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AN APPROACH FOR SIMULTANEOUSLY DETERMINING THE OPTIMAL TRAJECTORY AND CONTROL OF A VIBRATING SHELL

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Abstract. In this article, for a two-dimensional wave control system, we are going to present a new combinational solution path. First, by considering all necessary conditions, the problem is represented in a variational format in which the trajectory is shown by a trigonometric series with the unknown coefficients. Then the problem is converted into a new one that the unknowns are the mentioned coefficients and a positive Radon measure. It is proved that the optimal solution is existed and it is also explained how the optimal pair would be identified from the results deduced by a finite linear programming problem simultaneously. Two numerical examples are also given.

1. Introduction

Base on an idea of L. C. Young, in 1986 Rubio in [9] introduced a new method for solving optimal control problems, by transferring the problem into a theoretical measure optimization. The important properties of the method, like the global solution, the automatic existence theorem and introducing a linear treatment even for extremely nonlinear problems caused it to be applied for the wide variety of problems. Even in the recent decade, a considerable number of optimal control problems

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have been solved successfully based on the properties of the measures, like [1],[2],[3] and [4], but at least two important points have been not considered yet. Generally the method was not able to produce the acceptable optimal trajectory and control directly at the same time; moreover, the classical format of the system solution, usually is not taken into account. Therefore, it is not possible to use this important fact and its related literature in analysis of the system.

In this article, we try to bring the attention to these two facts, by introducing a new solution method for an optimal control problem governed by a two-dimensional wave equation system (a vibrating shell) with initial and boundary conditions and an integral criterion. Regarding a general classical trigonometric series format for the solution, the problem is presented in a variational form; then, by doing a deformation, it is converted into a measure theoretical one with some positive unknown coefficients. Next, by extending the underlying space, using some density properties and applying some discretization scheme, the optimal pair of trajectory and control is determined simultaneously as a result of a finite linear programming. The approach would be improved if the number of nods in discretization is exceeded.

2. The control System

For all $t \in [0, T] \subset R$, let the deflection of a vibrating shell at an arbitrary point x in time t, is denoted by $u(t, x, y)$ which satisfies in (see [6] and [12]):

$$u_{tt} = c^2(u_{xx} + u_{yy}) \ (1)$$

where c^2 is a constant dependent on physical structure of the shell. Since the shell is fixed at its boundary, there is no vibration at these points and hence we have the following boundary conditions:

$$u(t, 0, y) = u(t, a, y) = u(t, x, b) = u(t, x, 0) = 0, \ \forall \ 0 \leq x \leq a, \ 0 \leq y \leq b. \ (2)$$

If the initial deflection and velocity of the vibrating shell are denoted by $f(x,y)$ and $g(x,y)$, then the initial conditions of the system are defined as:

$$f(x,y) = u(0,x,y); \; g(x,y) = u_t(0,x,y) \; (3)$$

Regarding [7], $u(t,x,y)$ belongs to the class of homogeneous Cauchy problems. Thus, it can have a unique bounded classic solution on $D = [0,T] \times [0,a] \times [0,b]$, if $f(x,y)$, $g(x,y)$ and the different orders of their partial derivatives are continuous. Moreover, as mentioned in [13], the two-dimensional wave equation problem have the following Fourier series as the solution:

$$u(t,x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}cos\lambda_{mn}t + B_{mn}sin\lambda_{mn}t)sin\frac{m\pi x}{a}sin\frac{n\pi y}{b}, \; (4)$$

where $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$, $A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)sin\frac{m\pi x}{a}sin\frac{n\pi y}{b}dxdy$ and $B_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)sin\frac{m\pi x}{a}sin\frac{n\pi y}{b}dxdy$ for $m = 1,2,...;\; n = 1,2,....$ Convergence of the above series (to a bounded solution of the problem) indicates that one can approximate the solution by a finite number terms of the series.

To Control the two-dimensional wave system, we need to introduce some power to the system somehow. This fact can be done by inserting a shock on a specified place of the shell. Since the amount of the power to the system is proportional to the value of its velocity, the amount of the velocity in the mentioned place, is regarded as a controller. Without losing the generality, suppose that the place for inserting the shock be $(\frac{a}{2},\frac{b}{2})$.

