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AN UPPER BOUND FOR THE RAMSEY NUMBER $r(C_6, K_9)$

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ABSTRACT: Let C_m denote a cycle of length *m* and K_n a complete graph of order *n*. In this paper we establish an upper bound for the Ramsey number of $r(C_6, K_9)$ by proving that $41 \le r(C_6, K_9) \le 46$.

1. INTRODUCTION

All graphs considered in this paper are undirected and simple. C_m, P_m, K_m and S_m stand for cycle, path, complete, and star graphs on *m* vertices, respectively. The open neighborhood of a vertex u is the set of all vertices of G that are adjacent to u, denoted by $N(u)$ and the closed neighborhood of u is $N[u] = \{u\} \cup N(u)$. The minimum degree of all vertices in G is denoted by $\delta(G)$. If $V_1 \subset V(G)$ and V_1 is a nonempty set, the subgraph of G whose vertex set is V_1 and whose edge set is the set of these edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $< V_1 >_G$ or $G[V_1]$. If $V_1, V_2 \subseteq V(G)$, we use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The set $A \subseteq V(G)$ is called an independent set if any two vertices of A are non adjacent. The independence number of a graph G is the order of the largest independent set, and is denoted by $\alpha(G)$.

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The cycle-complete graph Ramsey number $r(C_n, K_m)$ is the smallest integer N such that every graph of order N contain a cycle C_n on n vertices or Ğ contains the complete graph K_m, where Ğ is the complement of G. Chartrand and Schuster [5] proved For all $n > 3$, $r(C_n, K_3) = 2n-1$. Bondy and Erdȍs [3] proved that For all $n \geq m^2 - 2$, $r(C_n, K_m) = (n-1)(m-1)+1$. In 1978, Erdȍs, Faudree, Roussean and Schelp [6] conjectured the following.

$r(C_n, K_m) = (n-1)(m-1)+1$ for all $n \geq m \geq 3$, and $(n,m) \neq (3,3)$.

The conjecture was confirmed for *n=3, 4, 5, 6 and 7* (see [15], [4], [13], and [16]). Nikiforov [11] proved the conjecture for all $n \geq 4m^2 + 2$, $m \geq 3$. In related work, Radziszowski and Tse [12] showed that $r(C_4, K_7) = 22$ and $r(C_4, K_8) = 26$. In [10] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. In [14] Schiermeyer proved that $r(C_5, K_7) = 25$. In [2] and [9] Bani abedalruhman and Jaradat proved that $r(C_7, K_7) = 37$ and $r(C_8, K_7) = 43$. In [7] Jaradat and Alzaleq proved that $r(C_8, K_8) = 50$. In 2009, Yaojun Chen, Edwin Cheng, and Ran Xu [17] proved the following theorem.

Theorem 1.1. $r(C_6, K_8) = 36$.

2. The Main Results

In this paper we establish an upper bound for the Ramsey number of $r(C_6, K_9)$ by proving that $41 \leq r(C_6, K_9) \leq 46$. It worth mentioning that, in proving our theorem, we follow the same proof that used in [8] by jaradat and Alzaleq. Our proof consists of a series of seven lemmas.

Lemma 2.1. Let G be a graph of order 46 that contains neither C_6 nor an independent set of 9-elements. Then $\delta(G) \geq 10$.

Proof. Suppose not, that is, G contains a vertex, say u of degree less than 10. Then $|V(G) - N[u]| \geq 46 - 10 = 36.$ By Theorem 1.1, $r(C_6, K_8) = 36$, as a result $G - N[u]$ has an independent set consisting of *8* vertices. This set together with the vertex u is an independent set consisting of 9-vertices. So we have $\alpha(G) \geq 9$, and this contradicts the fact that $\alpha(G) < 9$.

In the following five lemmas we assume that G is a graph of order 46 that contains no cycle of length 6 as a subgraph with minimum degree $\delta(G) \geq 10$ and $\alpha(G) < 9$.

