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EUCLIDEAN PARTIAL SEMIRINGS

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Abstract. The partial functions under disjoint-domain sums and functional composition is a partial semiring, an algebraic structure possessing a natural partial ordering, an infinitary partial addition and a binary multiplication, subject to a set of axioms. In this paper we study the Euclidean partial semirings.

1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The study of $pfn(D, D)$(the set of all partial functions of a set D to itself), $Mfn(D, D)$(the set of all multi functions of a set D to itself) and $Mset(D, D)$(the set of all total functions of a set D to the set of all finite multi sets of D) play an important role in the theory of computer science, and to abstract these structures Manes and Benson[1] introduced the notion of sum ordered partial semirings (so-rings). In this paper we introduced the notion of left Euclidean norm and Dale norm on a partial semiring and we generalize the results of Euclidean semirings studied by Golan[3] to partial semirings.

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2. Preliminaries

In this section we collect some important definitions, results and examples for our use in this paper.

Definition 2.1. [5] *A partial semiring is a quadruple* $(R, \Sigma, \cdot, 1)$*, where* (R, Σ) *is a partial monoid,* $(R, \cdot, 1)$ *is an monoid with multiplicative operation* $\cdot$ *and unit* 1*, and the additive and multiplicative structures obey the following distributive laws. If* $\Sigma(x_i : i \in I)$ *is defined in* R*, then for all* y *in* R*,* $\Sigma(y \cdot x_i : i \in I)$ *and* $\Sigma(x_i \cdot y : i \in I)$ *are defined and*

$y \cdot [\Sigma_i x_i] = \Sigma_i(y \cdot x_i)$*;* $[\Sigma_i x_i] \cdot y = \Sigma_i(x_i \cdot y)$.

Definition 2.2. [2] *Let* R *be a partial semiring. A subset* N *of* R *is said to be a partial ideal of* R *if the following are satisfied*

(1). if $(x_i : i \in I)$ *is summable family in* R *and* $x_i \in N$ *for every* $i \in I$ *then* $\Sigma x_i \in N$*,*

(2). if $x \in N$ *and* $r \in R$ *then* $xr, rx \in N$.

Remark 2.3. [2] *The set of all partial ideals of a partial semiring is a complete lattice, in which meet and join of a family*

$\{I_\alpha \mid \alpha \in \Delta\}$, $\Sigma I_\alpha = \{x \in R \mid x = \Sigma_\alpha r_\alpha x_\alpha s_\alpha, x_\alpha \in I, r_\alpha, s_\alpha \in R\}$.

Example 2.4. [2] *Consider the partial semiring* $pfn(D, D)$*. Let* a *be a fixed element in* D*. Then* $N = \{f \in Pfn(D, D) \mid dom(f) \subseteq \{a\}\}$ *is a right partial ideal of* $Pfn(D, D)$*.*

Definition 2.5. [2] *Let* N *and* P *be partial ideals of a partial semiring* R*. Then we define* $NP = \{x \in R \mid x = \Sigma_i a_i b_i$ *for some* $a_i \in N, b_i \in P\}$.

3. Euclidean partial semirings

We denote the set of all right divisors of $'a'$ in the partial monoid $(R, \cdot)$ by $RD(a)$. i.e., $RD(a) = \{b \in R \mid a \in Rb\} = \{b \in R \mid Ra \subseteq Rb\}$. We denote the set $\{b \in R \mid a \cdot b = 1 = b \cdot a\}$ by $U(R)$ and the set $\{a \in R \mid a \cdot a = a\}$ by $I^{\times}(R)$.

Remark 3.1. *If R is a partial semiring then*
(i). $b \in RD(a)$ if and only if $RD(b) \subseteq RD(a)$,
(ii). $U(R) \subseteq RD(1_R) \subseteq RD(a)$.

Proof. (i). Suppose $b \in RD(a)$. Then $a \in Rb$. Now for any $x \in RD(b)$, $Rb \subseteq Rx$ and hence $a \in Rx$. $\Rightarrow x \in RD(a)$. Hence $RD(b) \subseteq RD(a)$.

