

CERTAIN CLASSES OF UNIVALENT FUNCTIONS HAVING NEGATIVE COEFFICIENTS

VANITA JAIN

ABSTRACT. Eight classes of univalent analytic functions with negative coefficients, involving Ruscheweyh derivative and a linear operator, have been studied in regard to coefficient inequality, distortion theorems, extreme points, radii of starlikeness and convexity.

1. INTRODUCTION

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $U : |z| < 1$. Let T denote the subclass of S consisting of functions whose non-zero coefficients from second on, are negative; that is, an analytic and univalent functions f is in T if and only if it can be expressed as

$$(1.1) \quad f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

The class T together with its subclasses $TS^*(\alpha)$ and $TC(\alpha)$, respectively, of starlike and convex functions of order α , were introduced and studied by Silverman [12].

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Further, if f is given by (1.1) and g in S is given by $g(z) = z + \sum_{n=2}^{\infty} |b_n|z^n$, then the convolution (Hadamard product) of f and g is given by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} |a_n b_n| z^n.$$

The subclasses TS_s^* and TC_s , respectively, starlike functions and convex functions with symmetric points of T were introduced and studied by Sakaguchi [11] and Dass and Singh [1], respectively, see also [9]; whereas the subclasses $TS^*(\alpha, \beta)$ and $TC(\alpha, \beta)$, respectively, of starlike and convex functions of order α and type β were introduced and studied in Gupta and Jain [3], see also [4].

In this paper, involving two operators, namely, Ruschewey derivative operator $R(\alpha, \beta, \mu)$, introduced by Ruscheweyh [10] (Section 2), and the linear operator $L(a, c)$, introduced by Carlson and Shaffer (c.f.[9]) (Section 3), eight different classes of univalent analytic functions with negative coefficients have been considered. Results on coefficients inequalities, distortion theorems, extreme points and radii of starlikeness and convexity for these classes have been derived.

2. CLASSES OF FUNCTIONS INVOLVING RUSCHEWEYH DERIVATIVE

A function $f \in T$ is in $TS_s^*R(\alpha, \beta, \mu)$, the class of starlike functions of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to symmetric points, if and only if

$$(2.1) \quad \left| \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - 1 \right] \middle/ \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} + (1 - 2\alpha) \right] \right| < \beta, |z| < 1$$

where the operator $D^\mu f$, the Ruscheweyh derivative of f , is defined by

$$D^\mu f(z) = \frac{z(z^{\mu-1}f(z))'}{\mu!} = \frac{z}{(1-z)^{\mu+1}} * f(z)$$

$$(2.2) \quad = z - \sum_{n=2}^{\infty} A_n(\mu) |a_n| z^n,$$

with

$$A_n(\mu) = \binom{n+\mu-1}{\mu} = \frac{(\mu+1)(\mu+2)\dots(\mu+n-1)}{(n-1)!}.$$

Further, $f \in T$ is in $TC_sR(\alpha, \beta, \mu)$, the class of convex functions of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to symmetric points involving the Ruscheweyh derivative if and only if $zf' \in TS_s^*R(\alpha, \beta, \mu)$.

Note that $TS_s^*R(\alpha, \beta, 0) = TS_s^*R(\alpha, \beta)$ and $TC_sR(\alpha, \beta, 0) = TC_s(\alpha, \beta)$ which were considered and discussed in [5], [6]. We first give the results for the class $TS_s^*R(\alpha, \beta, \mu)$.

Theorem 2.1. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS_s^*R(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1+\beta) + (1 - (-1)^n)(\beta(1-2\alpha) - 1)\} A_n(\mu) |a_n| \leq \beta|3-4\alpha| - 1.$$

This result is sharp.

The following is the distortion result.

Theorem 2.2. *If $f \in TS_s^*R(\alpha, \beta, \mu)$, then*

$$(2.2.1) \quad r - \frac{\beta|3-4\alpha| - 1}{2(\beta+1)(1+\mu)} r^2 \leq |f(z)| \leq r + \frac{\beta|3-4\alpha| - 1}{2(\beta+1)(1+\mu)} r^2, \quad |z| = r$$

$$(2.2.2) \quad 1 - \frac{\beta|3-4\alpha| - 1}{(\beta+1)(1+\mu)} r \leq |f'(z)| \leq 1 + \frac{\beta|3-4\alpha| - 1}{(\beta+1)(1+\mu)} r \quad |z| = r$$

and

$$(2.2.3) \quad r - \frac{\beta|3-4\alpha|-1}{2(\beta+1)}r^2 \leq \left| D^\mu f(z) \right| \leq r + \frac{\beta|3-4\alpha|-1}{2(\beta+1)}r^2, \quad |z| = r$$

with equalities for $f(z) = z - \frac{\beta|3-4\alpha|-1}{2(1+\beta)(1+\mu)}z^2$, ($z = \pm r$).