Lemma 2.2. If G contains $K_5 - P_3$, then $|V(G)| \geq 55$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of $K_5 - P_3$ where the induced subgraph on $\{u_1, u_2, u_3, u_4\}$ is isomorphic to K_4. Without loss of generality, assume that $u_1u_5, u_2u_5 \in E(G)$ Define $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Since $\delta(G) \geq 10$, $|U_i| \geq 6$ for all $1 \leq i \leq 5$. Since G contains no C_6, we have $U_i \cap U_j = \phi$ for all $1 \leq i < j \leq 5$ except possibly for $i = 1$ and $j = 2$. $E(U_i, U_j) = \phi$ for all $1 \leq i < j \leq 5$, and $N_R(U_i) \cap N_R(U_j) = \phi$ for all $1 \leq i < j \leq 5$. From above, we have

$\sum_{i=3}^{i=5} |U_i \cup N_R(U_i) \cup \{u_i\}| \geq \sum_{i=3}^{i=5} (7 + 3 + 1) = 33.$ Since G contains no cycle of order 6, we have $U_1 \cap N_R(x) = U_2 \cap N_R(y) = \phi$ for all $x \in U_2$ and $y \in U_1$.

Let $A = (U_1 \cup N_R(U_1) \cup \{u_1\}) \cup (U_2 \cup N_R(U_2) \cup \{u_2\})$. It suffices to show that $|A| \geq 22$.

Now we consider two cases.

Case1. $U_1 - U_2 \neq \phi$ and $U_2 - U_1 \neq \phi$. Then

$$|A| \geq |(U_1 - U_2 \cup N_R(U_1 - U_2) \cup \{u_1\}) \cup (U_2 - U_1 \cup N_R(U_2 - U_1) \cup \{u_2\})|$$

$$= |(U_1 - U_2) \cup N_R(U_1 - U_2) \cup \{u_1\}| + |(U_2 - U_1) \cup N_R(U_2 - U_1) \cup \{u_2\}|$$

$$\geq 11 + 11 = 22.$$

Case2. $U_1 - U_2 = \phi$ or $U_2 - U_1 = \phi$. Then $|U_1 \cap U_2| \geq 6$. Since G contains no cycle of length 6, we have for any x *and* $y \in U_1 \cap U_2$, $N_G(x) \cap N_G(y) = \phi$. And for any $x \in U_1 \cap U_2$, $|N_G(x)| \geq 8$. So, we have.

$$\sum_{x \in U_1 \cap U_2} |N_G(x) - \{u_1, u_2\}| \geq 8|U_1 \cap U_2| \geq 48.$$

Lemma 2.3. If G contains K_4, then G contains $K_5 - P_3$.

Proof. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of K_4. Define $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 4$. Since $\delta(G) \geq 10$, $|U_i| \geq 7$ for all $1 \leq i \leq 4$. Now we consider the following cases.

Case 1. $U_i \cap U_j \neq \phi$ for some $1 \leq i < j \leq 4$.

Let $w \in U_i \cap U_j$ where $1 \leq i < j \leq 4$. Then G Contains $K_5 - P_3$, as required.

Case 2. $U_i \cap U_j = \phi$ for all $1 \leq i < j \leq 4$. Since G contains no cycle of length 6, we have for all $1 \leq i < j \leq 4$, $E(U_i, U_j) = \phi$, $N_R(U_i) \cap N_R(U_j) = \phi$ and $E(N_R(U_i), N_R(U_j)) = \phi$. If $\alpha(< U_1 >_G) \geq 3$ and $< U_i >_G$ are not complete for all $i = 2,\ 3$ *and* 4, then we have $\alpha(G) \geq 9$. So we need to consider that at least one of $< U_i >_G$ is complete where $i = 2,\ 3$ *and* 4, or $\alpha(< U_1 >_G) \leq 2$. Now, if one of $< U_i >_G$

for $i = 2,\ 3$ *and* 4, is complete, say, $<U_2>_G$ is complete, then $<U_2>_G$ contains K_5 because $|U_2| \geq 7$. So G contains $K_5 - P_3$, and we are done. If $\alpha(<U_1>_G) \leq 2$. We know that $\delta(G) \geq 10$, and $|U_1| \geq 7$, since $r(K_3, K_4 - e) = 7$ and $\alpha(<U_1>_G) \leq 2$ then $<U_1>_G$ contains $K_4 - e$, and so $<U_1 \cup \{u_1\}>_G$ contains $K_5 - e$, and so G contains $K_5 - P_3$, as required.