Conversely suppose $RD(b) \subseteq RD(a)$. Since $b \in RD(b)$, $b \in RD(a)$.
(ii). Let $x \in U(R)$. Then $\exists\, y \in R \ni xy = 1 = yx \in Rx$ and hence $x \in RD(1_R)$. Now let $x \in RD(1_R)$ then $1 \in Rx$. $\Rightarrow Rx = R$. $\Rightarrow a \in Rx$ and hence $x \in RD(a)$. Hence the remark. □

Definition 3.2. *Let R be a partial semiring and $a \in R$. Then 'a' is said to be irreducible from right if and only if it satisfy*
(i). $a \notin U(R)$, and
(ii). $RD(a) = U(R) \bigcup \{a\}$.

In the partial semirings $\mathbb{N}$ and $pfn(D, D)$, every nonzero element is irreducible from right.

Example 3.3. *Consider the partial semiring $Mat_D(R)$, the set of $D \times D$ matrices over R. Take $D = \{a, b\}$ and $R = \mathbb{N}$. Then the only elements of $Mat_D(R)$ having determinant 1 which are irreducible from right are $[a_{ij}]$ and $[b_{ij}]$ where*

$$a_{ij} = \begin{cases} 0, & if\ i = b \text{ and } j = a, \\ 1, & otherwise. \end{cases}$$

and

$$b_{ij} = \begin{cases} 0, & if\ i = a\ and\ j = b, \\ 1, & otherwise. \end{cases}$$

Definition 3.4. *Let A be a nonempty subset of a partial semiring R. Then the set of common right divisors of A is $CRD(A) = \bigcap\{RD(a) \mid a \in A\} = \{b \in R \mid RA \subseteq Rb\}$.*

Definition 3.5. *Let R be a partial semiring. Then an element $b \in CRD(A)$ is said to be a greatest common right divisor of A if and only if $CRD(A) = RD(b)$.*

Theorem 3.6. *If A is a nonempty subset of a partial semiring R then an element b of R is a greatest common right divisor of A if and only if the following conditions are satisfied:*
(i). $RA \subseteq Rb$,
(ii). if $c \in R$ satisfies $RA \subseteq Rc$ then $Rb \subseteq Rc$.

Proof. Suppose b is a greatest common right divisor of A.
(i). Since $b \in CRD(A)$, $b \in RD(a)\ \forall a \in A$. $\Rightarrow Ra \subseteq Rb\ \forall a \in A$ and hence $RA \subseteq Rb$.
(ii). Suppose $c \in R \ni RA \subseteq Rc$. Then $c \in CRD(A) = RD(b)$ and hence $Rb \subseteq Rc$.

Conversely suppose that the conditions (i) and (ii) are satisfied. By (i), $b \in CRD(A)$. Now for any $x \in RD(b)$, $b \in Rx$. $\Rightarrow b = rx$ and $b \in CRD(A)$. $\Rightarrow b = rx \in RD(a)\ \forall a \in A$. $\Rightarrow a \in Rrx \subseteq Rx\ \forall a \in A$. $\Rightarrow x \in RD(a)\ \forall a \in A$. $\Rightarrow x \in CRD(A)$ and hence $RD(b) \subseteq CRD(A)$. Now for any $c \in CRD(A)$, $RA \subseteq Rc$. $\Rightarrow Rb \subseteq Rc$ (by (ii)). $\Rightarrow c \in RD(b)$ and hence $CRD(A) \subseteq RD(b)$. Hence b is a greatest common right divisor of A. □

Corollary 3.7. *If every left partial ideal of a partial semiring R is principal then every nonempty subset of R has a greatest common right divisor.*

Proof. Let A be a nonempty subset of R. Since RA is a left partial ideal of R, we have $RA = Rb$ for some $b \in R$. Now let $c \in R \ni RA \subseteq Rc$. Then $Rb \subseteq Rc$. Then by theorem 3.6, b is the greatest common right divisor of A. □

