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INCLUSION PROPERTIES OF AN INTEGRAL OPERATOR INVOLVING HADAMARD PRODUCT

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Abstract. In this paper, new subclasses of analytic functions associated with an integral operator are introduced. Inclusion properties of these subclasses are investigated.

1. Introduction

Let A be the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

defined on the open unit disk $U = \{z : |z| < 1\}$. Let $S, S^*(\alpha), C(\alpha), Q(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of A consisting of functions that are univalent, starlike of order α, convex of order α, and close-to-convex of order α in U, respectively.

Let f and g be analytic in U. We say that the function f is subordinate to g, written by $f \prec g$ or $f(z) \prec g(z)$, $z \in U$ if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ such that $f(z) = g(w(z))$, $z \in U$.

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For functions f given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in U$. Let $(f * g)(z)$ denote the Hadamard product (convolution) of $f(z)$ and $g(z)$, defined by :

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let N be the class of all functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\Re\{\phi(z)\} > 0$ for $z \in U$.

Making use of the principle of subordination between analytic functions, many authors investigated the subclasses $S^*(\phi)$, $C(\phi)$, and $Q(\phi, \psi)$ of the class A for ϕ, $\psi \in \mathsf{N}$ (cf.[[5],[9]]), which are defined by

$$S^*(\phi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \ \ in \ U \right\},$$

$$C(\phi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \ \ in \ U \right\},$$

$$Q(\phi, \psi) := \left\{ f \in A : \exists g \in S^*(\phi) \ s.t. \frac{zf'(z)}{g(z)} \prec \psi(z) \ \ in \ U \right\}.$$

For $\phi(z) = \psi(z) = (1+z)/(1-z)$ in the definitions defined above, we have the well known classes S^*, C, and Q, respectively. Furthermore, for the function classes $S^*[A, B]$ and $C[A, B]$ investigated by Janowski [8], it is easily seen that

$$S^*\left(\frac{1+Az}{1+Bz}\right) = S^*[A, B] \quad (-1 \le B < A \le 1),$$

$$C\left(\frac{1+Az}{1+Bz}\right) = C[A, B] \quad (-1 \le B < A \le 1).$$

For numbers $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ the function $h(a, c)(z)$ is defined by

$$h(a, c)(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in U) \tag{1.2}$$

where $(x)_n$ is the pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n=0 \\ x(x+1)....(x+n-1), & n=1,2,.... \end{cases}$$

For $f \in A$ of the form (1.1), the authors in [12] have recently introduced a new generalized differential operator $D^k_{\alpha,\beta,\lambda,\delta}$, as follows:

$$D^0 f(z) = f(z)$$

$$D^1_{\alpha,\beta,\lambda,\delta} f(z) = [1-(\lambda-\delta)(\beta-\alpha)]\ f(z) + (\lambda-\delta)(\beta-\alpha)\ zf'(z)$$

$$= z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]\, a_n z^n$$

$$\vdots$$

$$D^k_{\alpha,\beta,\lambda,\delta} f(z) = D^1_{\alpha,\beta,\lambda,\delta}\left(D^{k-1}_{\alpha,\beta,\lambda,\delta} f(z)\right)$$

(1.3)
$$D^k_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k\, a_n z^n,$$

for $\alpha \geq 0,\ \beta \geq 0,\ \lambda > 0,\ \delta \geq 0,\ \lambda > \delta,\ \beta > \alpha$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark 1. (i) When $\alpha = 0,\ \delta = 0,\ \lambda = 1,\ \beta = 1$ we get the Sălăgean differential operator (see[16]).

(ii) When $\alpha = 0$ we get Darus and Ibrahim differential operator (see[6]).

(iii) And when $\alpha = 0$, $\delta = 0$, $\lambda = 1$ we get Al- Oboudi differential operator (see [1]).

