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GENERALIZED CLOSED SETS IN IDEAL $\mathcal{M}$-SPACES

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Abstract. Dontchev et al. [2] introduced and investigated the notion of $\mathcal{I}$-g-closed sets in ideal topological spaces as a modification of g-closed sets due to Levine [5]. The concept of ideal m-spaces was introduced by Al-Omari and Noiri [1]. In this paper, we introduce and study the concept of generalized closed ($\mathcal{I}_{g^*}$-closed) sets in an ideal m-space.

1. Introduction

The notion of ideal topological spaces was first studied by Kuratowski [4]. Jankovic and Hamlett [3] obtained the further properties of ideal topological spaces. In 1970, Levine [5] initiated the investigations of generalized closed (g-closed) sets in topological spaces. As a modification of g-closed sets, Dontchev et al. [2] introduced the notion of $\mathcal{I}$-g-closed sets in an ideal topological space $(X, \tau, \mathcal{I})$, where τ is a topology and $\mathcal{I}$ is an ideal.

Popa and Noiri [7] called a subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X a minimal structure, if $\emptyset$, $X \in m$. Recently, Ozbakir and Yildirim [6] have defined the minimal local function A^*_m in an ideal minimal space $(X, m, \mathcal{I})$. As an analogous

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notion to $\mathcal{I}$-g-closed sets in $(X, \tau, \mathcal{I})$, they defined and studied m-$\mathcal{I}_g$-closed sets in $(X, m, \mathcal{I})$.

Quite recently, the present authors called a subcollection $\mathcal{M}$ of $\mathcal{P}(X)$ a minimal structure on X if (1) $\emptyset$, $X \in \mathcal{M}$ and (2) $\mathcal{M}$ is closed under finite intersections. They defined the local function A_* in an ideal minimal space $(X, \mathcal{M}, \mathcal{I})$. Then $Cl_*(A) = A \cup A_*$ is a Kuratowski closure operator which generates a new topology $\mathcal{M}_*$ containing the minimal structure $\mathcal{M}$. In this paper, by using the local function A_* we introduce and investigate the notion of $\mathcal{I}_{g^*}$-closed sets in $(X, \mathcal{M}, \mathcal{I})$. In the last section, we introduce the notion of T_*-spaces and investigate the relationship between T_*-spaces and $T_{\frac{1}{2}}$-spaces.

2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [3, 4]) and is simply denoted by A^* instead of $A^*(\mathcal{I}, \tau)$.

Definition 2.1. [1] A subfamily $\mathcal{M}$ of the power set $\mathcal{P}(X)$ of a nonempty set X is called an m-structure on X if $\mathcal{M}$ satisfies the following conditions:

(1) $\mathcal{M}$ contains $\emptyset$ and X,

(2) $\mathcal{M}$ is closed under the finite intersection.

The pair $(X, \mathcal{M})$ is called an m-space. An m-space $(X, \mathcal{M})$ with an ideal $\mathcal{I}$ on X is called an ideal m-space and is denoted by $(X, \mathcal{M}, \mathcal{I})$.

A. Al-Omari and T. Noiri [1] introduced the following definitions and results

Definition 2.2. A set $A \in \mathcal{P}(X)$ is called an m-open set if $A \in \mathcal{M}$. $B \in \mathcal{P}(X)$ is called an m-closed set if $X - B \in \mathcal{M}$. We set $mInt(A) = \cup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $mCl(A) = \cap\{F : A \subseteq F, X - F \in \mathcal{M}\}$.

Definition 2.3. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. For a subset A of X, we define the following set: $A_*(\mathcal{I}, \mathcal{M}) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \mathcal{M}(x)\}$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$. In this case there is no confusion $A_*(\mathcal{I}, \mathcal{M})$ is briefly denoted by A_* and is called the $\mathcal{M}$-local function of A with respect to $\mathcal{I}$ and $\mathcal{M}$.

Lemma 2.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B any subsets of X. Then the following properties hold:*

(1) $(\emptyset)_* = \emptyset$,

(2) $(A_*)_* \subset A_*$,

(3) $A_* \cup B_* = (A \cup B)_*$.