Theorem 3.8. *Let a, b and c be elements of a partial semiring R. If d is a greatest common right divisor of $\{a, b\}$ and e is a greatest common right divisor of $\{c, d\}$ then e is a greatest common right divisor of $\{a, b, c\}$.*

Proof. By definition, $RD(e) = RD(d) \bigcap RD(c) = RD(a) \bigcap RD(b) \bigcap Rd(c)$ $= CRD(\{a, b, c\})$. Hence the theorem. □

Remark 3.9. *If a and b are elements of a partial semiring R and (a, b) is a summable family in R then $CRD(\{a, b\}) \subseteq CRD(\{a + b, b\})$.*

Proof. For any $x \in CRD(\{a, b\}) = RD(a) \bigcap RD(b)$, $a \in Rx$ and $b \in Rx$.
$\Rightarrow a + b \in Rx$ and hence $x \in RD(a + b) \bigcap RD(b) = CRD(\{a + b, b\})$. Hence the remark. □

Theorem 3.10. *Let R be a partial semiring. Then the following are equivalent*
(i). $CRD(\{a, b\}) = CRD(\{a + b, b\})$ for all $a, b \in R \ni a + b$ exists in R,
(ii). every principal left partial ideal of R is subtractive.

Proof. (i)⇒(ii): Suppose $CRD(\{a, b\}) = CRD(\{a + b, b\})$ for all $a, b \in R \ni a + b$ exists in R. Let Rd be a principal left partial ideal of R and let $x, x + y \in Rd$. Then $d \in RD(x)$ and $d \in RD(x + y)$. $\Rightarrow d \in CRD(\{x + y, x\}) = CRD\{x, y\}$.
$\Rightarrow d \in RD(y)$ and hence $y \in Rd$. Hence Rd is subtractive.
(ii)⇒(i): Suppose every principal left partial ideal of R is subtractive. Let $a, b \in R \ni a + b$ exists in R and let $x \in CRD(\{a + b, b\})$. Then $x \in RD(a + b)$ and $x \in RD(b)$. $\Rightarrow a + b \in Rx$ and $b \in Rx$. $\Rightarrow a \in Rx$ and hence $x \in RD(a) \bigcap RD(b) = CRD(\{a, b\})$. Hence $CRD(\{a, b\}) = CRD(\{a + b, b\})$ for all $a, b \in R \ni a + b$ exists in R. □

Definition 3.11. *A partial semiring R is said to be PLIS-semiring if it satisfies any one of the following equivalent conditions:*
(i). $CRD(\{a, b\}) = CRD(\{a + b, b\})$ for all $a, b \in R \ni a + b$ exists in R,
(ii). every principal left partial ideal of R is subtractive.

Note that the partial semirings $\mathbb{N}, \mathbb{R}$ are PLIS-semirings. The following is an example of a partial semiring which is not PLIS-semiring.

Example 3.12. *Let $S = (\mathbb{R}^+ \times \{0\}) \bigcup (\{0\} \times \mathbb{R}^+)$. Define Σ on S by*

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ (a + a', 0),\ if\ x_i = (a, 0),\ x_j = (a', 0)\ \&\ x_k = 0\ \forall\ k \neq i, j \\ (0, b + b'),\ if\ x_i = (0, b)\ or\ (b, 0),\ x_j = (0, b')\ \&\ x_k = 0\ \forall\ k \neq i, j \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$, defined on R by $(a, 0)\cdot(a', 0) = (aa', 0) = (0, a)\cdot(a', 0)$ and $(0, b)\cdot(o, b') = (0, bb')$ $= (b, 0) \cdot (0, b')$ $\forall a, a', b, b' \in \mathbb{R}^+$. Then $R = S \times \mathbb{N}$ is a partial semiring. Now $H = \{0\} \times \mathbb{R}^+ \times \{0\}$ is a principal left partial ideal of R. Since $(0, b, 0) \in H$, $(b, 0, 0) + (0, b, 0) = (0, 2b, 0) \in H$. But $(b, 0, 0) \notin H$. Hence H is not subtractive. Hence R is not a PLIS-semiring.

