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BOUNDEDNESS OF CONVOLUTION OPERATORS ON TRIEBEL-LIZORKIN SPACES VIA FOURIER TRANSFORM ESTIMATES

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Abstract. In this paper, we study the boundedness of some convolution operators defined by $Tf = \sum_{k\in\mathbb{Z}} \sigma_k * f$ on the homogeneous Triebel-Lizorkin spaces by Fourier transform estimates. As applications, we improve some known results, by proving the boundedness for singular integral operators with rough kernels on homogeneous Triebel-Lizorkin spaces.

1. Introduction

Let

$$T_{\Omega,h}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} h(|y|) f(x-y)dy \ , \tag{1.1}$$

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where h is a measurable function on $\mathbb{R}_+$ and Ω is a homogeneous function of degree zero such that $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0 \ . \tag{1.2}$$

If $h = 1$, then we denote $T_{\Omega,h}$ by T_Ω. This operator was first studied by Calderon and Zygmund in [2] and [3]. In [8], Fefferman generalized this Calderon-Zygmund singular integral by replacing the kernel $\Omega(x)|x|^{-n}$ by $h(|x|)\Omega(x)|x|^{-n}$, where h is a function in L^∞. This allows the kernel to be rough not only on the sphere, but also in the radial direction. Using a method which is different from Calderon and Zygmund's, Fefferman showed in [8] that if Ω satisfies a Lipschitz condition then $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Afterwards, in [6], using Littlewood-Paley theory and Fourier transform method, Duoandikoetxea, Rubio de Francia extended the result of Fefferman[8], and obtained the L^p boundedness of $T_{\Omega,h}$ for $1 < p < \infty$ when h satisfies

$$\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^2 dt < \infty$$

and $\Omega \in L^q(S^{n-1})$ for some $q > 1$.

On the other hand, the Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ which is denoted by the following, is a unified setting of many well-known function spaces including Lebesgue spaces $L^p(\mathbb{R}^n)$, Hardy spaces $H^p(\mathbb{R}^n)$ and Sobolev spaces $L_p^\alpha(\mathbb{R}^n)$. It is natural interest to extend the above mentioned results to the more general Triebel-Lizorkin spaces. Recently, Chen, Jia and Jiang[5] obtained that a convolution operator T is bounded on $\dot{F}_p^{s,q}(\mathbb{R}^n)$ if T is bounded on $L^q(\mathbb{R}^n, \omega(x)dx)$ for all $\omega \in A_1$. However, many convolution operators are not weighted bounded on $L^q(\mathbb{R}^n)$. As a result, we give a method which is different from the one given in [5], to prove $\dot{F}_p^{s,q}$-boundedness for convolution operator. This paper is motivated by the method in [6]. Indeed, we study the mapping properties on the homogeneous Triebel-Lizorkin spaces for general convolution operator T by using the Fourier transform estimates. The operator T can be decomposed by $Tf = \sum_{k\in\mathbb{Z}} \sigma_k * f$, where $\{\sigma_k\}_{k\in\mathbb{Z}}$ is a sequence of measures on $\mathbb{R}^n$. As applications, in section 3, we obtain the boundedness for singular integral operators with rough kernels on homogeneous Triebel-Lizorkin spaces . Our results extend some known results([6],[7], [8],[10]) on the singular integral operator.

Let us recall the definition of the Triebel-Lizorkin spaces.

Fix a radial function $\phi \in C^\infty(\mathbb{R}^n)$ satisfying that supp $\phi \subset \{x : 1/2 \le |x| \le 2\}$, $0 \le \phi(x) \le 1$, and $\phi(x) > c > 0$ if $3/5 \le |x| \le 5/3$. Let $\phi_j(x) = \phi(2^j x)$. Define $\widehat{S_j f}(\xi) = \phi_j(\xi)\widehat{f}(\xi)$. For $1 < p, q < \infty$ and $s \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ is the set of all distributions f satisfying

$$\|f\|_{\dot{F}_p^{s,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{-jsq} |S_j f|^q \right)^{1/q} \right\|_{L^p} < \infty \ . \tag{1.3}$$

It is also known space (see[12]) that the choice of ϕ in the definition of $\dot{F}_p^{s,q}$ is quite flexible. For instance, for the above ϕ and any fixed number $\rho \in [1/2,\ 1]$, let $\phi_{j,\rho} = \phi(2^j \rho x)$ and $\widehat{S_{j,\rho} f}(\xi) = \phi_{j,\rho}(\xi)\widehat{f}(\xi)$. Then using $S_{j,\rho}$ instead of S_j in (1.3), we obtain a Triebel-Lizorkin norm equivalent to the norm in (1.3). Also the ratio of these two norms is between two positive constants C_1 and C_2 that are independent of $\rho \in [1/2,\ 1]$. Furthermore, Let us recall the following Lemma.