Theorem 2.3. If $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n|z^n$ are in $TS_s^*R(\alpha, \beta, \mu)$, then $h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n|z^n$ is also in $TS_s^*R(\alpha, \beta, \mu)$.

Theorem 2.4. Let

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{\beta|3-4\alpha|-1}{\{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\}A_n(\mu)}z^n, \quad n \geq 2.$$

Then $f \in TS_s^*R(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Theorem 2.5. Let $f \in TS_s^*R(\alpha, \beta, \mu)$. Then f is starlike in the disc

$|z| < r = r(\alpha, \beta)$, where,

$$r(\alpha, \beta) = \inf_n \left[\frac{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)}{n(\beta|3-4\alpha|-1)} A_n(\mu) \right]^{\frac{1}{n-1}}$$

Now, we give results for the class $TC_sR(\alpha, \beta, \mu)$.

Theorem 2.6. A function $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ is in class $TC_sR(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=2}^{\infty} n \{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu) |a_n| \leq \beta|3-4\alpha|-1.$$

This result is sharp.

Theorem 2.7. If $f \in TC_sR(\alpha, \beta, \mu)$, then

$$(2.7.1) \quad r - \frac{\beta|3-4\alpha|-1}{4(\beta+1)(1+\mu)}r^2 \leq |f(z)| \leq r + \frac{\beta|3-4\alpha|-1}{4(\beta+1)(1+\mu)}r^2, \quad |z| = r$$

$$(2.7.2) \quad 1 - \frac{\beta|3-4\alpha|-1}{2(\beta+1)(1+\mu)}r \leq |f'(z)| \leq 1 + \frac{\beta|3-4\alpha|-1}{2(\beta+1)(1+\mu)}r \quad |z| = r$$

and

$$(2.7.3) \quad r - \frac{\beta|3-4\alpha|-1}{4(\beta+1)}r^2 \leq \left| D^\mu f(z) \right| \leq r + \frac{\beta|3-4\alpha|-1}{4(\beta+1)}r^2, \quad |z| = r$$

with equalities for $f(z) = z - \frac{\beta|3-4\alpha|-1}{4(1+\beta)(1+\mu)}z^2$, ($z = \pm r$).

Theorem 2.8. If $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n|z^n$ are in $TC_sR(\alpha, \beta, \mu)$, then $h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n|z^n$ is also in $TC_sR(\alpha, \beta, \mu)$.

Theorem 2.9. Let

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{\beta|3-4\alpha|-1}{n[n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)]A_n(\mu)}z^n, (n \geq 2).$$

Then $f \in TC_sR(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Theorem 2.10. Let $f \in TC_sR(\alpha, \beta, \mu)$. Then f is convex in the disc

$|z| < r = r(\alpha, \beta)$, where

$$r(\alpha, \beta) = \inf_n \left[\frac{(n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1))A_n(\mu)}{n^2(\beta|3-4\alpha|-1)} \right]^{\frac{1}{n-1}}, n \geq 2.$$

Similar to (2.1), we can introduce the classes $TS_c^*R(\alpha, \beta, \mu)$ and $TC_cR(\alpha, \beta, \mu)$, the class of starlike functions and the class of convex functions, respectively, of order α and type β with respect to conjugate points involving Ruscheweyh derivative. A function $f \in T$ is in $TS_c^*R(\alpha, \beta, \mu)$ if and only if

$$\left| \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) + \overline{D^\mu f(\bar{z})}} - 1 \right] \middle/ \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) + \overline{D^\mu f(\bar{z})}} + (1-2\alpha) \right] \right| < \beta, |z| < 1$$

and $f \in T$ is in $TC_cR(\alpha, \beta, \mu)$ if and only if $zf' \in TS_c^*R(\alpha, \beta, \mu)$.