Now, let $V \subseteq R$ be a bounded set, and $\vartheta = \vartheta(t) : [0,t] \to V$ be a Lebesgue-measurable control function. Moreover suppose $f_0 = f_0(t,x,y,\vartheta(t)) : D \times V \to R$ be a continues function then, the aim is to find the optimal pair of trajectory and control functions, simultaneously, as an optimal solution of the following control problem:

$Min: \; I(P) \equiv \int_D f_0(t,x,y,\vartheta)dA$

$S.\,to: \; u_{tt} = c^2(u_{xx}+u_{yy}); \; (5-1)$

$u(t,0,y) = u(t,a,y) = u(t,x,b) = u(t,x,0) = 0; \; (5-2)$

$f(x,y) = u(0,x,y); \; (5-3) \; (5)$

$g(x,y) = u_t(0,x,y); \; (5-4)$

$u_t|_{(\frac{a}{2},\frac{b}{2})} = \vartheta(t). \; (5-5)$

We remind that the objective functional $\int_D f_0(t,x,y,\vartheta)dA$ can explain the energy or expected error of the system or so on; indeed, the deflection of the shell is regarded indirectly in the criterion.

Definition: A pair $P \equiv (u,\vartheta)$ is called admissible if u be a bounded solution of (5-1) and the conditions (5-2)−(5-5) are satisfied. The set of all admissible pairs is denoted by **P** .

Therefore, we wish to find the admissible minimizer pair for the functional $I(P)$ over **P**. It is necessary to indicate that the controllability and the observability of the above system were discussed in many references such as [4]. Thus, we can suppose that **P** is nonempty. In the next, we will try to find the solution of (5) according to the trigonometrical series and use of the embedding method. For reaching to our purposes, first we need to represent the problem in a new formulation.

3. New Representation of the Problem

For a fixed $M, \acute{M} \in N$, the optimal trajectory of (5) can be approximated by the first $M \times \acute{M}$ terms of a trigonometric series; i.e.:

$$u(t,x,y) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} (A_{mn} cos\lambda_{mn} t + B_{mn} sin\lambda_{mn} t) sin\tfrac{m\pi x}{a} sin\tfrac{n\pi y}{b}, \quad (6)$$

where A_{mn} and B_{mn}, for $m = 1, 2, ..., M$ and $n = 1, 2, ..., \acute{M}$, are unknown real coefficients that must be determined under the conditions (5-2)-(5-5). Since the coefficients are unknown, the amount of the eliminated part of the solution in (4) (the tail of the series), can be accounted in the unknowns calculation. Moreover, it could caused more stability of the solution. By defining:

$$\overline{u}^{mn}(t,x,y) = (A_{mn} cos\lambda_{mn} t + B_{mn} sin\lambda_{mn} t) sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}, \quad (7)$$

since

$$\overline{u}^{mn}_{xxxyyyt}(t,x,y) = \frac{m^2 n^2 \pi^4}{a^2 b^2} \overline{u}^{mn}_{xyt}(t,x,y),$$

we have:

$$u_{xxxyyyt}(t,x,y) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \overline{u}^{mn}_{xxxyyyt}(t,x,y) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \frac{m^2 n^2 \pi^4}{a^2 b^2} \overline{u}^{mn}_{xyt}(t,x,y);$$

then, by integrating over $[0,T] \times [0, \frac{a}{2}] \times [0, \frac{b}{2}]$, we have:

$$\int_0^T \int_0^{\frac{a}{2}} \int_0^{\frac{b}{2}} u_{xxxyyyt} dydxdt = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \frac{m^2 n^2 \pi^4}{a^2 b^2} \int_0^T \int_0^{\frac{a}{2}} \int_0^{\frac{b}{2}} \overline{u}^{mn}_{xyt}(t,x,y) dydxdt.$$

Now, continuity of $\overline{u}^{mn}(t,x,y)$ and its partial derivatives, allow us to change the order of the integration. In this manner, by doing some simple calculations, the constraint

(5-5) can be appeared in the following new format:

$$\int_0^T \vartheta(t)dt = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} \lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}(B_{mn} sin\lambda_{mn}T + A_{mn}(cos\lambda_{mn}T - 1)). \quad (8)$$

Therefore, based on the equations (6) and (8), the problem (5) can be represented by the new following exhibition:

$Min: \ I(P) = \int_D f_0(x, y, t, \vartheta(t))dA$

$S.\ to: \ f(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} A_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}$

$g(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} \lambda_{mn} B_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}$ (9)

$\int_0^T \vartheta(t)dt = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} \lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}(B_{mn} sin\lambda_{mn}T + A_{mn}(cos\lambda_{mn}T - 1))$.