Definition 3.13. *Let R be a partial semiring. Then a mapping $\delta : R \setminus \{0\} \to \mathbb{N}$ is said to be a left Euclidean norm on R if it satisfies the following condition:*

If a and b are elements of R with $b \neq 0$ and $\delta(a) \geq \delta(b)$ then $\exists\ q, r \in R \ni a = qb+r$ with $r = 0$ or $\delta(r) < \delta(b)$.

Definition 3.14. *A partial semiring R is said to be left Euclidean if and only if there exists a left Euclidean norm defined on R.*

The partial semiring $\mathbb{N}$ is left Euclidean if we define the left Euclidean norm δ by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$.

Remark 3.15. *If δ is a left Euclidean norm on a partial semiring R, then we can extend δ to δ' from R to $\mathbb{N}\bigcup\{\infty\}$ by defining $\delta'(0) = \infty$ and $\delta'(a) = \delta(a)\ \forall a \in R\setminus\{0\}$. Conversely if $\delta' : R \to \mathbb{N}\bigcup\{\infty\}$ is a function satisfying the condition: for any $a, b \in R$ $\ni \delta'(a) \geq \delta'(b)\ \exists\ q, r \in R \ni a = qb+r$ with $r = 0$ or $\delta'(r) < \delta'(b)$, then its restriction is a left Euclidean norm on R.*

Theorem 3.16. *If δ is a left Euclidean norm defined on a partial semiring R then there exists a left Euclidean norm δ^* satisfying*
(i). $\delta^*(a) \leq \delta(a)\ \forall a \in R \setminus \{0\}$, and
(ii). $\delta^*(b) \leq \delta(rb)\ \forall b, r \in R \ni rb \neq 0$.

Proof. Define $\delta^* : R \setminus \{0\} \to \mathbb{N}$ by $\delta^*(a) = \min\{\delta(ra) \mid ra \neq 0\}\ \forall 0 \neq a \in R$. Then $\delta^*(a) \leq \delta(a)\ \forall a \in R \setminus \{0\}$, and $\delta^*(b) \leq \delta(rb)\ \forall b, r \in R \ni rb \neq 0$.

Now we prove that δ^* is a left Euclidean norm:
Let $a, b \in R \setminus \{0\} \ni \delta^*(a) \geq \delta^*(b) = \min\{\delta(rb) \mid rb \neq 0\}$. Then $\exists\ s \in R \ni \delta^*(b) = \delta(sb)$. By (i), $\delta(a) \geq \delta^*(a) \geq \delta^*(b) = \delta(sb)$. $\Rightarrow \exists\ q, r \in R \ni a = q(sb) + r$ where $r = 0$ or $\delta(r) < \delta(sb)$. Suppose $\delta(r) < \delta(sb)$. Then $\delta^*(r) \leq \delta(r) < \delta(sb) = \delta^*(b)$. $\Rightarrow$ $\exists\ qs, r \in R \ni a = (qs)b + r$ where $r = 0$ or $\delta^*(r) < \delta^*(b)$. Hence the theorem. ☐

Definition 3.17. *Let (R, δ) be a left Euclidean partial semiring. Then δ is said to be submultiplicative norm if it satisfies the following condition:*

$\delta(b) \leq \delta(rb)\ \forall\ 0 \neq b \in R,\ r \in R \ni rb \neq 0.$

Definition 3.18. *A left Euclidean norm δ defined on a partial semiring R is said to be multiplicative norm if and only if $\delta(ab) = \delta(a)\delta(b)\ \forall a, b \in R \ni ab \neq 0$.*

In the left Euclidean partial semiring $\mathbb{N}$, δ defined by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$ is a submultiplicative and multiplicative norm.