Lemma 2.4. If G contains $K_1 + p_4$, then G contains K_4.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of $K_1 + p_4$, where $p_4 = u_2u_3u_4u_5$ is a path and $V(P_4) \subseteq N(u_1)$. Define $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Observe that $|U_i| \geq 6$ for all $1 \leq i \leq 5$ because $\delta(G) \geq 10$. Since G contains no cycle of order 6 we have, $U_i \cap U_j = \phi$, $E(U_i, U_j) = \phi$, $N_R(U_i) \cap N_R(U_j) = \phi$ and $E(N_R(U_i), N_R(U_j)) = \phi$ for all $2 \leq i < j \leq 5$. If $\alpha(<U_5 \cup N_R(U_5)>_G) \geq 3$ and $<U_i>_G$ are not complete for all $i = 2,\ 3$ *and* 4, then we have $\alpha(G) \geq 9$. So we need to consider the case in which at least one of $<U_i>_G$ for $i = 2,\ 3$ *and* 4, is complete, or $\alpha(<U_5 \cup N(U_5)>_G) \leq 2$. If one of $<U_i>_G$ for $i = 2,\ 3$ *and* 4, is complete, say, $<U_2>_G$ is complete, then $<U_2>_G$ contains K_5, because $|U_2| \geq 6$ So G contains K_4. If $\alpha(<U_5 \cup N_R(U_5)>_G) \leq 2$, we know that $\delta(G) \geq 10$ and $|U_5 \cup N_R(U_5)| \geq 9$, since $r(K_3, K_4) = 9$ we have $<U_5 \cup N_R(U_5)>_G$ contains K_4, and so G contains K_4, as required.

Lemma 2.5. If G contains $K_1 + p_3$, then G contains $K_1 + p_4$ or K_4.

Proof**.** Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of $K_1 + p_3$, where $p_3 = u_2u_3u_4$ is a path and $V(P_3) \subseteq N(u_1)$. If $u_2u_4 \in E(G)$, then $<U>_G$ is K_4 and hence we are done. So we need to consider the case in which $u_2u_4 \notin E(G)$. We define $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \le i \le 4$. Since $\delta(G) \ge 10$, $|U_i| \ge 7$ for $i = 1$ and $i = 3$, $|U_i| \ge 8$ for $i = 2$ and $i = 4$. Now we consider the following cases:

Case 1. $U_i \cap U_j = \phi$ for all $2 \le i < j \le 4$.

Since G contains no cycle of order 6 we have, $U_i \cap U_j = \phi$, $E(U_i, U_j) = \phi$ and $N_R(U_i) \cap N_R(U_j) = \phi$ for all $2 \le i < j \le 4$. If $\alpha(< U_i >_G) \ge 3$ for all $i = 2,\ 3\ \textit{and}\ 4$, then we have $\alpha(G) \ge 9$. So we need to consider the case in which $\alpha(< U_i >_G) \le 2$ for at least one i , where $i = 2,\ 3\ \textit{and}\ 4$. Say $i = 2.$ We know that $\delta(G) \ge 10$ and $|U_2| \ge 6$, since $r(K_3, K_3) = 6$ and $\alpha(< U_2 >_G) \le 2$ then $< U_2 >_G$ contains K_3, and so $< U_2 \cup \{u_2\} >_G$ contains K_4, and so G contains K_4, as required.

Case 2. $U_1 \cap U_2 \ne \phi$, then G would have $K_1 + p_4$, as required.

Case 3. $U_1 \cap U_3 \ne \phi$.

Let $u_5 \in U_1 \cap U_3$. Now, if $u_2u_5 \in E(G)$, then G contains K_4, hence we are done. So we assume that $u_2u_5 \notin E(G)$. Define $U^{'} = \{u_1, u_2, u_3, u_4, u_5\}$, $R^{'} = G - U^{'}$ and $U^{'}_i = N(u_i) \cap V(R^{'})$ for each $1 \le i \le 5$. Since G contains no cycle of length6, we have, $U^{'}_2 \cap U^{'}_5 = \phi$, $U^{'}_2 \cap U^{'}_4 = \phi$, $U^{'}_4 \cap U^{'}_5 = \phi$, $E(U^{'}_2, U^{'}_5) = \phi$, $E(U^{'}_2, U^{'}_4) = \phi$ and $E(U^{'}_4, U^{'}_5) = \phi$. If $\alpha(< U_i^{'} >_G) \ge 3$ for all $i = 2,\ 4\ \textit{and}\ 5$, then we have $\alpha(G) \ge 9$. So we need to consider the case in which $\alpha(< U_i^{'} >_G) \le 2$ for at least one i , where $i = 2,\ 4\ \textit{and}\ 5$, we know that $\delta(G) \ge 10$. $|U_5^{'}| \ge 8$, $|U_2^{'}| \ge 8$ and $|U_4^{'}| \ge 8$ because $u_2u_5 \notin E(G)$ and $u_2u_4 \notin E(G)$. Since $r(K_3, K_3) = 6$ and $\alpha(< U_i^{'} >_G) \le 2$ then $< U_i^{'} >_G$ contains K_3, and so $< U_i^{'} \cup \{u_i\} >_G$ contains K_4, and so G contains K_4, as required.