Analogous to $D^k_{\alpha,\beta,\lambda,\delta}$ $z \in U$, we define an integral operator $I^{k,\mu}_{\alpha,\beta,\lambda,\delta} : A \to A$ as follows. Let

$$F_k = \sum_{n=1}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k z^n, \tag{1.4}$$

where $\alpha \geq 0,\ \beta \geq 0,\ \lambda > 0,\ \delta \geq 0,\ \lambda > \delta,\ \beta > \alpha$, and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. And let $F_k^{(-1)}$ be defined such that

$$F_k * F_k^{(-1)} = \frac{z}{(1-z)^{\mu+1}}. \tag{1.5}$$

It is well known that for $\mu > -1$, we have

$$\frac{z}{(1-z)^{\mu+1}} = \sum_{n=0}^{\infty} \frac{(\mu+1)_n}{n!} z^{n+1} \quad (z \in U). \tag{1.6}$$

Then we obtain

$$I^{k,\mu}_{\alpha,\beta,\lambda,\delta} = F_k^{(-1)} * f(z). \tag{1.7}$$

Now the explicit form of the function $F_k^{(-1)}$ is given. Putting (1.4) and (1.6) in (1.5), we get

$$\sum_{n=1}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k z^n * F_k^{(-1)} = \sum_{n=1}^{\infty} \frac{(\mu+1)_{n-1}}{(n-1)!} z^n.$$

Therefore the function $F_k^{(-1)}$ has the following form

$$F_k^{(-1)} = \sum_{n=1}^{\infty} \frac{(\mu+1)_{n-1}}{[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k (n-1)!} z^n \qquad (z \in U).$$

Note that

$$I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{n=1}^{\infty} \frac{a_n (\mu+1)_{n-1}}{[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k (n-1)!} z^n \quad (z \in U), \tag{1.8}$$

where $\alpha \geq 0,\ \beta \geq 0,\ \lambda > 0,\ \delta \geq 0,\ \lambda > \delta,\ \beta > \alpha$, and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now we define an operator: $I^{k,\mu}_{\alpha,\beta,\lambda,\delta} : A \to A$ by

$$I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z) = h(a,c) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z), \tag{1.9}$$

such that $I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)$ is given by (1.8), where $\alpha \geq 0,\ \beta \geq 0,\ \lambda > 0,\ \delta \geq 0,$ $\lambda > \delta,\ \beta > \alpha$, and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Similar operation can also be found in [2].

Remark 2. Let $f \in A$, then

(i) $I^{0,\mu}_{\alpha,\beta,\lambda,\delta}(1,\mu+1)\,f(z) = f(z)$.

(ii) $I^{0,1}_{\alpha,\beta,\lambda,\delta}(a,a,)\,f(z) = zf'(z)$.

(iii) $z\left[I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z)\right]' =$

$$(\mu+1)\,J^{k,\mu+1}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z) - \mu\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,b,c)\,f(z).$$

(iv) $z\left[I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z)\right]' =$

$$aI^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a+1,c)\,f(z) - (a-1)\,\mu\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a+1,c)\,f(z).$$

By using the integral operator $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)$, we introduce the following classes of analytic functions for $\phi,\ \psi \in \mathsf{N}$, $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$:

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi) = \left\{ f \in A : I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z) \in S^*(\phi) \right\},$$

$$C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi) = \left\{ f \in A : I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z) \in C(\phi) \right\}, \tag{1.10}$$

$$Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi,\psi) = \left\{ f \in A : I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)\,f(z) \in Q(\phi,\psi) \right\}.$$

Note that

$$f(z) \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi) \Leftrightarrow zf'(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi). \tag{1.11}$$

In particular, we set

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,(1+Az)/(1+Bz)) = S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,[A,\ B]), \tag{1.12}$$

$$C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,(1+Az)/(1+Bz)) = C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,[A,\ B]), \tag{1.13}$$

where $-1 \leq B < A \leq 1$.

In this paper, we investigate some inclusion properties of classes $S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$, $C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$ and $Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi,\psi)$ associated with the integral operator $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)$. The work here reflects results given by Cho [4]. In fact the techniques are similar. In addition the integral preserving properties in connection with this operator are also considered. Furthermore, relevant connection of the results presented here with those obtained in earlier works are pointed out.

For this paper, we need the following results in the sequel.

Lemma 1.1. *(*[14, pages 60-61]*) Let $c \geq a > 0$. If $c \geq 2$ or $a+c \geq 3$, then the function*

$$h(a,c)(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in U), \tag{1.14}$$

belongs to the class C of convex functions.

Lemma 1.2. [15] *Let $f \in C$ and $g \in S^*$, then for each analytic function H in U,*

$$\frac{(f*Hg)}{(f*g)}(U) \subset \overline{co}H(U), \tag{1.15}$$

where $\overline{co}H(U)$ denotes the closed convex hull of $H(U)$.