Theorem 3.19. *Let R be a partial semiring and $\delta : R\backslash\{0\} \to \mathbb{N}$ be a submultiplicative Euclidean norm. If $M_\delta = \{r \in R \mid \delta(r) \leq \delta(a) \forall 0 \neq a \in R\}$ is a minimal element of $im(\delta)$ then*
(i). $1_R \in M_\delta$,
(ii). if $a \in M_\delta$ then $\exists\ q \in R \ni 1 = qa$,
(iii). $M_\delta \bigcap I^\times(R) = \{1_R\}$,
(iv). $U(R) \subseteq M_\delta$, with equality holding if R is commutative.

Proof. (i). Since δ is submultiplicative norm, $\delta(1_R) \leq \delta(a)\ \forall 0 \neq a \in R$ and hence $1_R \in M_\delta$.
(ii). Let $a \in M_\delta$. Then $\delta(a) \leq \delta(b)\ \forall 0 \neq b \in R$. $\Rightarrow \delta(a) \leq \delta(1_R)$. $\Rightarrow \exists\ q, r \in R \ni 1_R = qa + r$ with $r = 0$ or $\delta(r) < \delta(a)$. Since $a \in M_\delta$, $\delta(a) \leq \delta(r)$ for $0 \neq r \in R$ and hence $r = 0$. Hence $1_R = qa$.
(iii). Let $c \in M_\delta \bigcap I^\times(R)$. Then $c \in M_\delta$ and $c^2 = c$. By (ii), $\exists\ q \in R \ni 1_R = qc = qc^2 = 1_R c = c$. Hence $M_\delta \bigcap I^\times(R) = \{1_R\}$.
(iv). Let $a \in U(R)$. Then $\exists\ b \in R \ni 1_R = ba$. Since δ is submultiplicative norm, $\delta(a) \leq \delta(ba) = \delta(1_R)$ and by (i), $\delta(1_R) \leq \delta(a)$. $\Rightarrow \delta(a) = \delta(1_R) \leq \delta(b)\ \forall 0 \neq b \in R$ and hence $a \in M_\delta$.

Suppose R is commutative and let $a \in M_\delta$. By (ii), $\exists\ q \in R \ni 1 = qa = aq$ and hence $a \in U(R)$. Hence $M_\delta = U(R)$. □

Theorem 3.20. *If $\gamma : R \to S$ is an epimorphism of partial semirings R, S and δ is a left Euclidean norm on R then $\exists$ a left Euclidean norm δ' on S defined by $\delta'(c) = min\{\delta(a) \mid a \in \gamma^{-1}(c)\}\ \forall 0 \neq c \in S$.*

Proof. Define $\delta' : S \setminus \{0\} \to \mathbb{N}$ by $\delta'(c) = \min\{\delta(a) \mid a \in \gamma^{-1}(c)\}\ \forall 0 \neq c \in S$. Let $c, d \in S \ni d \neq 0$ with $\delta'(c) \geq \delta'(d)$. $\Rightarrow \exists\ a, 0 \neq b \in R \ni \gamma(a) = c$, $\gamma(b) = d$ where b is such that $\delta(b) = \min\{\delta(y) \mid y \in \gamma^{-1}(d)\}$. Since $\delta'(c) \geq \delta'(d)$, $\min\{\delta(x) \mid x \in \gamma^{-1}(c)\}$ $\geq \min\{\delta(y) \mid y \in \gamma^{-1}(d)\}$. $\Rightarrow \delta(a) \geq \delta(b)$. $\Rightarrow \exists\ q, r \in R \ni a = qb + r$ where

$r = 0$ or $\delta(r) < \delta(b)$. $\Rightarrow c = \gamma(a) = \gamma(qb + r) = \gamma(q)d + \gamma(r)$ where $\gamma(r) = 0$ or $\delta'(\gamma(r))$ =min$\{\delta(a) \mid a \in \gamma^{-1}(\gamma(r))\} \leq \delta(r) < \delta(b) = \delta'(d)$. Hence δ' is a Euclidean norm on S. □

Theorem 3.21. *If R is a left Euclidean partial semiring then every subtractive left partial ideal of R is principal.*