Definition 2.4. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. For any subset A of X, we put $Cl_*(A) = A \cup A_*$. Then the operator Cl_* is a Kuratowski closure operator. The topology generated by Cl_* is denoted by $\mathcal{M}_*$, that is $\mathcal{M}_* = \{U \subseteq X : Cl_*(X - U) = X - U\}$. The closure and the interior of A with respect to $\mathcal{M}_*$ are denoted by $Cl_*(A)$ and $Int_*(A)$, respectively.

Theorem 2.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then $\mathcal{M}_*$ is a topology containing the minimal structure $\mathcal{M}$.*

Lemma 2.2. *Let $(X, \mathcal{M})$ be an m-space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X, and let A, B be subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $A_* \subseteq B_*$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $A_*(\mathcal{I}) \supseteq A_*(\mathcal{J})$.*
(3) *$A_* = mCl(A_*) \subseteq mCl(A)$*
(4) *If $A \subseteq A_*$, then $A_* = mCl(A_*) = mCl(A)$.*
(5) *If $A \in \mathcal{I}$, then $A_* = \emptyset$.*

3. $\mathcal{I}_{g^*}$-CLOSED SETS

In this section 3 we investigate the class of generalized m-closed sets in an ideal m-space.

Definition 3.1. A subset A of an ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is said to be $\mathcal{I}_{g^*}$-closed (resp. mg-closed) if $A_* \subseteq U$ (resp. $mCl(A) \subseteq U$) whenever $A \subseteq U$ and $U \in \mathcal{M}$. The complement of an $\mathcal{I}_{g^*}$-closed (resp. mg-closed) set is said to be $\mathcal{I}_{g^*}$-open (resp. mg-open).

Definition 3.2. [5] Let (X, τ) be a topological space. A subset A of X is called a g-closed set if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 3.3. [2] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called an $\mathcal{I}$-g-closed set if $A^* \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. The complement of an $\mathcal{I}$-g-closed set is said to be $\mathcal{I}$-g-open.

Remark 1. Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. If we take the m-structure $\mathcal{M} = \tau$, then $\mathcal{I}_{g^*}$-closed (resp. mg-closed) sets coincide with $\mathcal{I}$-g-closed (resp. g-closed) sets.

Proposition 3.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then the following properties are hold:*

(1) *Every m-closed set is mg-closed.*

(2) *Every mg-closed set is $\mathcal{I}_{g^*}$-closed.*

Proposition 3.2. *The union of two $\mathcal{I}_{g^*}$-closed sets in an ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is $\mathcal{I}_{g^*}$-closed.*

Proof. Let A, B be two $\mathcal{I}_{g^*}$-closed sets, and $A \cup B \subseteq U$, where $U \in \mathcal{M}$. Since A and B are $\mathcal{I}_{g^*}$-closed sets, then $A_* \subseteq U$ and $B_* \subseteq U$. Hence by Lemma 2.1, $A_* \cup B_* = (A \cup B)_* \subseteq U$ and hence $A \cup B$ is $\mathcal{I}_{g^*}$-closed. □

Definition 3.4. A subset A of an ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is said to be $\mathcal{M}_*$-closed (resp. $\mathcal{M}_*$-dense in itself, $\mathcal{M}_*$-perfect) if $A_* \subseteq A$ (resp. $A \subseteq A_*$, $A_* = A$).

Proposition 3.3. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A be a subset of X. If A is $\mathcal{I}_{g^*}$-closed and m-open, then A is $\mathcal{M}_*$-closed.*

Proposition 3.4. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then every subset of X is $\mathcal{I}_{g^*}$-closed if and only if every m-open set is $\mathcal{M}_*$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{g^*}$-closed. If U is m-open, then it is $\mathcal{I}_{g^*}$-closed and hence $U_* \subseteq U$. Hence U is $\mathcal{M}_*$-closed. Conversely, suppose that every m-open set is $\mathcal{M}_*$-closed. If A is any subset of X and U is an m-open set such that $A \subseteq U$, then $A_* \subseteq U_* \subseteq Cl_*(U) = U$ and hence A is $\mathcal{I}_{g^*}$-closed.