Lemma 1.3. [7] *Let ϕ be analytic, univalent , convex in U, with $\phi(0) = 1$ and*

$$\Re\left(\eta\phi(z) + \mu\right) > 0 \quad (\eta, \mu \in \mathbb{C};\ z \in U).$$

If $p(z)$ is analytic in U, with $p(0) = \phi(0)$, then

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

2. Inclusion properties involving the operator $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)$

The following will be our main results.

Theorem 2.1. *Let $a_2 \geq a_1$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, and $\phi \in N$. If $a_2 \geq 2$ or $a_1 + a_2 \geq 3$, then*

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c, \phi) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c, \phi). \tag{2.1}$$

Proof. Let $f \in S^{k}_{\alpha,\beta,\lambda,\delta}(a_2, c, \phi)$. Then there exists an analytic function w in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$ such that

$$\frac{z\left(D^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c) f(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c) f(z)} = \phi(w(z)) \quad (z \in U). \tag{2.2}$$

Since we can write $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c) f(z) = h(a,c)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)$,

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c) f(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c) f(z)} = \frac{z\left(h(a_1, c)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)\right)'}{h(a_1, c)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)}$$

$$= \frac{z\left(h(a_2, c)(z) * h(a_1, a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)\right)'}{h(a_2, c)(z) * h(a_1, a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta} f(z)}$$

$$= \frac{h(a_{1,2})(z) * z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c) f(z)\right)'}{h(a_1, a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c) f(z)}$$

$$= \frac{h(a_1,a_2)(z) * \phi(w(z))\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c)\, f(z)}{h(a_1,a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c)\, f(z)}. \tag{2.3}$$

It follows from (2.2) and Lemma 1.1 that $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c)\, f(z) \in S^*$ and $h(a_1,a_2)(z) \in C$, respectively. Then by applying Lemma 1.2 to (2.3), we obtain

$$\frac{\left\{h(a_1,a_2)(z) * \phi(w(z))\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c)\, f\right\}}{\left\{h(a_1,a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c)\, f\right\}}(U) \subset \overline{co}\,(\phi(U)) \subset \phi(U). \tag{2.4}$$

Since ϕ is convex univalent, therefore, from the definition of subordination and (2.4), we have

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1,c)\, f(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1,c)\, f(z)} \prec \phi(z) \quad (z \in U), \tag{2.5}$$

or equivalently, $f \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1,c,\phi)$, which completes the proof of Theorem 2.1.

Theorem 2.2. *Let* $a \in \mathbb{R}$, $c_2 \geq c_1$, $k \in \mathbb{N}_0$ *and* $\phi \in N$. *If* $c_2 \geq 2$ *or* $c_1 + c_2 \geq 3$, *then*

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1,\phi) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_2,\phi) \tag{2.6}$$

Proof: Let $f \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1,\phi)$. Using a similar argument as in the proof of Theorem 2.1, we obtain

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_2)\, f(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_2)\, f(z)} = \frac{h(a_1,a_2)(z) * \phi(w(z))\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1)\, f(z)}{h(a_1,a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1)\, f(z)}, \tag{2.7}$$

where w is an analytic function in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$. Applying Lemma 1.1 and the fact that $I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1)\, f(z) \in S^*$, we see that

$$\frac{\left\{h(a_1,a_2)(z) * \phi(w)\, I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1)\, f\right\}}{\left\{h(a_1,a_2)(z) * I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c_1)\, f\right\}}(U) \subset \overline{co}\,(\phi(U)) \subset \phi(U), \tag{2.8}$$

since ϕ is convex univalent. Thus the proof of Theorem 2.2 is complete.