Proof. Let δ be the Euclidean norm defined on R and I be a subtractive left partial ideal of R. Take $\mathcal{C} = \{\delta(a) \mid a \in I\}$. Then by Zorn's lemma, $\mathcal{C}$ has a minimal element. Let it be $\delta(b)$. Suppose $I \neq Rb$. Then $\exists\ a \in I \ni a \notin Rb$. $\Rightarrow \delta(b) \leq \delta(a)$ (by the minimality of $\delta(b)$). $\Rightarrow \exists\ q, r \in R \ni a = qb + r$ with $r = 0$ or $\delta(r) < \delta(b)$. Suppose $r = 0$ then $a = qb \in Rb$, a contradiction. $\Rightarrow \delta(r) < \delta(b)$. Since $qb + r = a \in I$ and $b \in I$, we have $r \in I$. $\Rightarrow r \in I$ and $\delta(r) < \delta(b)$, a contradiction. Hence $I = Rb$ is a principal left partial ideal of R. □

Theorem 3.22. *The following conditions on a left Euclidean partial semiring are equivalent:*

(i). R is a PLIS-semiring,

(ii). there exists a left Euclidean norm δ defined on R satisfying the condition that if $a = qb + r$ for $r \in R \setminus \{0\}$ and $\delta(r) < \delta(b)$ then $a \notin Rb$.

Proof. (i)⇒(ii): Suppose R is a PLIS-semiring. Since R is left Euclidean partial semiring, $\exists$ a left Euclidean norm δ on R. By theorem 3.16, $\exists$ a left Euclidean norm δ^* defined on $R \ni \delta^*(b) \leq \delta(rb)\ \forall r, b \in R \ni rb \neq 0$. Now suppose $a = qb + r \in Rb$ for $r \in R \setminus \{0\}$ and $\delta^*(r) < \delta^*(b)$. Since R is PLIS-semiring, Rb is subtractive. $\Rightarrow r \in Rb$. $\Rightarrow r = cb$ for some $c \in R$. $\Rightarrow \delta^*(r) = \delta^*(cb) = \delta(cb) \geq \delta^*(b)$, a contradiction. Hence $a \notin Rb$.

(ii)⇒(i): Suppose the condition (ii) is valid and let $t \in CRD(\{a+b, b\})$. $\Rightarrow a+b = dt$ and $b = et$ for some $d, e \in R$. $\Rightarrow a + et = dt \in Rt$. Then by (ii), $\delta(r) \geq \delta(t)$

$\forall r \in R \setminus \{0\}$. $\Rightarrow \delta(a) \geq \delta(t)$. $\Rightarrow \exists\, q, r \in R \ni a = qt + r$ where $r = 0$ or $\delta(r) < \delta(t)$. $\Rightarrow dt = a + b = qt + r + et$. Suppose $\delta(r) < \delta(t)$. Then by (ii), $dt \notin Rt$, a contradiction. Hence $r = 0$. $\Rightarrow a = qt$. $\Rightarrow t \in RD(a) \bigcap RD(b) = CRD(\{a, b\})$. Hence R is PLIS-semiring. □

Theorem 3.23. *If R is a left Euclidean PLIS-semiring then any nonempty finite subset A of R has a greatest common right divisor.*