Lemma 1.1 ([4])**.** *For any $\rho > 0$, we have*

$$C_1 \|f\|_{\dot{F}_p^{s,q}} \le \left\| \left(\sum_{j\in\mathbb{Z}} (2^j \rho)^{-sq} |S_{j,\rho} f|^q \right)^{1/q} \right\|_{L^p} \le C_2 \|f\|_{\dot{F}_p^{s,q}} \tag{1.4}$$

where $C_2 \ge C_1$ are independent of ρ.

2. Main Results

Theorem 2.1. *Let α, ρ be positive and $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a sequence of measures on $\mathbb{R}^n$ such that*

(i) $\|\sigma_k\| \le 1$ *for all* $k \in \mathbb{Z}$;

(ii) $\|\sigma^*(f)\|_{L^r} = \| \sup_k | \, |\sigma_k| * f| \, \|_{L^r} \le C\|f\|_{L^r}$, *for some* $1 < r < \infty$;

(iii) $|\widehat{\sigma}_k(\xi)| \le C min\{2^{k+1}\rho|\xi| \ , \ (2^k\rho|\xi|)^{-\alpha}\}$, *for all* $\xi \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$.

Suppose that $s \in \mathbb{R}$ *and* $1 < p, q < \infty$ *satisfy*

(a) $q \le p, \quad 1/q - 1/2 < r(1/q - 1/p) < 1/q$; or

(b) $p \le q, \quad 1/2 - 1/p < r(1/p - 1/q) < 1 - 1/q$; or

(c) $p = q = 2$.

Then, the operator

$$Tf(x) = \sum_{k=-\infty}^{\infty} \sigma_k * f(x) \tag{2.1}$$

is bounded on $\dot{F}_p^{s,q}(\mathbb{R}^n)$, *and* $\|Tf\|_{\dot{F}_p^{s,q}} \le C\|f\|_{\dot{F}_p^{s,q}}$, *where* C *is independent of* ρ.

Proof. Let ϕ be a C^∞ function and $\phi_{j,\rho}(x) = \phi(2^j\rho x)$ for $j \in \mathbb{Z}$, and assume that

$$\sum_{j=-\infty}^{\infty} [\phi_{j,\rho}(x)]^2 = 1 \qquad \forall x \ne 0.$$

Then for all $f \in \mathcal{S}(\mathbb{R}^n)$, define the multiplier operators $S_{j,\rho}$ by

$$\widehat{S_{j,\rho}f}(\xi) = \phi_{j,\rho}(\xi)\widehat{f}(\xi) \ , \qquad j \in \mathbb{Z} \ .$$

Decompose T as follows

$$\begin{aligned} Tf &= \sum_{k\in\mathbb{Z}} \sigma_k * \left(\sum_{j\in\mathbb{Z}} S_{j+k,\rho} S_{j+k,\rho} f \right) \\ &= \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} S_{j+k,\rho}(\sigma_k * S_{j+k,\rho} f) \right) := \sum_{j\in\mathbb{Z}} T_j f \ . \end{aligned} \tag{2.2}$$

Let $\Gamma_j = \{\xi \in \mathbb{R}^n : 2^{-j-1}\rho^{-1} \le |\xi| \le 2^{-j+1}\rho^{-1}\}$. By Plancherel's theorem,

$$\|T_j f\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) d\xi \ .$$

For $j \ge 2$ and $k \in \mathbb{Z}$,

$$|\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) \le C\left(2^{k+1}\rho|\xi|\right)^2 \chi_{\Gamma_{j+k}}(\xi) \le C\left(2^{k+1}/2^{j+k-1}\right)^2 \le C2^{-2(j-2)} \ .$$

Then, we have

$$\|T_j f\|_{L^2} \le C2^{-(j-2)} \left(\sum_{k\in\mathbb{Z}} \int_{\Gamma_{j+k}} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \le C2^{-(j-2)}\|f\|_{L^2} \ .$$