The following results for the classes $TS_c^*R(\alpha, \beta, \mu)$ and $TC_cR(\alpha, \beta, \mu)$ are analogous to those for classes $TS_s^*R(\alpha, \beta, \mu)$ and $TC_sR(\alpha, \beta, \mu)$ and can be derived in similar fashion.

Theorem 2.11. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS_c^*R(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} A_n(\mu) |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 2.12. *If $f \in TS_c^*R(\alpha, \beta, \mu)$, then*

$$(2.12.1) \quad r - \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)(1 + \mu)} r^2 \leq |f(z)| \leq r + \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)(1 + \mu)} r^2, \quad |z| = r$$

$$(2.12.2) \quad 1 - \frac{\beta|3 - 4\alpha| - 1}{2\beta(1 - \alpha)(1 + \mu)} r \leq |f'(z)| \leq 1 + \frac{\beta|3 - 4\alpha| - 1}{2\beta(1 - \alpha)(1 + \mu)} r, \quad |z| = r$$

and

$$(2.12.3) \quad r - \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)} r^2 \leq |D^\mu f(z)| \leq r + \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)} r^2, \quad |z| = r$$

with equalities for $f(z) = z - \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)(1+\mu)} z^2, (z = \pm r)$.

Theorem 2.13. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in $f \in TS_c^*R(\alpha, \beta, \mu)$, then $h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n| z^n$ is also in $f \in TS_c^*R(\alpha, \beta, \mu)$.*

Theorem 2.14. *Let*

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{\beta|3 - 4\alpha| - 1}{[n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)] A_n(\mu)} z^n, (n = 2, 3, \dots).$$

*Then $f \in TS_c^*R(\alpha, \beta, \mu)$ if and only if can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.*

Theorem 2.15. *Let $f \in TS_c^*R(\alpha, \beta, \mu)$. Then f is starlike in the disc*

$|z| < r = r(\alpha, \beta)$, where

$$r(\alpha, \beta) = \inf_n \left[\frac{\{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} A_n(\mu)}{n(\beta|3 - 4\alpha| - 1)} \right]^{\frac{1}{n-1}}, n \geq 2.$$

Theorem 2.16. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_cR(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} n \{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} A_n(\mu) |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 2.17. *If $f \in TC_cR(\alpha, \beta, \mu)$, then*

$$(2.17.1) \quad r - \frac{\beta|3 - 4\alpha| - 1}{8\beta(1 - \alpha)(1 + \mu)} r^2 \leq |f(z)| \leq r + \frac{\beta|3 - 4\alpha| - 1}{8\beta(1 - \alpha)(1 + \mu)} r^2, \quad |z| = r$$

$$(2.17.2) \quad 1 - \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)(1 + \mu)} r \leq |f'(z)| \leq 1 + \frac{\beta|3 - 4\alpha| - 1}{4\beta(1 - \alpha)(1 + \mu)} r, \quad |z| = r$$

and

$$(2.17.3) \quad r - \frac{\beta|3 - 4\alpha| - 1}{8\beta(1 - \alpha)} r^2 \leq |D^\mu f(z)| \leq r + \frac{\beta|3 - 4\alpha| - 1}{8\beta(1 - \alpha)} r^2, \quad |z| = r$$

with equalities for $f(z) = z - \frac{\beta|3-4\alpha|-1}{8\beta(1-\alpha)(1+\mu)} z^2$, $(z = \pm r)$.

Theorem 2.18. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in $TC_cR(\alpha, \beta, \mu)$, then $h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n| z^n$ is also in $TC_cR(\alpha, \beta, \mu)$.*

Theorem 2.19. *Let*

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{\beta|3 - 4\alpha| - 1}{n[n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)] A_n(\mu)} z^n, \quad (n = 2, 3, \dots).$$

Then $f \in TC_cR(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Theorem 2.20. *Let $f \in TC_cR(\alpha, \beta, \mu)$. Then f is convex in the disk*

$|z| < r = r(\alpha, \beta)$, where

$$r(\alpha, \beta) = \inf_n \left[\frac{\{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} A_n(\mu)}{n^2(\beta|3 - 4\alpha|) - 1} \right]^{\frac{1}{n-1}}, \quad n \geq 2.$$