Proof. By theorem 3.8, it is enough to prove that $\exists$ a greatest common right divisor for any a, b in A. For $a = b = 0$, the greatest common right divisor is 0. Suppose $b \neq 0$. By theorem 3.22, $\exists$ a left Euclidean norm δ on $R \ni$ if $a = qb + r$ for $r \in R \setminus \{0\}$ and $\delta(r) < \delta(b)$ then $a \notin Rb$. Since δ is a left Euclidean norm on R, $\exists\, q_1, r_1 \in R \ni a = q_1 b + r_1$ where $r_1 = 0$ or $\delta(r_1) < \delta(b)$. If $r_1 = 0$ then $a = q_1 b \in Rb$, a contradiction. Hence $\exists\, q_1, 0 \neq r_1 \in R \ni a = q_1 b + r_1$ where $\delta(r_1) < \delta(b)$. Continuing this process, we get $q_1, q_2, ..., q_n, q_{n+1}$, $0 \neq r_1, 0 \neq r_2, ..., 0 \neq r_n \in R$ such that $a = q_1 b + r_1$, $b = q_2 r_1 + r_2$,..., $r_{n-2} = q_n r_{n-1} + r_n$, $r_{n-1} = q_{n+1} r_n$ and $\delta(b) > \delta(r_1) > ... > \delta(r_n)$. This process of selecting q_i, r_i is terminated after a finitely many steps. Then $r_{n-1} = q_{n+1} r_n$, $r_{n-2} = (q_n q_{n+1} + 1) r_n$, ..., $b = (q_2 q_3 ... q_n q_{n+1} + ... + q_2 + q_{n+1}) r_n$. $\Rightarrow r_n \in RD(b)$. Now $a = q' r_n$ for some $q' \in R$ and hence $r_n \in RD(a)$. $\Rightarrow r_n \in RD(a) \bigcap RD(b) = CRD(\{a, b\})$. Let $d \in CRD(\{a, b\})$. Then $d \in CRD(\{q_1 b + r_1, b\})$. $\Rightarrow d \in CRD(\{r_1, b_1\})$ and hence $d \in RD(r_1)$. Similarly $d \in RD(r_2)$,..., $d \in RD(r_n)$. Hence $CRD(\{a, b\}) = RD(r_n)$. Therefore r_n is the greatest common right divisor of $\{a, b\}$. Hence the theorem. □

Remark 3.24. *If R is a partial semiring then $P(R) = \{0_R\} \bigcup \{r + 1_R \mid r \in R\}$ is a partial subsemiring of R.*

Proof. Clearly $0_R, 1_R \in P(R)$. Let $(r_i : i \in I)$ be a summable family in $R \ni r_i \in P(R), i \in I$. Then $\Sigma_{i \in I} r_i$ exists and $r_i = s_i + 1_R$ for some $s_i \in R, i \in I$. $\Rightarrow \Sigma_{i \in I} r_i = \Sigma_{i \in I}(s_i + 1_R) = (\Sigma_{i \in I} s_i + \Sigma_{i \neq k} 1_R) + 1_R \in P(R)$. Hence $\Sigma_{i \in I} r_i \in P(R)$.

Let $r_1, r_2 \in P(R)$. Then $r_1 = s_1 + 1_R, r_2 = s_2 + 1_R$ for some $s_1, s_2 \in R$.
$\Rightarrow r_1 r_2 = (s_1 + 1_R)(s_2 + 1_R) = (s_1 s_2 + s_1 + s_2) + 1_R \in P(R)$. Hence $P(R)$ is a partial subsemiring of R. □

Definition 3.25. *A partial semiring R is said to be antisimple if $P(R) = R$.*

The partial semiring $\mathbb{N}$ is antisimple whereas $pfn(D, D)$ is not a antisimple partial semiring.

Definition 3.26. *Let R be a commutative antisimple partial semiring. Then a function $\delta : R \to \mathbb{N}$ is said to be Dale norm if and only if the following conditions are satisfied:*
(i). $\delta(a) = 0$ if and only if $a = 0_R$,
(ii). If $\Sigma_{i \in I} a_i$ exists then $\delta(\Sigma_{i \in I} a_i) \geq \delta(a_i)$ for any $i \in I$,
(iii). $\delta(ab) = \delta(a)\delta(b)$ for all $a, b \in R$,
(iv). If $a \in R$ and $0 \neq b \in R$ then there exists $q, r \in R \ni a = qb + r$, where $r = 0$ or $\delta(r) < \delta(b)$.

The functions defined by $n \mapsto n$ or $n \mapsto n^2$ is a Dale norm on the partial semiring $\mathbb{N}$.