Let $x_0 = 0, x_1, x_2, ..., x_l$ and $y_0 = 0, y_1, y_2, ..., y_{\acute{l}}$ be belong to a dense subset of $[0, a]$ and $[0, b]$ respectively.If $l, \acute{l} \to \infty$ then obviously the solution of the following problem converges to the solution of (9). Hence, for a suitable numbers l and $\acute{l}$, the solution of the problem (9) can be approximated by the solution of the following one:

$Min: \ I(P) = \int_0^T [\sum_{i=1}^{l}\sum_{j=1}^{\acute{l}} \int\int f_0(t, x_i, y_j, \vartheta(t))dxdy]dt \equiv \int_0^T F_0(t, \vartheta)dt$

$S.\ to: \ f(x_i, y_j) = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} A_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b}$

$g(x_i, y_j) = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} \lambda_{mn} B_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b}$ (10)

$\int_0^T \vartheta(t)dt = \sum_{m=1}^{M}\sum_{n=1}^{\acute{M}} \lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}(B_{mn} sin\lambda_{mn}T +$
$A_{mn}(cos\lambda_{mn}T - 1)).$
$i = 1, 2, ..., l,\ j = 1, 2, ..., \acute{l}$

4. Metamorphosis

To solve problem (10), we follow [2] and [9] by applying some new ideas. We do the metamorphosis step in this section, to deform the problem and redefine it in a new space in which it has many advantages.

Let $\Omega = [0, T]\times V$; we know that for each $(u, \vartheta) \in P$, the functional $\Lambda_\vartheta : C(\Omega) \to R$ defined by $\Lambda_\vartheta(h) = \int_0^T h(t, \vartheta)dt$, is a positive continuous linear functional. Based on the Riesz Representation Theorem ([11]), there exists a positive Radon measure $\mu_\vartheta \in M^+(\Omega)$ (the space of all positive Radon measures on Ω) so that for all $h \in C(\Omega),\ \mu_\vartheta(h) = \int_\Omega h\ d\mu = \Lambda_\vartheta(h)$. Therefore, problem (10) is changed into a new one in which its unknowns are the coefficients A_{mn} and B_{mn} $(m = 1, 2, ..., M;\ n = 1, 2, ..., \acute{M})$ and a positive Radon measure, say μ , produced by the Riesz Representation Theorem. Now, we are able to assure the global solution, like [2],[5] and [9], by enlarging the underlying space and seeking on a subset of $M^+(\Omega)$ which is defined just by the last equations of (10); this means that instead of searching for the optimal measure, say μ^*, between the introduced measures from the Riesz Representation Theorem, we seek in the set of all positive Radon measures in which they just satisfy in the last condition of (10); therefore, our minimization is global. In this manner, we try to solve the following problem:

$Min: \ \mu(F_0)$

$S.\ to: \ f(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} A_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b};$

$g(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \lambda_{mn} B_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b}, \ (11)$

$i = 1, 2, \ldots, l, \ j = 1, 2, \ldots, \acute{l};$

$\mu(\vartheta) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}(B_{mn} sin\lambda_{mn}T + A_{mn}(cos\lambda_{mn}T - 1));$

$\mu(\xi) = a_\xi, \ \forall\ \xi \in C^1(\Omega).$

Here, the unknown measure μ belongs to $M^+(\Omega)$, $C^1(\Omega)$ is a subset of functions in $C(\Omega)$ that depends only on variable t and a_ξ is the Lebesgue integral of ξ over $[0, T]$; indeed the last set of equations is added to the problem to guarantee that projection of an admissible measure on the real line is the Lebesgue measure (see for instance [1] and [9]).

Suppose A_{mn}'s and B_{mn}'s be obtained by solving the following linear equations:

$$f(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} A_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b}, \ i = 1, 2, ..., l; \ j = 1, 2, ..., \acute{l};$$

$$g(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \lambda_{mn} B_{mn} sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b} \ i = 1, 2, \ldots, l; \ j = 1, 2, \ldots, \acute{l};$$

then, by substituting the obtained coefficients in the third equation of (11), the problem is converted into one in which the unknown is just the measure $\mu \in M^+(\Omega)$. Let Q be the space of all measures in $M^+(\Omega)$ which satisfied the conditions of (11); as Rubio shown in [9] and [10], Q is compact in the sense of weak* topology. Moreover, the function $\mu \rightarrow \mu(F_0)$ is continuous. Since each continuous function has an infimum on a compact space, there exists an optimal measure which minimizes the objective function of (11). Thus , we have the following proposition:

proposition 1: Problem (11) has the optimal solution.