□

Theorem 3.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. For a subset A of X, the following properties are hold:*

1. *A is $\mathcal{I}_{g^*}$-closed if and only if $Cl_*(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}$.*
2. *If A is $\mathcal{I}_{g^*}$-closed, then the following equivalent properties hold:*
   - (a) *$Cl_*(A) - A$ contains no a nonempty m-closed set.*
   - (b) *$A_* - A$ contains no a nonempty m-closed set.*

Proof. (1) Suppose that A is $\mathcal{I}_{g^*}$-closed. Then $A_* \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}$ and hence $Cl_*(A) = A \cup A_* \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}$. The converse is obvious.

(2) Suppose $F \subseteq Cl_*(A) - A$ and F is m-closed. Since $F \subseteq X - A$, $A \subseteq X - F$ and $X - F \in \mathcal{M}$. Since A is $\mathcal{I}_{g^*}$-closed, $Cl_*(A) \subseteq X - F$ and $F \subseteq X - Cl_*(A)$. Therefore, $F \subseteq Cl_*(A) \cap (X - Cl_*(A)) = \emptyset$. Thus, (a) is proved.

(a) $\Leftrightarrow$ (b): This follows from the fact that $Cl_*(A) - A = A_* - A$. □

Corollary 3.1. *For a subset of an ideal m-space $(X, \mathcal{M}, \mathcal{I})$, the following diagram holds:*

$$\begin{array}{ccc} m\text{-closed} & \longrightarrow & \mathcal{M}_*\text{-closed} \\ \downarrow & & \downarrow \\ mg\text{-closed} & \longrightarrow & \mathcal{I}_{g^*}\text{-closed} \end{array}$$

None of these implications in Corollary 3.1 is reversible as shown by the below examples.

Example 3.1. *Let $X = \{a, b, c\}$, $\mathcal{M} = \{\emptyset, X, \{a\}, \{b\}, \{b, c\}\}$, and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $A = \{a, b\}$ is an mg-closed set but it is not $\mathcal{M}_*$-closed.*

Example 3.2. *Let $X = \{a, b, c, d\}$, $\mathcal{M} = \{\emptyset, X, \{a, c\}, \{d\}\}$, and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $A = \{a\}$ is an $\mathcal{M}_*$-closed set but it is not mg-closed.*

Remark 2. (1) By Lemma 2.2, since $I_* = \emptyset$, for every $I \in \mathcal{I}$, I is $\mathcal{I}_{g^*}$-closed for every $I \in \mathcal{I}$.

(2) By Lemma 2.1, since $(A_*)_* \subseteq A_*$, it follows that A_* is always $\mathcal{I}_{g^*}$-closed for every subset A of X.

Corollary 3.2. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A be an $\mathcal{I}_{g^*}$-closed set. Then the following properties are equivalent:*

(1) *A is an $\mathcal{M}_*$-closed set;*

(2) *$Cl_*(A) - A$ is an m-closed set;*

(3) *$A_* - A$ is an m-closed set.*

Proof. (1) $\Rightarrow$ (2): If A is $\mathcal{M}_*$-closed, then $Cl_*(A) = A \cup A_* = A$ and hence $Cl_*(A) - A = \emptyset$ is m-closed.
(2) $\Rightarrow$ (3): This follows from the fact that $Cl_*(A) - A = A_* - A$.
(3) $\Rightarrow$ (1): Let $A_* - A$ be m-closed. Since A is $\mathcal{I}_{g^*}$-closed, by Theorem 3.1, $A_* - A = \emptyset$ and hence $A_* \subseteq A$. Therefore $Cl_*(A) = A \cup A_* = A$ and A is $\mathcal{M}_*$-closed. □

Corollary 3.3. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A be a subset of X. Then A is $\mathcal{M}_*$-closed if and only if $A_* - A$ is m-closed and A is $\mathcal{I}_{g^*}$-closed.*

Proof. Let A be an $\mathcal{M}_*$-closed set. Then $Cl_*(A) = A_* \cup A = A$ and $A_* \subseteq A$. Since $A_* - A = \emptyset$, then $A_* - A$ is an m-closed set. By Corollary 3.1, every $\mathcal{M}_*$-closed set is $\mathcal{I}_{g^*}$-closed and hence A is $\mathcal{I}_{g^*}$-closed.
Conversely. Let $A_* - A$ be m-closed and A is $\mathcal{I}_{g^*}$-closed. Then by Corollary 3.2, A is $\mathcal{M}_*$-closed. □