Remark 3.27. *If R is a commutative antisimple partial semiring and δ is a Dale norm on R then R is entire.*

Proof. Let $a, b \in R \ni ab = 0_R$. Then $\delta(ab) = \delta(0_R) = 0$. $\Rightarrow \delta(a)\delta(b) = 0$. $\Rightarrow \delta(a) = 0$ or $\delta(b) = 0$. $\Rightarrow a = 0_R$ or $b = 0_R$ and hence R is entire. □

Clearly every Dale norm defined on a partial semiring R is a left Euclidean norm. The following is an example of a partial semiring R in which δ is a left Euclidean norm but not Dale norm.

Example 3.28. *Consider the partial semiring $R = \{0, a, b, 1\}$ in which Σ defined on R by*

$$\Sigma x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j, \ for \ some \ j, \\ 0, \ if \ x_i = x_j = a \ for \ some \ i, \ j \ \& \ x_k = 0 \ \forall \ k \neq \ i, j \\ 1, \ if \ x_i = a, \ x_j = b \ for \ some \ i, \ j \ \& \ x_k = 0 \ \forall \ k \neq \ i, j \\ a, \ if \ x_i = x_j = 1 \ or \ b \ for \ some \ i, \ j \ \& \ x_k = 0 \ \forall \ k \neq \ i, j \\ b, \ if \ x_i = 1, \ x_j = a \ for \ some \ i, \ j \ \& \ x_k = 0 \ \forall \ k \neq \ i, j \\ undefined, \ otherwise. \end{cases}$$

and $\cdot$ defined on R by the following table:

.	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	1	b
1	0	a	b	1

Then R is a commutative antisimple partial semiring. Now $\delta : R \setminus \{0\} \to \mathbb{N}$ defined by $\delta(1) = \delta(b) = 2$ and $\delta(a) = 3$ is a left Euclidean norm which cannot be converted to a Dale norm.

Theorem 3.29. *If R is a commutative antisimple partial semiring and δ is a Dale norm defined on R then*
(i). $U(R) = \{a \in R \mid \delta(a) = 1\}$,
(ii). R is a division partial semiring if and only if $\delta(R)$ is finite.

Proof. (i). Note that $\delta(1_R) = \delta(1_R \cdot 1_R) = \delta(1_R) \cdot \delta(1_R)$ and hence $\delta(1_R) = 1$. Let $a \in U(R)$. Then $\exists\ b \in R \ni ab = 1_R$. $\Rightarrow \delta(ab) = \delta(a)\delta(b) = \delta(1_R) = 1$. $\Rightarrow \delta(a) = 1$ and $\delta(b) = 1$ and hence $a \in \{c \in R \mid \delta(c) = 1\}$. Now let $a \in \{c \in R \mid \delta(c) = 1\}$. Then $\delta(a) = 1 = \delta(1_R)$. $\Rightarrow \exists\ q, r \in R \ni 1_R = qa + r$, where $r = 0_R$ or $\delta(r) < \delta(a)$. Suppose $\delta(r) < \delta(a) = 1$. Then $\delta(r) = 0$ and hence $r = 0_R$. $\Rightarrow 1_R = qa$ and hence $a \in U(R)$. Hence $U(R) = \{a \in R \mid \delta(a) = 1\}$.
(ii). Suppose R is a division partial semiring and let $0 \neq \delta(a) \in \delta(R)$. Then $0_R \neq a \in R$. $\exists\ b \in R \ni ab = 1_R$. $\Rightarrow \delta(ab) = \delta(1_R) = 1$. $\Rightarrow \delta(a) = 1$ and $\delta(b) = 1$. $\delta(R) = \{0, 1\}$, a finite set.

Conversely suppose that $\delta(R)$ is a finite subset of $\mathbb{N}$ and suppose $\exists$ a nonunit $r \in R \setminus \{0\}$. Then $\delta(r) > 1$ and r^k is nonunit for all $k \geq 1$. $\Rightarrow \delta(r^k) = \delta(r)\delta(r^{k-1}) > \delta(r^{k-1})\ \forall k > 1$ and hence $\delta(R)$ is not finite, a contradiction. Hence R is division partial semiring. □

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