By regarding the result of Rosenblooms work which is mentioned in [9], the optimal measure has the form

$$\mu^* = \sum_{r=1}^{L} \alpha_r \delta(z_r) \quad (12)$$

where $\delta(z_r)$ is a unitary atomic measure with the support of the singleton set $\{z_r\}$, α_r is a nonnegative real coefficient and z_r is a point belongs to Ω. Applying (12) in (11), changes the problem into a nonlinear one in which its unknowns are the coefficients $A_{mn},\ B_{mn}, \alpha_r$, and the supporting points z_r for $m = 1, 2, \ldots, M,\ n = 1, 2, \ldots, \acute{M}$ and $r = 1, 2, \ldots, L$. We know that, by doing a discretization on Ω with nodes $z_r = (t_r, \vartheta_r),\ r = 1, 2, \ldots, L$, in a dense subset of $\omega \subseteq \Omega$, the supporting points can be determined; hence the problem can be converted into a linear one. But, regarding the last set of equations in (11), the number of constrains are still infinite. It would be more convenient if somehow we could change the problem into a finite linear programming one. In the next step of approximation, by choosing a dense countable subset of $C^1(\Omega)$ and then selecting a finite number of its elements as ξ_k for $k = 1, 2, ..., K$, the total number of the constraints of the problem would be finite. Therefore, the solution of (11) can be approximated by the following linear programming problem with variables $\alpha_r, r = 1, 2, \ldots, L$, and $A^+_{mn}, A^-_{mn}, B^+_{mn}, B^-_{mn}$ that $A_{mn} = A^+_{mn} - A^-_{mn}$ and $B_{mn} = B^+_{mn} - B^-_{mn}$:

$Min:\ \sum_{r=1}^{L} \alpha_r F_0(t_r, \vartheta_r)$

$S.\ to:\ \ f(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} (A^+_{mn} - A^-_{mn}) sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b};$

$g(x_i, y_j) = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \lambda_{mn} (B^+_{mn} - B^-_{mn}) sin\frac{m\pi x_i}{a} sin\frac{n\pi y_j}{b};\ (11)$

$\sum_{r=1}^{L} \alpha_r \vartheta_r = \sum_{m=1}^{M} \sum_{n=1}^{\acute{M}} \lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}((B^+_{mn} - B^-_{mn}) sin\lambda_{mn}T +$
$(A^+_{mn} - A^-_{mn})(cos\lambda_{mn}T - 1));$

$\sum_{r=1}^{L} \alpha_r \xi_k(t_r, \vartheta_r) = a_k,\ \ k = 1, 2, ..., K.$

$\alpha_r \geq 0,\ \ r = 1, 2, \ldots, L; A^+_{mn}, A^-_{mn}, B^+_{mn}, B^-_{mn}, A^+_{mn} \geq 0,$

$m = 1, 2, \ldots, M,\ \ n = 1, 2, \ldots, \acute{M},\ \ i = 1, 2, \ldots, l,\ \ j = 1, 2, \ldots, \acute{l}.$

The density properties of the applied sets, indicate that if $N, l, \acute{l}, m, n, k$ tend to infinity, the optimal solution of (13) convergence into the solution of (10), or more precisely (5)(see [9]). Therefore, the optimal solution of (5) can be approximated by the results of the finite linear programming problem (13).

To set up (13), as mentioned in [10] and some other literature (like [1],[4]and [6]), for $k = 1, 2, \ldots, K-1$, we choose $J_k = [\frac{k-1}{T}, \frac{k}{T})$ and $J_K = [\frac{K-1}{T}, T]$; hence $[0, T] = \bigcup_{k=1}^{K} J_k$. Now for each $k = 1, 2, \ldots K$, we define:

$$\xi_k(t, \vartheta) = \begin{cases} 1,\ t \in J_k, \\ 0,\ otherwise. \end{cases}$$

Although these class of functions are not continuous, but when $k \to \infty$ every functions in $C^1(\Omega)$ can be approximated by a finite linear combination of these functions (see [5]). In this manner, for an arbitrary function ξ_k, we have $a_k = \int_{J_k} \xi_k\ dt$. Now, by solving the linear programming problem (13), one can obtain the optimal coefficients α_r^*, A_{mn}^* and B_{mn}^* at the same time. Then, according to (6) and the explained method in [9], the optimal trajectory and control functions can be determined simultaneously, which is one of the main aim of this paper.

5. Numerical Examples

Based on the explained approach, we incline to find the optimal pair of trajectory and control for given vibrating systems in the following numerical examples. In the first example, more than the obtained results, the procedure of the new approach was also described. But in the second one, just the results was mentioned.