Case 4. $U_1 \cap U_4 \ne \phi$, then G would have $K_1 + p_4$, as required.

Case 5. $U_2 \cap U_3 \neq \phi$, then G would have $K_1 + p_4$, as required.

Case 6. $U_3 \cap U_4 \neq \phi$, then G would have $K_1 + p_4$, as required.

Case 7. $U_2 \cap U_4 \neq \phi$.

Let $u_5 \in U_2 \cap U_4$. Now, if $u_5 u_3 \in E(G)$, then G contains $K_1 + p_4$ where $K_1 = u_3$ and $p_4 = u_5 u_4 u_1 u_2$, hence we are done. So we may assume that $u_5 u_3 \notin E(G)$. Define $U^{'} = \{u_1, u_2, u_3, u_4, u_5\}$, $R^{'} = G - U^{'}$ and $U^{'}{}_i = N(u_i) \cap V(R^{'})$ for each $1 \leq i \leq 5$. Since G contains no cycle of length 6, we have, $U^{'}{}_2 \cap U^{'}{}_3 = \phi$, $U^{'}{}_2 \cap U^{'}{}_5 = \phi$, $U^{'}{}_3 \cap U^{'}{}_5 = \phi$, $E(U^{'}{}_2, U^{'}{}_3) = \phi$, $E(U^{'}{}_2, U^{'}{}_5) = \phi$ and $E(U^{'}{}_3, U^{'}{}_5) = \phi$. If $\alpha(< U_i^{'} >_G) \geq 3$ for all $i = 2,\ 3$ *and* 5, then we have $\alpha(G) \geq 9$. So we need to consider the case in which $\alpha(< U_i^{'} >_G) \leq 2$ for at least one i, where $i = 2,\ 3$ *and* 5, we know that $\delta(G) \geq 10$. Thus, $|U_2^{'}| \geq 7$, $|U_3^{'}| \geq 7$ and $|U_5^{'}| \geq 7$ because $u_2 u_4 \notin E(G)$ and $u_3 u_5 \notin E(G)$. Since $r(K_3, K_3) = 6$ and $\alpha(< U_i^{'} >_G) \leq 2$ then $< U_i^{'} >_G$ contains K_3, and so $< U_i^{'} \cup \{u_i\} >_G$ contains K_4, and so G contains K_4, as required.

Lemma 2.6. If G contains K_3, then G contains $K_1 + p_3$ or K_4.

Proof. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of K_3. Define $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 3$. Since $\delta(G) \geq 10$, $|U_i| \geq 8$ for all $1 \leq i \leq 3$. Now we consider the following cases:

Case 1. $U_i \cap U_j \neq \phi$ for some $1 \leq i < j \leq 3$, then G contains $K_1 + p_3$, as required.

Case 2. $U_i \cap U_j = \phi$ for all $1 \leq i < j \leq 3$. Let $y_i \in U_i$ $1 \leq i \leq 3$. Let $Y = \{y_1, y_2, y_3\}$. Define $R^{'} = G - (Y \cup U)$ and $Y_i = N(y_i) \cap U(R^{'})$. Then $|Y_i| \geq 8$ for all $1 \leq i \leq 3$.

Since G contains no cycle of length6, we have, $Y_i \cap Y_j = \phi$ and $E(Y_i, Y_j) = \phi$ for all $1 \le i < j \le 3$. If $\alpha(< Y_i >_G) \ge 3$ for all $1 \le i \le 3$, then we have $\alpha(G) \ge 9$. So we need to consider the case in which $\alpha(< Y_i >_G) \le 2$ for at least one i, where $i = 1$,2 *and* 3. We know that $|Y_i| \ge 8$, since $r(K_3, K_3) = 6$ we have $< Y_1 >_G$ contains K_3 . $< Y_1 \cup \{y_1\} >_G$ contains K_4. Hence G contains either K_4, as required.

Lemma 2.7. Let G be a graph of order 46 with $\delta(G) \ge 10$ and $\alpha(G) < 9$. Suppose that G does not have C_6 as a subgraph. Then G contains K_3.

