b) are hold. Let $0 \neq x \in c_0$. Given $\varepsilon > 0$, take a positive number K be such that

$$\sum_{j=1}^{\infty} |a_{ij}| \leq K$$

For each i=1,2,..., then there exists N_0 such that

$$|x_j| < \frac{\in}{2K} \qquad for\ all\ \ j > N_0$$

where $x = \{x_1, x_2, \ldots, x_j, \ldots\} \in c_0$.

By condition (b),

$$\lim_{i\to\infty} a_{ij} = 0$$

For each $j = 1,2,\ldots$, then there exists a number M_0 such that

$$\sum_{j=1}^{N_0} |a_{ij}| < \frac{\varepsilon}{2\|x\|_0},$$

for each $i > M_0$ where $\|x\|_0 = \sup_j |x_j|$.

By condition (a) makes the series

$$Ax_i = \sum_{j=1}^{\infty} a_{ij} x_j$$

is well-defined for each i.

Now,

$$\begin{aligned}\left|Ax_i\right| &= \left|\sum_{j=1}^{N_0} a_{ij}x_j + \sum_{j=N_0+1}^{\infty} a_{ij}x_j\right| \\ &\le \sum_{j=1}^{N_0}\left|a_{ij}\right|\left|x_j\right| + \sum_{j=N_0+1}^{\infty}\left|a_{ij}\right|\left|x_j\right| \\ &< \frac{\varepsilon}{2\|x\|_0}\sup_j\left|x_j\right| + \frac{\varepsilon}{2K}\sup_i\sum_{j=N_0+1}^{\infty}\left|a_{ij}\right| \\ &= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \qquad \forall i > M_0 .\end{aligned}$$

Hence $Ax \in c_0$.

Therefore A maps c_0 into c_0.

Conversely: Assume that for every $x = \{x_j\} \in c_0$ then $Ax \in c_0$.

Fix i=1,2,…

Let

$$f_i(x) = \sum_{j=1}^{\infty} a_{ij}x_j ,$$

for any $x = \{x_j\}_1^{\infty} \in c_0$ then $f_i \in c_0^{'}$ (where $c_0^{'}$ is the topological dual of c_0), and

$$\| f_i\|_0 \le \sum_{j=1}^{\infty}\left|a_{ij}\right| . \qquad (1)$$

Choose x as $x_j = \operatorname{sgn} a_{ij}$, $j = 1,2,\ldots$, then $\|x\|_0 \le 1$ and $f_i(x) = \sum_{j=1}^{\infty}\left|a_{ij}\right|$.

So that $$\sum_{j=1}^{\infty}\left|a_{ij}\right| \le \| f_i\|_0 \|x\|_0 \le \| f_i\|_0 \qquad (2)$$

From (1) and (2) we get

$$\| f_i\|_0 = \sum_{j=1}^{\infty}\left|a_{ij}\right| \quad ,\forall i = 1,2,\ldots$$

Now, for every $x \in c_0$, then $f_i(x) = Ax_i \in c_0$ means that the sequence $\{f_i(x)\} \to 0$ as $i \to \infty$, so A is a bounded linear operator, then by Banach-Steinhaus theorem we have

$$\|A\|_0 = \sup_i \sum_{j=1}^{\infty} |a_{ij}| = \sup_i \|f_i\| < \infty$$

Hence condition (a).

Let $e_j = (0,0,...,0,1,0,...) \in c_0$,where 1 occur only in the jth entry, for $j = 1,2,...$ then $Ae_j \in c_0$ and sequently $Ae_j^{(i)} \in c_0$.

Since $a_{ij} = Ae_j^{(i)}$ for $j = 1,2,3,...$ then

$$\lim_{i\to\infty} a_{ij} = \lim_{i\to\infty} Ae_j^{(i)} = 0$$

for $j = 1,2,...$.

That is $\lim_{i\to\infty} a_{ij} = 0$.

Hence condition (b).

Proposition 1: The multiplication by the infinite-dimensional Leslie matrix $L = (a_{ij})_{i,j=1}^{\infty}$, with the top row $\{\alpha_j\}_1^{\infty} \in c_0$ and the subdiagonal $\{\omega_j\}_{j=1}^{\infty} \in c_0$,defines a compact linear operator from c_0 into c_0 .

Proof: Consider the operator $L : c_0 \to c_0$ defined by $y = Lx$ such that

$$Lx_i = y_i = \sum_{j=1}^{\infty} a_{ij} x_j \qquad \textit{for each} \quad i = 1,2,..., \qquad (1)$$

where $(a_{ij})_{i,j=1}^{\infty}$ is an infinite-dimensional Leslie whenever $x = \{x_j\}_{j=1}^{\infty} \in c_0$. Thus by the above lemma we have $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$.