Corollary 2.1. *Let* $a_2 \geq a_1 > 0,\ c_2 \geq c_1 > 0,\ k \in \mathbb{N}_0$ *and* $\phi \in \mathsf{N}$. *If* $a_2 \geq \min\{2, 3 - a_1\}$ *and* $c_2 \geq \min\{2, 3 - c_1\}$, *then*

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, \phi) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, \phi). \tag{2.9}$$

Theorem 2.3. *Let* $a_2 \geq a_1 > 0,\ c_2 \geq c_1 > 0,\ k \in \mathbb{N}_0$ *and* $\phi \in \mathsf{N}$. *If* $a_2 \geq \min\{2, 3 - a_1\}$ *and* $c_2 \geq \min\{2, 3 - c_1\}$, *then*

$$C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, \phi) \subset C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi) \subset C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, \phi). \tag{2.10}$$

Proof. Applying (1.11) and Corollary 2.1, we observe that

$$\begin{aligned}
f(z) \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, \phi) &\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1) f(z) \in C(\phi) \\
&\Leftrightarrow z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1) f(z)\right)' \in S^*(\phi) \\
&\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1)\, z(f(z))' \in S^*(\phi) \\
&\Leftrightarrow zf'(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, \phi) \\
&\Rightarrow zf'(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi) \\
&\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2)\, z(f(z))' \in S^*(\phi) \\
&\Leftrightarrow z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2) f(z)\right)' \in S^*(\phi) \\
&\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2) f(z) \in C(\phi) \\
&\Leftrightarrow f(z) \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi),
\end{aligned}$$

$$\begin{aligned}
f(z) \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi) &\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2) f(z) \in C(\phi) \\
&\Leftrightarrow I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2)\, z(f(z))' \in S^*(\phi) \\
&\Leftrightarrow zf'(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, \phi) \\
&\Leftrightarrow z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2) f(z)\right)' \in S^*(\phi) \\
&\Leftrightarrow f(z) \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, \phi),
\end{aligned}$$

which evidently proves Theorem 2.3.

Taking $\phi(z) = (1 + Az)/(1 + Bz) \quad (-1 \le B < A \le 1; z \in U)$ in Corollary 2.1 and Theorem 2.3, we have the following:

Corollary 2.2. *Let $a_2 \ge a_1 > 0$, $c_2 \ge c_1 > 0$, $k \in \mathbb{N}_0$, $(-1 \le B < A \le 1)$ and $\phi \in \mathsf{N}$. If $a_2 \ge \min\{2, 3 - a_1\}$ and $c_2 \ge \min\{2, 3 - c_1\}$, then*

$$S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, [A, B]) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, [A, B]) \subset S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, [A, B]),$$

(2.11)

$$C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, [A, B]) \subset C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, [A, B]) \subset C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1, c_2, [A, B]).$$

Theorem 2.4. *Let $k \in \mathbb{N}_0$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, and $\phi \in \mathsf{N}$. If $f \in S^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c, \phi)$, then $F_\varepsilon(f)(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c, \phi)$.*

Notice that F_ε is the generalized Libera integral [10] *given by*

$$F_\varepsilon(f) = \frac{\varepsilon + 1}{z^\varepsilon} \int_0^z t^{\varepsilon - 1} f(t)\, dt. \tag{2.12}$$

Proof. Let $f \in S^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c, \phi)$, and set

$$p(z) := \frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) F_\varepsilon(f)(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) F_\varepsilon(f)(z)}, \tag{2.13}$$

where $p(z)$ is analytic in U with $p(0) = 1$. From (2.12), we obtain

$$z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) F_\varepsilon(f)(z)\right)'$$

$$= (\varepsilon + 1) I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) f(z) - \varepsilon I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) F_\varepsilon(f)(z). \tag{2.14}$$

Then by using (2.13) and (2.14), we have

$$p(z) + \varepsilon = (\varepsilon + 1) \frac{I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) f(z)}{I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a, c) F_\varepsilon(f)(z)}. \tag{2.15}$$

Taking the logarithm differentiation on both sides of (2.15) and multiplying by z, we have

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a,c)f(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a,c)f(z)} = p(z) + \frac{zp'(z)}{p(z)+\varepsilon}. \tag{2.16}$$

Applying Lemma 1.3 to (2.16), it follows that $p \prec \phi$, that is,
$F_\varepsilon(f)(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\gamma}(a,c,\phi)$.

To prove the theorems below, we need the following lemma.

Lemma 2.1. *Let $\phi \in N$. If $f \in C$ and $q \in S^*(\phi)$, then $f * q \in S^*(\phi)$.*

Proof. Let $q \in S^*(\phi)$. Then

$$zq'(z) = q(z)\,\phi(w(z)), \tag{2.17}$$

where w is an analytic function in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$. Thus we have

$$\frac{z(f(z)*q(z))'}{f(z)*q(z)} = \frac{f(z)*zq'(z)}{f(z)*q(z)} = \frac{f(z)*\phi(w(z))\,q(z)}{f(z)*q(z)} \quad (z \in U). \tag{2.18}$$

By using similar arguments to those used in the proof of Theorem 2.1, we conclude that (2.18) is subordinated to ϕ in U and so $f * q \in S^*(\phi)$.