Proof. Suppose not, that is G contains no K_3. Let $u \in V(G)$. Since $\delta(G) \ge 10$ then $|N(u)| \ge 10$ .Since G contains no K_3, then the induced subgraph $< N(u) >_G$ is a null graph of order at least 10. Hence, $\alpha(G) \ge 10$. Contradiction with $\alpha(G) < 9$.

Theorem 2.1. $41 \le r(C_6, K_9) \le 46$.

Proof. We prove it by contradiction. Suppose that G is a graph of order 46 which contains neither C_6 nor a 9-elements independent set. Then by Lemma 2.1, $\delta(G) \ge 10$. By Lemma 2.7, G contains K_3. Thus, by Lemma 2.6, 2.5, 2.4, 2.3 and 2.2, $|V(G)| \ge 55$, contradiction. Thus $r(C_6, K_9) \le 46$. To prove $r(C_6, K_9) \ge 41$. Let $H = 8K_5$ observe that $\alpha(H) = 8$ and does not contain a cycle of length 6 as a subgraph. Thus $r(C_6, K_9) \ge 41$, and hence $41 \le r(C_6, K_9) \le 46$. The proof is complete.

References

[1] Alzaalig A. M., "On the Ramsey number of Graphs", M.Sc. Thesis, Yarmouk University, July, (2010).

[2] Baniabedalruhman A. “On Ramsey Numbers for cycle-complete Graphs”. M.Sc. Thesis, Yarmouk University, (2006).

[3] Bondy J. A. and Erd s P., Ramsey Numbers for Cycles in Graphs, Journal of Combinatorial Theory, Series B, 14 (1973), 46-54.

[4] Bollobás B., Jayawardene C. J., Yang Jian Sheng, Huang Yi Ru, Rousseau C. C, and Zhang Ke Min, On a Conjecture Involving Cycle-Complete Graph Ramsey Numbers, Australasian Journal of Combinatorics, 22 (2000), 63-71.

[5] Chartrand G. and Schuster S., On the existence of specified cycles in complementary graphs, Bulletin of the American Mathematical Society, 77 (1971), 995-998.

[6] Erd s P., FaudreeR.J., Rousseau C. C. and Schelp R. H., On Cycle-Complete Graph Ramsey Numbers, Journal of Graph Theory, 2 (1978), 53-64.

[7] Jaradat M. and Alzaleq B., The cycle-complete graph Ramsey number $r(C_8, K_8)$,. SUT Journal of Mathematics, Vol. 43, No. 1 (2007), 85-98.

[8] Jaradat M. and Alzaleq B., cycle-complete graph Ramsey number $r(C_6, K_8) \leq 38$, SUT Journal of Mathematics, Vol. 44, No. 2 (2008), 257-263.

[9] Jaradat M. and Baniabedalruhman, A. M., The cycle-complete graph Ramsey number $r(C_8, K_7)$, International Journal of Pure and Applied Mathematics, 41 (2007), 667-677.

[10] Jayawardene C. J. and Rousseau C., the Ramsey numbers for a cycle of length five versus a complete graph of order six, Journal of Graph Theory, 35 (2000), 99-108.

[11] Nikiforov V., The Cycle-Complete Graph Ramsey Numbers, Combinatorics, Probability and Computing,14 (2005), 349-370.

[12] Radziszowski S. P. and Tse K.-K., A computational approach for the Ramsey number $r(C_4, K_n)$, J. Comb. Math. Comb. Comput., 42 (2002), 195-207.

[13] Schiermeyer I., All Cycle-Complete Graph Ramsey Numbers $r(C_m, K_6)$, Journal of Graph Theory, 44 (2003), 251-260.

[14] Schiermeyer I., All Cycle-Complete Graph Ramsey Numbers $r(C_5, K_7)$, Discussiones Mathematics Grah Theory 25 (2005), 129-139.

[15] Yang Jian Sheng, Huang Yi Ru and Zhang Ke Min, The Value of the Ramsey Number $r(C_n, K_4)$ is $3n-2$ ($n \geq 4$), Australasian Journal of Combinatorics, 20 (1999), 205-206.

[16] Yaojun Chen, Edwin Cheng T. C. and Yunqing Zhang, The Ramsey Numbers $r(C_m, K_7)$ and $r(C_7, K_8)$, European Journal of Combinatorics, 29 (2008), 1337-1352.

[17] Yaojun Chen, T.C. Edwin Cheng and Ran Xu, The Ramsey Number for a Cycle of Length Six versus a Clique of Order Eight, Discrete Applied Mathematics, 157 (2009), 8-12.

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