Example 1: Consider the following vibrating shell:

$$u_{tt} = u_{xx} + u_{yy}$$

$$u(t, 0, y) = u(t, a, y) = u(t, x, b) = u(t, x, 0) = 0;$$

$$u(0, x, y) = x + y;$$

$$u_t(0, x, y) = xy,$$

with the performance criterion defined by $F_0(t, \vartheta) = (\vartheta - t^2)^2$; indeed, here was supposed that $c = 1,\ t \in [0, 1],\ D = [0, 2] \times [0, 2] \times [0, 1], U = [0, 1],\ f(x, y) = x + y$, and $g(x, y) = xy$. Also we choose $M = \acute{M} = 6,\ K = 10$ and $l = \acute{l} = 3$. Therefore, to solve the problem, a similar linear programming problem like (13) with 1044 variables and 29 constraints was established as follow:

$$Min:\ \sum_{r=1}^{900} \alpha_r(\vartheta - t^2)^2$$

$$S.\ to:\ x_i + y_j = \sum_{m=1}^{6} \sum_{n=1}^{6} (A_{mn}^+ - A_{mn}^-) sin\frac{m\pi x_i}{2} sin\frac{n\pi y_j}{2};$$

$x_i y_j = \sum_{m=1}^{6}\sum_{n=1}^{6}\lambda_{mn}(B_{mn}^{+} - B_{mn}^{-}) sin\frac{m\pi x_i}{2} sin\frac{n\pi y_j}{2};$

$\sum_{r=1}^{900}\alpha_r\vartheta_r - \sum_{m=1}^{6}\sum_{n=1}^{6}\lambda_{mn} sin\frac{m\pi}{2} sin\frac{n\pi}{2}((A_{mn}^{+} - A_{mn}^{-})(cos\lambda_{mn}T - 1) +$
$(B_{mn}^{+} - B_{mn}^{-}) sin\lambda_{mn}T);$

$$i = 1, 2, 3;\ j = 1, 2, 3.$$

$\alpha_1 + \alpha_2 + \ldots + \alpha_{90} = 0.1;\ \alpha_{91} + \alpha_{92} + \ldots + \alpha_{180} = 0.1;\ \alpha_{181} + \alpha_{182} + \ldots + \alpha_{270} = 0.1;$

$\alpha_{271} + \alpha_{272} + \ldots + \alpha_{360} = 0.1;\ \alpha_{361} + \alpha_{362} + \ldots + \alpha_{450} = 0.1;\ \alpha_{451} + \alpha_{452} + \ldots + \alpha_{540} = 0.1;$

$\alpha_{541} + \alpha_{542} + \ldots + \alpha_{600} = 0.1;\ \alpha_{601} + \alpha_{602} + \ldots + \alpha_{720} = 0.1;\ \alpha_{721} + \alpha_{722} + \ldots + \alpha_{780} = 0.1;$

$$\alpha_{781} + \alpha_{782} + \ldots + \alpha_{900} = 0.1;$$

$A_{mn}^{+}, A_{mn}^{-}, B_{mn}^{+}, B_{mn}^{-}, \alpha_r \geq 0,\ r = 1, \ldots, 900,\ m = n = 1, 2, \ldots, 6.$

We applied the subroutine **DLPRS** from **IMSL** library of **Compaq Visual Fortran** to solve the above linear programming problem by Revised Simplex Method. The optimal value of the objective function was obtained as 0.00000000841. The optimal nonzero's value of the variables were as follows:

$A_{23}^{*} = -0.86125305181532;\ A_{26}^{*} = 0.2073658162327 \times 10^{-6};$
$A_{32}^{*} = -0.86253251456649;\ A_{33}^{*} = 8.145973600226562;$
$A_{35}^{*} = 2.514406849562812;\ A_{53}^{*} = 2.5144070418132100;$
$A_{55}^{*} = 0.8828391960869988;\ A_{56}^{*} = 0.15414731179242598;$
$A_{65}^{*} = 0.1541471182493354;$
$B_{25}^{*} = 3.605543346961127;\ B_{33}^{*} = 1.289129664170388;$
$B_{35}^{*} = 3.706182792805683;\ B_{36}^{*} = 0.1365719754782218;$
$B_{53}^{*} = 0.455939940928165;\ B_{55}^{*} = 2.840023316025101;$
$B_{56}^{*} = 0.05883815637717339;\ B_{63}^{*} = 0.136571975478222;$

$B^*_{65} = 2.51892580651433$ $B^*_{66} = 0.01875655299702234;$

$$\alpha^*_{61} = \alpha^*_{121} = \alpha^*_{212} = \alpha^*_{304} = \alpha^*_{366} = \alpha^*_{521} = \alpha^*_{552} = \alpha^*_{646} = \alpha^*_{741} = \alpha^*_{900} = 0.1 \ .$$

Base on these values, the nearly optimal piecewise-constant control was calculated via the explained manner in [9]. Also, by regarding (6) and the above optimal coefficient, the trajectory function $u^*(t, x, y)$ was determined as:

$$u^*(t, x, y) = \sum_{m=1}^{6}\sum_{n=1}^{6}(A^*_{mn} cos\lambda_{mn}t + B^*_{mn} sin\lambda_{mn}t) sin\frac{m\pi x}{2} sin\frac{n\pi y}{2}.$$

The obtained nearly optimal control and trajectory functions are plotted in figures 1 and 2 respectively (since the optimal trajectory is a function of four variables, it was plotted for some specified times) .