Theorem 2.5. *Let $a_2 \geq a_1 > 0$, $c_2 \geq c_1 > 0$, $k \in \mathbb{N}_0$ and $\phi \in N$. If*
$a_2 \geq \min\{2, 3-a_1\}$ and $c_2 \geq \min\{2, 3-c_1\}$, then

$$Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_1,\phi,\psi) \subset Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_2,\phi,\psi) \subset Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_1,c_2,\phi,\psi). \tag{2.19}$$

Proof. First of all, we show that

$$Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_1,\phi,\psi) \subset Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_2,\phi,\psi). \tag{2.20}$$

Let $f \in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_1, \phi, \psi)$ .Then there exist a function $q_2 \in S^*(\phi)$, such that

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_1)f(z)\right)'}{q_2(z)} \prec \psi(z). \tag{2.21}$$

From (2.21), we obtain

$$z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_1)f(z)\right)' = \psi(w(z))\,q_2(z), \tag{2.22}$$

where w is an analytic function in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$. By virtue of Lemmas 1.1 and 2.1, we see that $h(a_1, a_2)(z) = q_2(z) \equiv q_1(z)$ belongs to $S^*(\phi)$. Then we have

$$\begin{aligned}\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_2)f(z)\right)'}{q_1(z)} &= \frac{h(c_1,c_2)(z) * z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2,c_1)f(z)\right)'}{h(c_1,c_2)(z) * q_2(z)}\\ &= \frac{h(c_1,c_2)(z) * \psi(w(z))\,q_2(z)}{h(c_1,c_2)(z) * q_2(z)}\\ &\prec \psi(z) \quad (z \in U),\end{aligned}$$

which implies that $f \in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a_2, c_2, \phi, \psi)$.

Moreover, the proof of the second part is similar to that of the first part and so we omit the details involved.

3. Inclusion properties involving various operators

In the next theorem we will show that the classes $S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi)$, $C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi)$, and $Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi, \psi)$ are invariant under convolution with convex functions.

Theorem 3.1. *Let* $a > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $k \in \mathbb{N}_0$ *and* $\phi, \psi \in N$. *and let* $g \in C$. *Then*

(i)$f \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi) \Rightarrow g * f \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi)$,

(ii)$f \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi) \Rightarrow g * f \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi)$,

(iii)$f \in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi, \psi) \Rightarrow g * f \in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a, c, \phi, \psi)$.

Proof. Let $f \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$. Then we have

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)(g*f)(z)\right)'}{I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)(g*f)(z)} = \frac{g(z)*z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)f(z)\right)'}{g(z)*I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)f(z)}. \tag{3.1}$$

By using the same techniques as in the proof of Theorem 2.1, we obtain (i).

(ii) Let $f \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$. Then by (1.11) and from Theorem 3.1(i), we have

$$\begin{aligned} f \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi) \Leftrightarrow z f'(z) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi), \\ &\Rightarrow g*(zf'(z)) \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi), \\ &\Leftrightarrow z(g*f)' \in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi), \\ &\Leftrightarrow g*f \in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi). \end{aligned}$$

(iii) Let $f \in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi,\psi)$. Then there exist a function $q \in S^*(\phi)$, such that

$$z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)f(z)\right)' = \psi(w(z))\,q(z) \quad (z \in U), \tag{3.2}$$

where w is an analytic function in U with $|w(z)| < 1 \ (z \in U)$ and $w(0) = 0$. From Lemma 2.1, we have that $g*q \in S^*(\phi)$. Since

$$\frac{z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)(g*f)(z)\right)'}{(g*q)(z)} = \frac{g(z)*z\left(I^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c)f(z)\right)'}{g(z)*q(z)}$$

$$= \frac{g(z)*\psi(w(z))\,q(z)}{g(z)*q(z)} \prec \psi(z), \tag{3.3}$$

we obtain (iii).