Thus there exists a positive number M and $i_0 \in N$ such that $\sum_{j=1}^{\infty} |a_{ij}| \le M$, for any positive integer $i \ge i_0$.

The series in (1) defines a linear functionals $f_1, f_2,...$ on c_0 given by

$$f_i(x) = y_i = \sum_{j=1}^{\infty} a_{ij} x_j \quad , \quad i = 1,2,\ldots.$$

Thus $\{f_i\}_{i=1}^{\infty} \to 0$ as $i \to \infty$.

Therefore for any $x \in c_0$ we have

$$\begin{aligned} |f_i(x)| &\le \sum_{j=1}^{\infty} |a_{ij}||x_j| \\ &\le \sup_j |x_j| \sum_{j=1}^{\infty} |a_{ij}| \\ &\le M.\|x\|_0 . \end{aligned}$$

Thus $\|f_i\|_0 \le M$.

Therefore $\|y\|_0 = \|Lx\|_0 \le M\|x\|_0$.

Hence L is a bounded operator from c_0 into c_0 and $\|L\| \le M$.

Since every sequence $\{x_n\}_1^{\infty}$ in c_0 contains a null subsequence $\{x_{n_k}\}_1^{\infty}$ then $\{Lx_{n_k}\}_1^{\infty}$ converges in c_0. Consider any subset B of c_0. Let $\{y_n\}_1^{\infty}$ be a sequence in $L(B)$. Then $y_n = Lx_n$ for some sequence x_n in B and $\{x_n\}_1^{\infty}$ is a bounded sequence.

So $\{Lx_n\}$ contains a null subsequence therefore the closure $\overline{L(B)}$ is compact and hence L is a compact linear operator from c_0 into c_0.

In the theory of matrices and graphs irreducibility of a matrix is known to be equivalent to strong connectedness of its directed path (cf [2]).

Definition 2 ([3]) **:** A directed graph is called strongly connected if for any pair of its vertices there exists a finite directed path from one vertex to the other.

If $A = (a_{ij})$ be an infinite-dimensional matrix with non-negative entries $(a_{ij} \geq 0)$, then the directed graph $D(A)$ associated with A is given as $D(A) = \{(i, j) \in \mathbf{N}\times\mathbf{N} \,/\, a_{ij} \neq 0 \}$ and it is strongly connected if for every pair $(i, j) \in \mathbf{N}\times\mathbf{N}$ a directed path $(i, k_1), (k_1, k_2), ..., (k_n, j)$ in $D(A)$.

Now since the top row and the subdiagonal of Leslie matrix L are non-zero elements; so that the direct graph with L is strongly connected and hence the infinite-dimensional Leslie matrix is irreducible.

Lemma 2 ([9]) **:** Let $A = (a_{ij}) \in$ be an irreducible matrix. Then for all $(i, j) \in \mathbf{N}\times\mathbf{N}$ there exists $n \in \mathbf{N}$ such that $a_{ij}^{(n)} > 0$ where $A^n = (a_{ij}{}^{(n)}) \forall i \leq n$.

Theorem 1: Let $L = (a_{ij})_{i,j=1}^{\infty}$ be an infinite-dimensional Leslie matrix with $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\omega_j\}_{j=1}^{\infty}$ are in c_0 and set $B = \{(i, j) \in \mathbf{N}\times\mathbf{N} \,/\, a_{ij} > 0 \}$ is infinite then there exists an eigenvalue $\lambda_1 \in \mathbf{R}_{+0}$ of L and corresponding eigenvector $x^1 \in c_0$ with $Lx^1 = \lambda_1 x^1$ and $x_i^1 > 0 \; \forall i \in N$.

To prove this result we need the following theorem due to Krein-Rutmann [4]

Theorem 2 : Let X be a real Banach space and $K \subset X$ be a convex closed cone with $K \cup (-K) = X$ and $K \cap (-K) = \{0\}$.

(a) $T \in £\,(X, X)$ be a compact operator leaving the cone K invariant, that is $Tx \in K \; \forall x \in K$.

(b) Suppose $x^0 \in K - \{0\}, \|x^0\| = 1$ then $\exists n \in \mathbf{N}$ and $s \in \mathbf{R}_{+0}$ such $T^n x^0 - s x^0 \in K$.

Then the operator T has an eigenvalue $\lambda_1 \in \mathbf{R}_{+0}$ and an eigenvector $x^1 \in K$ associated with λ_1.