Theorem 3.2. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. If A is $\mathcal{M}_*$-dense in itself and $\mathcal{I}_{g^*}$-closed in X, then A is mg-closed.*

Proof. Suppose A is an $\mathcal{M}_*$-dense in itself and $\mathcal{I}_{g^*}$-closed subset of X. If $U \in \mathcal{M}$ and $A \subseteq U$, then by Theorem 3.1, $Cl_*(A) = A_* \cup A = A_* \subseteq U$. Since A is $\mathcal{M}_*$-dense in itself, by Lemma 2.2 $mCl(A) = A_* \subseteq U$ and hence A is mg-closed. □

Theorem 3.3. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B be subsets of X. If $A \subseteq B \subseteq Cl_*(A)$ and A is $\mathcal{I}_{g^*}$-closed, then B is $\mathcal{I}_{g^*}$-closed.*

Proof. Let $B \subseteq U$ and $U \in \mathcal{M}$. Since $A \subseteq B \subseteq U$ and A is $\mathcal{I}_{g^*}$-closed, then by Theorem 3.1, $Cl_*(A) \subseteq U$ and hence $Cl_*(B) \subseteq Cl_*(Cl_*(A)) = Cl_*(A) \subseteq U$. Therefore, by Theorem 3.1, B is $\mathcal{I}_{g^*}$-closed. □

Corollary 3.4. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B be subsets of X. If $A \subseteq B \subseteq A_*$ and A is $\mathcal{I}_{g^*}$-closed, then A and B are mg-closed.*

Proof. Let $A \subseteq B \subseteq A_*$. Then by Lemmas 2.1 and 2.2, we have $A_* \subseteq B_* \subseteq (A_*)_* \subseteq A_*$ and hence $A_* = B_*$. Therefore, A and B are $\mathcal{M}_*$-dense in itself. Since $A \subseteq B \subseteq A_* \subseteq Cl_*(A)$, then by Theorem 3.3, B is $\mathcal{I}_{g^*}$-closed. Therefore, by Theorem 3.2, A and B are mg-closed. □

Corollary 3.5. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $\mathcal{I} = \emptyset$. Then A is $\mathcal{I}_{g^*}$-closed if and only if A is mg-closed.*

Proof. The proof follows from the fact that for $\mathcal{I} = \emptyset$, $A \subseteq mCl(A) = A_*$ and hence every subset of X is $\mathcal{M}_*$-dense in itself. Therefore, by Theorem 3.2 every $\mathcal{I}_{g^*}$-closed set is mg-closed. □

The following theorem gives a characterization of $\mathcal{I}_{g^*}$-open sets.

Theorem 3.4. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A be a subset of X. Then A is $\mathcal{I}_{g^*}$-open if and only if $F \subseteq Int_*(A)$ whenever F is m-closed and $F \subseteq A$.*

Proof. Suppose A is $\mathcal{I}_{g^*}$-open. If F is m-closed and $F \subseteq A$, then $X - A \subseteq X - F$ and so $Cl_*(X - A) \subseteq X - F$. Therefore, $F \subseteq Int_*(A)$. Conversely, suppose the condition holds. Let $U \in \mathcal{M}$ such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq Int_*(A)$ which implies that $Cl_*(X - A) \subseteq U$. Therefore, $X - A$ is $\mathcal{I}_{g^*}$-closed and so A is $\mathcal{I}_{g^*}$-open. □

Theorem 3.5. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B be subsets of X. If A is $\mathcal{I}_{g^*}$-open and $Int_*(A) \subseteq B \subseteq A$, then B is $\mathcal{I}_{g^*}$-open.*

Proof. This is an immediate consequence of Theorems 3.3 and 3.4. □

Theorem 3.6. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A be a subset of X. Then for the following statements, (1) implies (2) and (2) is equivalent to (3).*

(1) *A is $\mathcal{I}_{g^*}$-closed.*
(2) *$A \cup (X - A_*)$ is $\mathcal{I}_{g^*}$-closed.*
(3) *$A_* - A$ is $\mathcal{I}_{g^*}$-open.*