3. CLASSES OF FUNCTIONS INVOLVING A LINEAR OPERATOR

Let us consider the linear operator $L(a, c)f$, introduced by Carlson and Shaffer (cf. [9]), which is given by

$$\begin{aligned} L(a, c)f(z) &= \phi(a, c; z) * f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| z^n \end{aligned}$$

$$\text{Here } \phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n; \text{ for } c \neq 0, -1, -2, \dots; a \neq -1; z \in U$$

where $(\lambda)_n$ is Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(n + \lambda)}{\Gamma(\lambda)} = \begin{cases} 1 & n = 0 \\ \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + n - 1) & n \in N \end{cases}$$

In analogy with the classes considered relating to the operator $R(\alpha, \beta, \mu)$ in Section 2, we consider here the classes relating to the linear operator $L(a, c)$; namely, $TS_s^*L(\alpha, \beta, a, c)$, $TS_c^*L(\alpha, \beta, a, c)$, $TC_sL(\alpha, \beta, a, c)$, and $TC_cL(\alpha, \beta, a, c)$:

(i) A function $f \in T$ is in $TS_s^*L(\alpha, \beta, a, c)$, the class of starlike of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to symmetric points, if and only if

$$\left| \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) - L(a, c)f(-z)} - 1 \right] \middle/ \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) - L(a, c)f(-z)} + (1 - 2\alpha) \right] \right| < \beta, |z| < 1.$$

(ii) A function $f \in T$ is in $TS_c^*L(\alpha, \beta, a, c)$, the class of starlike of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to conjugate points, if and only if

$$\left| \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) + \overline{L(a, c)f(\bar{z})}} - 1 \right] \middle/ \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) + \overline{L(a, c)f(\bar{z})}} + (1 - 2\alpha) \right] \right| < \beta, |z| < 1.$$

A function $f \in T$ is in $TC_sL(\alpha, \beta, a, c)$ and $TC_cL(\alpha, \beta, a, c)$, respectively, if and only if $zf' \in TS_s^*L(\alpha, \beta, a, c)$ and $zf' \in TS_c^*L(\alpha, \beta, a, c)$.

Analogous to Theorems 2.1 to 2.20, we can now have theorems for these four classes involving the operator $L(a, c)$. In what follows, we only give results concerning coefficient inequalities for each of these classes and the others can be stated analogously which are omitted for the sake of brevity.

Theorem 3.1. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS_s^*L(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 3.2. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_sL(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} n \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 3.3. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS_c^*L(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 3.4. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_cL(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} n \{n(1 + \beta) + 2(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

4. CLASSES OF FUNCTIONS WITH SYMMETRIC CONJUGATE POINTS INVOLVING RUSCHEWEYH DERRIVATIVE AND A LINEAR OPERATOR.

In this section, we consider the classes $TS_{sc}^*R(\alpha, \beta, \mu)$ and $TC_{sc}R(\alpha, \beta, \mu)$ involving Ruscheweyh derivative $R(\alpha, \beta, \mu)$ and the classes $TS_{sc}^*L(\alpha, \beta, a, c)$, $TC_{sc}L(\alpha, \beta, a, c)$ involving the linear operator $L(\alpha, \beta, a, c)$.

A function $f \in T$ is in $TS_{sc}^*R(\alpha, \beta, \mu)$, the class of starlike functions of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to symmetric conjugate points, if and only if

$$(4.1) \quad \left| \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - \overline{D^\mu f(-z)}} - 1 \right] / \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - \overline{D^\mu f(-z)}} + 1 - 2\alpha \right] \right| < \beta, |z| < 1.$$

Further, $f \in T$ is in $TC_{sc}R(\alpha, \beta, \mu)$, the class of convex function with symmetric conjugate points, if and only if $zf' \in TS_{sc}^*R(\alpha, \beta, \mu)$.

Similarly, a function $f \in T$ is in $TS_{sc}^*L(\alpha, \beta, a, c)$, the class of starlike functions of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) with respect to symmetric conjugate points, if and only if

$$(4.2) \quad \left| \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) - \overline{L(a, c)f(-z)}} - 1 \right] / \left[\frac{z(L(a, c)f(z))'}{L(a, c)f(z) - \overline{L(a, c)f(-z)}} + 1 - 2\alpha \right] \right| < \beta, |z| < 1.$$

Further, $f \in T$ is in $TC_{sc}L(\alpha, \beta, a, c)$, the class of convex functions with symmetric conjugate points, if and only if $zf' \in TS_{sc}^*L(\alpha, \beta, a, c)$.