Proof **:**

Define $K = \{x = \{x_n\} \in c_0 / x_n \geq 0 \quad \forall n \in \mathbf{N}\}$, then $K \cap (-K) = \{0\}$ and $K \cup (-K) = c_0$.

To show that K is convex: Let $x, y \in K$ thus $x, y \in c_0$ with $x_i \geq 0$ and $y_i \geq 0$ $\forall i \in$ **N**.

For any $\beta \geq 0$ and $\gamma \geq 0$ such that $\beta + \gamma = 1$,then $\beta x \in c_0$ and $\gamma y \in c_0$ this implies that $\beta x_i \geq 0$ and $\gamma y_i \geq 0$ $\forall i$.

Thus $\beta x_i + \gamma y_i \geq 0$ $\forall i$ and hence $\beta x + \gamma y \in K$.

Therefore K is convex set.

To show that K is cone: Let $x \in K$ then $x \in c_0$ with $x_i \geq 0$ $\forall i$ .Let $\lambda \geq 0$,then $\lambda x_i \geq 0$ $\forall i \in \mathbf{N}$, which implies that $\lambda x \in c_0$ and hence $\lambda x \in K$ .There fore K is cone.

To prove that K is a closed set; consider any $x_0 = \left\{x_j^{(0)}\right\}_{j=1}^{\infty} \in \overline{K}$, where $\overline{K}$ is the closure of K, then there are $x_n = \left\{x_j^{(n)}\right\}_{j=1}^{\infty} \in K$ such that $x_n \to x_0$ as $n \to \infty$.

Hence for any $\varepsilon > 0$, find a positive integer n such that $\|x_n - x_0\| < \dfrac{\varepsilon}{2}$, so we have $\left|x_j^{(n)} - x_j^{(0)}\right| \leq \|x_n - x_0\| < \dfrac{\varepsilon}{2}$ for all j.

Since c_0 is closed then $x_0 \in c_0$ and $\left|x_j^{(0)}\right| \leq \|x\| < \varepsilon$.

Now ,as $x_n \geq 0$ thus $x_j^{(0)} \geq 0$ $\forall j \in \mathbf{N}$,

thus $x_0 \in K$ and since $x_0 \in \overline{K}$ was arbitrary, therefore K is closed set.

Let $L : c_0 \to c_0$ be a compact linear operator defined by $Lx = y$ such that $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$, $i = 1,2,3,\ldots$, where $\{x_j\}_1^\infty$ and $\{y_j\}_1^\infty$ are null sequences $(i.e.\ in\ c_0)$ and $(a_{ij})_{i,j=1}^\infty$ be an infinite-dimensional Leslie matrix whose terms are non-negative.

Then for any $x \in K$ implies $x_j \geq 0\ \forall i \in$ **N**,

Thus $a_{ij} x_j \geq 0\ \forall i, j$, hence $y_i \geq 0\ \forall i \in$ **N**.

That is $Lx \in K$. Therefore L is a compact linear operator leaving the cone K invariant, which is part (a) of Theorem 2.

Since the infinite-dimensional Leslie matrix L is irreducible, then by Lemma 2, for $i = 1$ and $j = 1$ $\exists n \in \mathbf{N}$ with $a_{11}^{(n)} > 0$.

Let $x^0 \in K/\{0\}$ with $\|x^0\| = 1$ and $s = a_{11}^{(n)} > 0$ thus $L^n x^0 - s x^0 = L^n x^0 - a_{11}^{(n)} x^0 \in K$.

Hence (b) of Theorem 2. Therefore Theorem 2 implies that there exists an eigenvalue $\lambda_1 \in \mathbf{R}_{+0}$ of L and corresponding eigenvector $x^1 \in K$ with $Lx^1 = \lambda_1 x^1$.

To show that $x_j^1 > 0\ \forall j \in$ **N**, assume that there exists $j \in$ **N** such that $x_j^1 = 0$ then the $j-th$ coordinate of $L^n x^1 = 0\ \forall n \in$ **N**.

On the other hand $\exists k \in$ **N** with $x_k^1 > 0$ then $\exists n \in$ **N** such that $a_{i_k}^{(n)} > 0$ and $0 = \sum_{j=1}^{\infty} a_{ij}^{(n)} x_j^1 \geq a_{i_k}^{(n)} x_k^1 > 0$**,** which is impossible.

Therefore $x_j^1 > 0\ \forall j \in$ **N**.

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Hadhramout University of Science and Technology, Faculty of Science, Mathematics Department ,Hadhramout, Yemen.

E-mail address: Mukherij@yahoo.com