Proof. (1) $\Rightarrow$ (2): Suppose A is $\mathcal{I}_{g^*}$-closed. If $U \in \mathcal{M}$ and $(A \cup (X - A_*)) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A_*)) = A_* - A$. Since A is $\mathcal{I}_{g^*}$-closed, by Theorem 3.1, it follows that $X - U = \emptyset$ and hence $X = U$. Since X is the only m-open set containing $A \cup (X - A_*)$, clearly, $A \cup (X - A_*)$ is $\mathcal{I}_{g^*}$-closed.
(2) $\Leftrightarrow$ (3): This follows from the fact that $A \cup (X - A_*) = X - (A_* - A)$. □

Definition 3.5. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and A, B be subsets of X such that $B \subseteq A$. Then

(1) The family $\{U \subseteq A : U = V \cap A \text{ for some } V \in \mathcal{M}\}$ is an $\mathcal{M}$-structure on A and is denoted by $\mathcal{M}_A$.
(2) The family $\{I \subseteq A : I \in \mathcal{I}\}$ is an ideal on A and is denoted by $\mathcal{I}_A$.

(3) For the ideal m-space $(A, \mathcal{M}_A, \mathcal{I}_A)$, the local function $B_{*(A)}$ is defined as follows: $B_{*(A)} = \{x \in A : B \cap U \notin \mathcal{I}_A$ for any $U \in \mathcal{M}_A(x)\}$, where $\mathcal{M}_A(x) = \{U \in \mathcal{M}_A : x \in U\}$.

Lemma 3.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $B \subseteq A \subseteq X$. Then $B_{*(A)} = B_* \cap A$ holds.*

Proof. First we prove $B_{*(A)} \subseteq B_* \cap A$. Let $x \notin B_* \cap A$. We consider the following two cases:

Case 1. $x \notin A$. Since $B_{*(A)} \subseteq A$, then $x \notin B_{*(A)}$.

Case 2. $x \in A$. In this case $x \notin B_*$. There exists a set $V \in \mathcal{M}$ such that $x \in V$ and $V \cap B \in \mathcal{I}$. Since $x \in A$, we have a set $A \cap V \in \mathcal{M}_A$ such that $x \in A \cap V$ and $(B \cap V) \cap A \in \mathcal{I}_A$. Consequently $x \notin B_{*(A)}$.

Secondly. we prove $B_* \cap A \subseteq B_{*(A)}$. Let $x \notin B_{*(A)}$. Then, there exists $V \in \mathcal{M}$ such that $x \in V \cap A \in \mathcal{M}_A$ and $(V \cap A) \cap B \in \mathcal{I}_A$. Since $B \subseteq A$, then $V \cap B \in \mathcal{I}_A \subseteq \mathcal{I}$, thus $V \cap B \in \mathcal{I}$ for some $V \in \mathcal{M}$ containing x. This shows that $x \notin B_*$. Therefore, we obtain $x \notin B_* \cap A$. □

Theorem 3.7. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Let $B \subseteq A \subseteq X$, where A is an $\mathcal{I}_{g^*}$-closed and m-open set. Then B is $\mathcal{I}_{g^*}$-closed in $(A, \mathcal{M}_A, \mathcal{I}_A)$ if and only if B is $\mathcal{I}_{g^*}$-closed in $(X, \mathcal{M}, \mathcal{I})$.*

Proof. We first note that since $B \subseteq A$ and A is both $\mathcal{I}_{g^*}$-closed and m-open, then $A_* \subseteq A$ and thus $B_* \subseteq A_* \subseteq A$. By Lemma 3.1, $A \cap B_* = B_{*(A)}$ and we have $B_* = B_{*(A)} \subseteq A$.

Necessity. Suppose that B is $\mathcal{I}_{g^*}$-closed in A. If U is an m-open subset of X such that $B \subseteq U$, then $B = B \cap A \subseteq U \cap A$, where $U \cap A$ is m-open in A. Since B is $\mathcal{I}_{g^*}$-closed in A, $B_* = B_{*(A)} \subseteq U \cap A \subseteq U$. Therefore B is $\mathcal{I}_{g^*}$-closed in X.