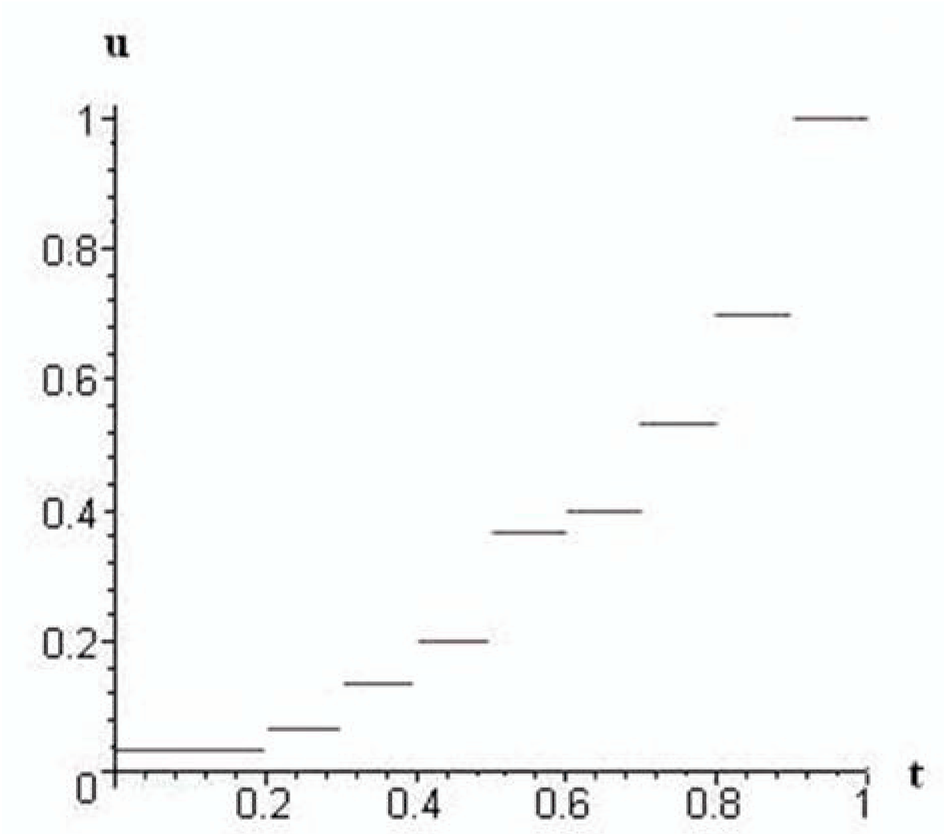

FIGURE 1. The Optimal Control of example (1)