Given below, without the details of the proofs, are the coefficient inequalities for above four classes whereas the results concerning distortion theorems, extreme points, radii of starlikeness and convexity etc. for these classes can be formulated accordingly.

Theorem 4.1. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS_{sc}^*R(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} A_n(\mu) |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 4.2. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_{sc}R(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} n \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} A_n(\mu) |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 4.3. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_{sc}L(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Theorem 4.4. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TC_{sc}L(\alpha, \beta, a, c)$ if and only if*

$$\sum_{n=2}^{\infty} n \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq \beta|3 - 4\alpha| - 1.$$

This result is sharp.

Remark 1. If the definitions for the classes $TS_s^*R(\alpha, \beta, \mu)$ given in (2.1) and $TS_{sc}^*R(\alpha, \beta, \mu)$ given in (4.1) are compared, it appears that these classes are identical. This is further justified by observing that the coefficient inequality for the class $TS_{sc}^*R(\alpha, \beta, \mu)$ obtained in (4.1) is exactly the same as that for the class $TS_s^*R(\alpha, \beta, \mu)$ obtained in Theorem 2.1. Similar remark is also applied to the classes $TC_{sc}R(\alpha, \beta, \mu)$, $TS_{sc}^*L(\alpha, \beta, a, c)$ and $TC_{sc}L(\alpha, \beta, a, c)$.

5. PROOFS OF THE RESULTS

Proof of Theorem 2.1. Let $|z| = 1$. Then

$$\begin{aligned}
& |z(D^\mu f(z))' - D^\mu f(z) + D^\mu f(-z)| - \beta |z(D^\mu f(z))' + (1 - 2\alpha)(D^\mu f(z) - D^\mu f(-z))| \\
&= \left| z + \sum_{n=2}^{\infty} (n - 1 + (-1)^n) A_n(\mu) |a_n| z^n \right| - \beta \left| (3 - 4\alpha)z \right. \\
&\quad \left. - \sum_{n=2}^{\infty} [n + (1 - (-1)^n)(1 - 2\alpha)] A_n(\mu) |a_n| z^n \right| \\
&\leq [1 - \beta|3 - 4\alpha|] + \sum_{n=2}^{\infty} [n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)] A_n(\mu) |a_n| \leq 0.
\end{aligned}$$

hence, by the maximum modulus theorem, $f \in TS_S^* R(\alpha, \beta, \mu)$.

For the converse, assume that

$$\left| \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - 1 \right] \middle/ \left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} + (1 - 2\alpha) \right] \right| < \beta.$$

Since $|Re(z)| \leq |z|$, for all z , we have

$$Re \left\{ \frac{z + \sum_{n=2}^{\infty} (n - 1 + (-1)^n) A_n(\mu) |a_n| z^n}{(3 - 4\alpha)z - \sum_{n=2}^{\infty} [n + (1 - 2\alpha)(1 - (-1)^n)] A_n(\mu) |a_n| z^n} \right\} < \beta.$$

Choose values of z on the real axis so that $\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)}$ is real and then let $z \rightarrow 1$, through real values, we obtain

$$\sum_{n=2}^{\infty} \{n(\beta + 1) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\} A_n(\mu) |a_n| - [\beta|3 - 4\alpha| - 1] \leq 0,$$

which gives the required condition.

Finally, the function

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(\beta|3 - 4\alpha| - 1)}{\{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\}A_n(\mu)} z^n$$

is an extremal function for the inequality. \square

Proof of Theorem 2.2. We note from (1.1) and (2.2) that

$$(2.2.4) \quad r - r^2 \sum_{n=2}^{\infty} |a_n| \leq |f(z)| \leq r + r^2 \sum_{n=2}^{\infty} |a_n|,$$

$$(2.2.5) \quad 1 - r \sum_{n=2}^{\infty} n|a_n| \leq |f'(z)| \leq 1 + r \sum_{n=2}^{\infty} n|a_n|$$

and

$$(2.2.6) \quad r - r^2 \sum_{n=2}^{\infty} A_n(\mu)|a_n| \leq |D^\mu f(z)| \leq r + r^2 \sum_{n=2}^{\infty} A_n(\mu)|a_n|.$$