Sufficiency. Suppose that B is $\mathcal{I}_{g^*}$-closed in X. Let U be an m-open subset of A such that $B \subseteq U$. Then $U = V \cap A$ for some m-open subset V of X. Since $B \subseteq V$ and B is $\mathcal{I}_{g^*}$-closed in X, $B_* \subseteq V$. Thus $B_{*(A)} = B_* \cap A \subseteq V \cap A = U$. Therefore B is $\mathcal{I}_{g^*}$-closed in A.

□

4. T_*-SPACES

Proposition 4.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. For $x \in X$, the set $X - \{x\}$ is $\mathcal{I}_{g^*}$-closed or m-open.*

Proof. Suppose $X - \{x\}$ is not m-open. Then X is the only m-open set containing $X - \{x\}$. This implies that $(X - \{x\})_* \subseteq X$. Hence $X - \{x\}$ is $\mathcal{I}_{g^*}$-closed. □

Definition 4.1. An ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is called a T_*-space if every $\mathcal{I}_{g^*}$-closed set in $(X, \mathcal{M}, \mathcal{I})$ is $\mathcal{M}_*$-closed.

Theorem 4.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then the following properties are equivalent:*

(1) *X is a T_*-space.*

(2) *Every singleton of X is either m-closed or $\mathcal{M}_*$-open.*

Proof. (1) ⇒ (2): Let $x \in X$. If $\{x\}$ is not m-closed. Then $X - \{x\}$ is not m-open and hence by Proposition 4.1 $X - \{x\}$ is $\mathcal{I}_{g^*}$-closed. Since $(X, \mathcal{M}, \mathcal{I})$ is a T_*-space, $X - \{x\}$ is $\mathcal{M}_*$-closed and thus $\{x\}$ is $\mathcal{M}_*$-open.

(2) ⇒ (1): Let A be an $\mathcal{I}_{g^*}$-closed subset of $(X, \mathcal{M}, \mathcal{I})$ and $x \in A_*$. We show that $x \in A$.

Case 1. If $\{x\}$ is m-closed and $x \notin A$, then $A \subseteq X - \{x\} \in \mathcal{M}$. Since A is $\mathcal{I}_{g^*}$-closed, $A_* \subseteq X - \{x\}$. This is contrary to $x \in A_*$. Hence $x \in A$.

Case 2. If $\{x\}$ is $\mathcal{M}_*$-open, since $x \in A_* \subseteq Cl_*(A)$, then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$. Thus in both cases we have $x \in A$. Therefore, $A_* \subseteq A$ and hence A is $\mathcal{M}_*$-closed. This shows that X is a T_*-space. □

We recall that a topological space (X, τ) is called a $T_{\frac{1}{2}}$-space [5] if every g-closed set of X is closed in X.

Proposition 4.2. *If an ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is a T_*-space, then the topological space $(X, \mathcal{M}_*)$ is a $T_{\frac{1}{2}}$-space.*

Proof. Let A be any g-closed set of $(X, \mathcal{M}_*)$. Suppose that $A \subseteq U$ and $U \in \mathcal{M}$. Then $U \in \mathcal{M}_*$ and hence $Cl_*(A) \subseteq U$. Therefore, A is $\mathcal{I}_{g^*}$-closed and by the hypothesis A is $\mathcal{M}_*$-closed. This shows that $(X, \mathcal{M}_*)$ is a $T_{\frac{1}{2}}$-space. □

Definition 4.2. [2] An ideal topological space $(X, \tau, \mathcal{I})$ is called a $T_{\mathcal{I}}$-space if every $\mathcal{I}$-g-closed set of X is τ^*-closed.

Corollary 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following implications hold:*

$$(X, \tau) \text{ is } T_{\frac{1}{2}} \longrightarrow (X, \tau, \mathcal{I}) \text{ is } T_{\mathcal{I}} \longrightarrow (X, \tau^*) \text{ is } T_{\frac{1}{2}}$$

Proof. The first implication follows from Corollary 3.4 of [2]. By putting $\tau = \mathcal{M}$ in Proposition 4.2, we obtain the second implication. □

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