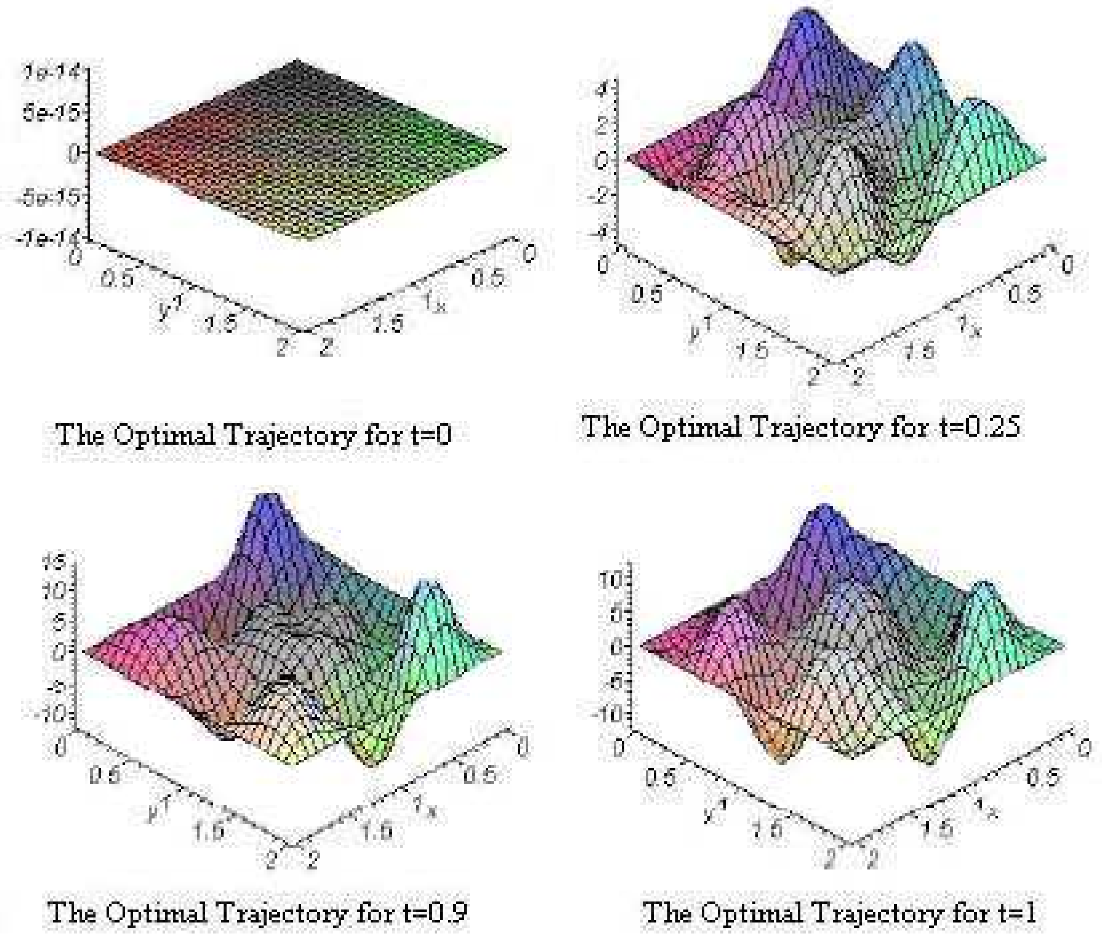

Figure 2. The Optimal Trajectory of example (1)

Example 2: Consider the following vibrating shell system:

$$u_{tt} = u_{xx} + u_{yy}$$

$$u(t, 0, y) = u(t, a, y) = u(t, x, b) = u(t, x, 0) = 0;$$

$$u(0, x, y) = (0.1)sin^2\pi x \; sin^2\pi y;$$

$$u_t(0, x, y) = 0,$$

with the performance criterion defined by $F_0(t, \vartheta) = (\vartheta - t^2)^2$; indeed, here was supposed that $c = 1,\ t \in [0, 1],\ D = [0, 2] \times [0, 2] \times [0, 1], U = [0, 1]$,
$f(x, y) = (0.1)sin^2\pi x\ sin^2\pi y$, and $g(x, y) = 0$. With the same discretization scheme as the previous example, we transfer the problem into a finite programming with 1044 variables and 29 constraints. By solving this problem and obtaining the results in the explained manner as mentioned in example 1, the optimal value of the objective function was determined as 0.00000000841 .The obtained optimal trajectory and control of this system are plotted in below figures 3 and 4 respectively.