By Theorem 2.1, we have

$$\begin{aligned} \sum_{n=2}^{\infty} |a_n| &\leq \frac{\beta|3 - 4\alpha| - 1}{2(1 + \beta)(1 + \mu)}, \\ \sum_{n=2}^{\infty} n|a_n| &\leq \frac{\beta|3 - 4\alpha| - 1}{(1 + \beta)(1 + \mu)} \end{aligned}$$

and

$$\sum_{n=2}^{\infty} A_n(\mu)|a_n| \leq \frac{\beta|3 - 4\alpha| - 1}{2(1 + \beta)}$$

Using these inequalities in (2.2.4), (2.2.5) and (2.2.6), respectively, the results (2.2.1), (2.2.2) and (2.2.3) are established. \square

Proof of Theorem 2.3. The proof follows directly by appealing to Theorem 2.1. In fact, f and g being in $TS_s^*R(\alpha, \beta, \mu)$, we have

$$(2.3.1) \quad \sum_{n=2}^{\infty} \{n(1 + \beta) + (1 - (-1)^n)(\beta(1 - 2\alpha) - 1)\}A_n(\mu)|a_n| \leq [\beta|3 - 4\alpha| - 1]$$

and

$$(2.3.2) \quad \sum_{n=2}^{\infty} \{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu) |b_n| \leq [\beta|3-4\alpha|-1].$$

It is sufficient, for h to be a member of $TS_s^*R(\alpha, \beta, \mu)$, to show

$$\frac{1}{2} \sum_{n=2}^{\infty} \{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu) |a_n + b_n| \leq [\beta|3-4\alpha|-1].$$

which follows immediately by the use of (2.3.1) and (2.3.2). \square

Proof of Theorem 2.4. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{(\beta|3-4\alpha|-1)\lambda_n}{\{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu)} z^n, \\ &= z - \sum_{n=2}^{\infty} t_n z^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} \left\{ \frac{[n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)] A_n(\mu)}{\beta|3-4\alpha|-1} \right\} t_n \\ = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1 \end{aligned}$$

Thus, by Theorem 2.1, $f \in TS_s^*R(\alpha, \beta, \mu)$.

Conversely, suppose $f \in TS_s^*R(\alpha, \beta, \mu)$. Again by Theorem 2.1, we have

$$|a_n| \leq \frac{\beta|3-4\alpha|-1}{\{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu)}, \quad n = 2, 3, \dots$$

Setting

$$\lambda_n = \frac{\{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\} A_n(\mu)}{(\beta|3-4\alpha|-1)} |a_n|, \quad n = 2, 3, \dots$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,$$

we have

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

which completes the proof. \square

Proof of Theorem 2.5. Noting that

$$\left| \frac{zf'(z)}{f(z) - f(-z)} - 1 \right| \leq \frac{|z| + \sum_{n=2}^{\infty} [n - 1 + (-1)^n] |a_n| |z|^n}{2|z| - \sum_{n=2}^{\infty} [1 + (-1)^n] |a_n| |z|^n}$$

we find that $\left| \frac{zf'(z)}{f(z) - f(-z)} \right| < 1$ for $|z| < 1$ if

$$\sum_{n=2}^{\infty} n |a_n| |z|^{n-1} < 1.$$

Hence f is starlike if

$$|z| \leq \left\{ \frac{\{n(1 + \beta) + [1 - (-1)^n][\beta(1 - 2\alpha) - 1]\} A_n(\mu)}{n(\beta|3 - 4\alpha| - 1)} \right\}^{\frac{1}{n-1}}, \quad n = 2, 3, \dots$$

which completes the proof. \square

Proofs of Theorem 2.6 to 2.10. Using the fact that $f \in T$ is in $TC_s R(\alpha, \beta, \mu)$ if and only if $zf' \in TS_s^* R(\alpha, \beta, \mu)$ and the results proved for the class $TS_s^* R(\alpha, \beta, \mu)$ in Theorem 2.1 to 2.5, the proofs follow. \square

Proofs of Theorem 2.11 to 2.20 Using arguments similar to those for Theorems 2.1 to 2.10, the proofs can be derived. \square

Proofs of Theorems 3.1 to 3.4 The coefficient inequality in these theorems can be derived by using the arguments similar to those of Theorems 2.1, 2.6, 2.11 and 2.16. \square

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DEPARTMENT OF MATHEMATICS, MAHARAJA AGRASEN COLLEGE,, UNIVERSITY OF DELHI,
DELHI 110093, INDIA.

E-mail address: vanitajain_2011@yahoo.com