Now we consider the following operators [[13], [10]] given by

$$\Psi_1(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \quad (c \in \mathbb{C},\ \Re\{c\} > 0;\ z \in U),$$

(3.4)

$$\Psi_2(z)=\frac{1}{1-x}\log\left[\frac{1-xz}{1-z}\right]\quad (\log 1=0;\ |x|<1,\ z\in U).$$

It is well known [3] that for f in A the operators $\Gamma_i : A \to A$ defined by:

(i) $\Gamma_1(f)=\Psi_1 * f$ is the Bernard's operator [10],

(ii) $\Gamma_2(f)=\Psi_2 * f$ is the operator was first used by Pommerenke [11]. And Ψ_1, Ψ_2, are convex in U. Therefore, we have the following result, which can be obtained from Theorem 3.1 immediately.

Corollary 3.1. *Let $a>0$, $c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $k\in\mathbb{N}_0$ and $\phi,\psi\in N$. and let $\Psi_i\,(i=1,2)$ be defined by (3.4). Then*

(i) $f\in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)\Rightarrow \Psi_i * f\in S^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$,

(ii) $f\in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)\Rightarrow \Psi_i * f\in C^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi)$,

(iii) $f\in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi,\psi)\Rightarrow \Psi_i * f\in Q^{k,\mu}_{\alpha,\beta,\lambda,\delta}(a,c,\phi,\psi)$.

4. Conclusion

By using the principle of subordination between analytic functions, new subclasses of analytic functions defined by an integral operator are introduced. Some inclusion properties for theses classes are investigated. Many other results are yet to be solved using the operator particularly in the case of harmonic functions and logharmonic. It may not be easy, but it is worth of trying.

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References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator. *Int. J. Math. Math. Sci.* **27**, (2004), 1429–1436.

[2] K. Al-Shaqsi and M. Darus, An operator defined by convolution involving polylogaritms functions. *J. Math. Stat.*, **44**(1) (2008), 46–50.

[3] R. W. Barnard and Ch. Kellogg, Applications of convolution operators to problems in univalent funcion theory. *Michigan Mathematical J.* **27**(1), (1980), 81–94.

[4] N. E. Cho, Inclusion properties for certain subclasses of analytic functions defined by a linear operator. *Abstract and Applied Analysis* Volume (2008), Artical ID 246876, 8 pages.

[5] J. H. Choi, M. Saigo, And H. M. Srivastava, Some inclusion properties of a certain family of integral operators. *J. Math. Appli.*, **276**(1) (2002), 432–445.

[6] M. Darus and R. W. Ibrahim, On subclasses for generalized operators of complex order. *Far East J. Math. Sci:,(FJMS)* **33**(3) (2009), 299–308.

[7] P. Eenigenburg, S. Miller, P. Mocanu and M. Reade, On a Briot-Bouquet differential subordination. *General Inequl.* **3**,(Oberwolfach, 1981), 339–348, *Internat. Schriftenreihe Numer. Math.,* 64, Birkhauser, Basel, (1983).

[8] W. Janowski, Some extremal problems for certain families of analytic functions. I, *Academie Polonaise des Sciences. Serie des Sciences Mathematiques, Astronomiques et Physiques*, **21** (1973), 17-25.

[9] W. Ma and D. Minda, An internal geometric characterization of strongly starlike functions, *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), 125–128.

[10] S. Owa and H. M. Srivastava, Some applications of the generalized Libera integral operator, *Academie Polonaise des Sciences. Serie des Sciences Mathematiques, Astronomiques et Physiques*, **21** (1973), 17–25.

[11] Ch. Pommerenke, On close-to-convex analytic functions, *Trans. Amer. Math. Soc.* **114** (1965), 176–186.

[12] S. F. Ramadan and M. Darus, On the Fekete- Szegö inequality for a class of analytic functions defined by using generalized differential operator, *Acta Uni. Apul.* **26** (2011), 167–178.

[13] S. Ruscheweyh, New criteria for univalent functions, *Pro. Amer. Math. Soc.* **49** (1975), 109–115.

[14] S. Ruscheweyh, Convolutions in geometric function theory, *Presses de I Univ. de Montreal*, **83** (1982).

[15] S. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Polya-Schoenberg conjecture, *Comment. Math. Helv.* **48**(1) (1973), 119–135.

[16] G. S. Sălăgean, Subclasses of univalent functions, *Lecture Notes in Math. 1013, Springer, Verlag Berlin.* (1983), 362–372.

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