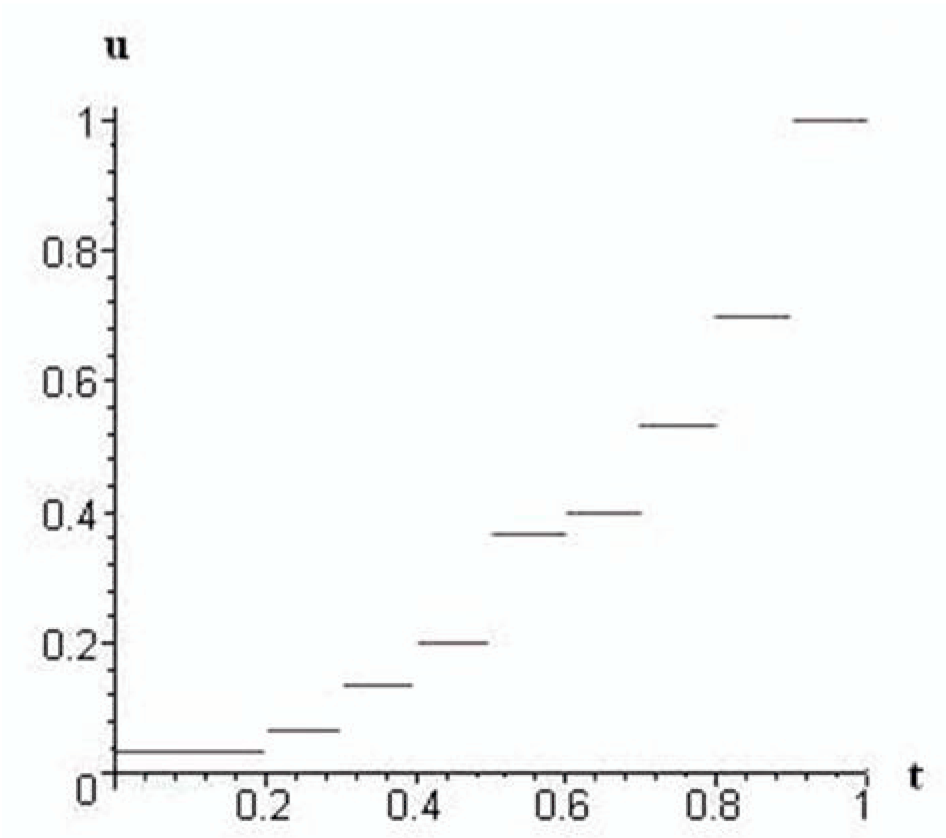

FIGURE 3. The Optimal Control of example (2)

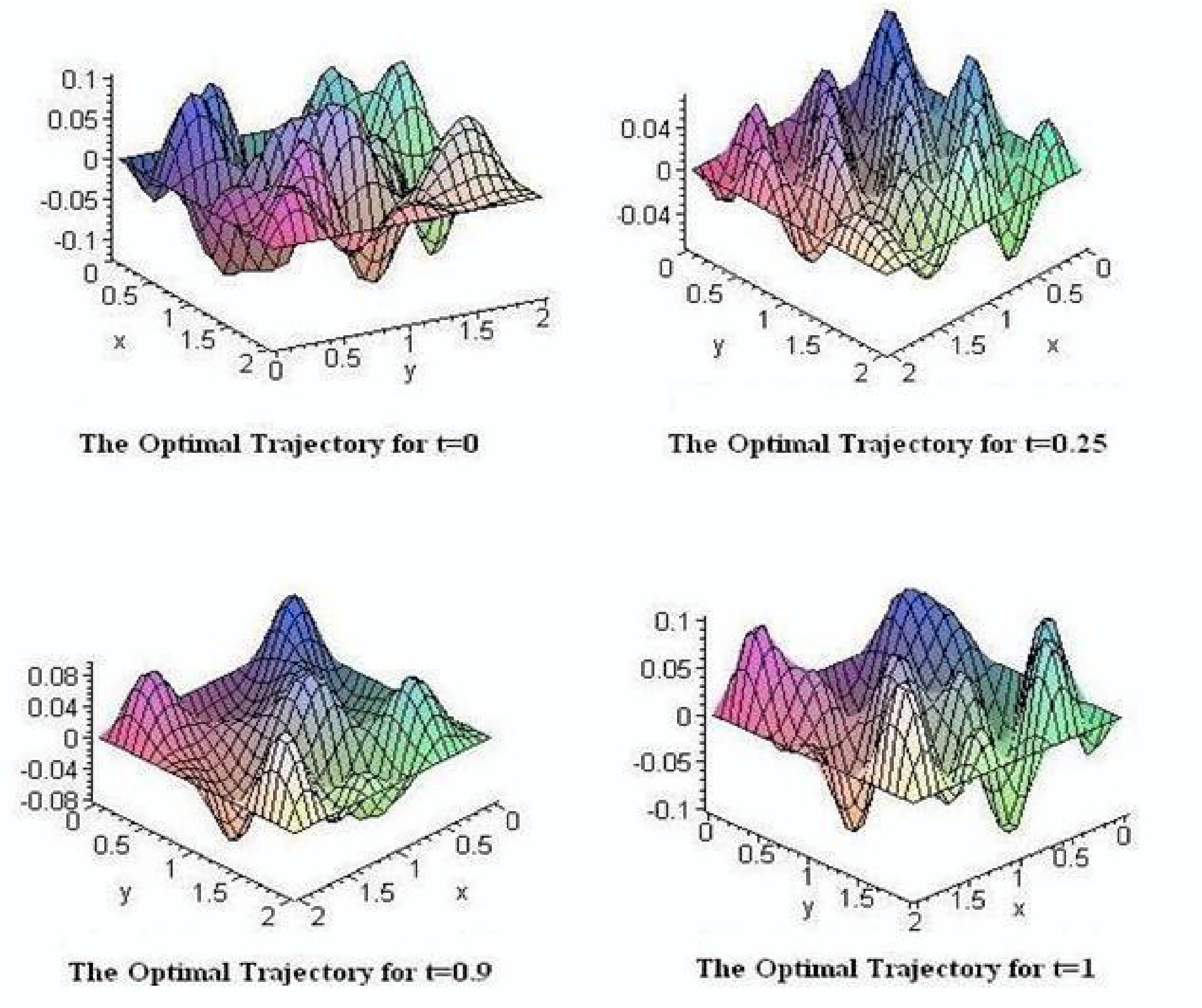

FIGURE 4. The Optimal Trajectory of example (2)

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