For $j \le -1$ and $k \in \mathbb{Z}$,

$$|\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) \le C\left(2^{k}\rho|\xi|\right)^{-2\alpha} \chi_{\Gamma_{j+k}}(\xi) \le C\left(2^{j+k+1}/2^{k}\right)^{2\alpha} \le C2^{2\alpha(j+1)} \ .$$

Then, $\|T_j f\|_{L^2} \le C2^{\alpha(j+1)}\|f\|_{L^2}$. It follows from $|\widehat{\sigma}_k(\xi)| \le \|\sigma_k\| \le 1$ that $\|T_j f\|_{L^2} \le C\|f\|_{L^2}$ for $j = 0, 1$. Consequently, we have $\|T_j f\|_{L^2} \le C2^{-\alpha|j|}\|f\|_{L^2}$ for all $j \in \mathbb{Z}$. Therefore, there is a constant C independent of ρ such that

$$\|T_j f\|_{\dot{F}_2^{0,2}} \le C2^{-\alpha|j|}\|f\|_{\dot{F}_2^{0,2}} \ . \tag{2.3}$$

Setting $1/r + 1/r' = 1$. If $1 < q \le p < \infty$ and $1/q - 1/2 < r(1/q - 1/p) < 1/q$, then $0 < r'/p - 1/q < (r'-1)/2$. We choose $1 < q_0 \le p_0 < \infty$ such that

$$0 < \theta = \frac{2(\frac{r'}{p} - \frac{1}{q})}{r'-1} < 1 \ , \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0} \ , \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_0} \ .$$

Then

$$\frac{p_0}{q_0} = \frac{1/q - \theta/2}{1/p - \theta/2} = r' > 1 \ .$$

For $s \in \mathbb{R}$, taking $s = (1-\theta)s_0$. For any $g \in \dot{F}_{p_0'}^{-s_0,\ q_0'}(\mathbb{R}^n)$, it follows from (1.4) that

$$\begin{aligned} |\langle T_j f,\ g\rangle| &= \left|\sum_{k\in\mathbb{Z}}\langle \sigma_k * S_{j+k,\rho}f,\ S_{j+k,\rho}^* g\rangle\right| \\ &\leq \|g\|_{\dot{F}_{p_0'}^{-s_0,\ q_0'}} \left\|\left(\sum_{k\in\mathbb{Z}}(2^{k+j}\rho)^{-s_0q_0}\left|\sigma_k * S_{j+k,\rho}f\right|^{q_0}\right)^{1/q_0}\right\|_{L^{p_0}} . \end{aligned}$$

Therefore,

$$\|T_j f\|_{\dot{F}_{p_0}^{s_0,q_0}} \leq \left\|\left(\sum_{k\in\mathbb{Z}}(2^{k+j}\rho)^{-s_0q_0}\left|\sigma_k * S_{j+k,\rho}f\right|^{q_0}\right)^{1/q_0}\right\|_{L^{p_0}} .$$

As $\|\sigma_k\| \leq 1$ we get at the end of the line, for all $1 < q < \infty$

$$\begin{aligned} |\sigma_k * S_{j+k,\rho}f(x)| &\leq \int_{\mathbb{R}^n}|S_{j+k,\rho}f(x-y)|\ d|\sigma_k(y)| \\ &\leq \left(\int_{\mathbb{R}^n}|S_{j+k,\rho}f(x-y)|^q d|\sigma_k(y)|\right)^{1/q}\|\sigma_k\|^{1/q'} \\ &\leq \left[|\sigma_k| * |S_{j+k,\rho}f|^q(x)\right]^{1/q} . \end{aligned}$$

Hence, we choose $u(x) \in L^{(\frac{p_0}{q_0})'}(\mathbb{R}^n) = L^r(\mathbb{R}^n)$ with $\|u\|_{L^r} = 1$ such that

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} \left|\sigma_k * S_{j+k,\rho} f\right|^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}}^{q_0} \\
= \;& \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} \left|\sigma_k * S_{j+k,\rho} f(x)\right|^{q_0} u(x) dx \\
\leq \;& \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} |\sigma_k| * |S_{j+k,\rho} f|^{q_0}(x) u(x) dx \\
\leq \;& \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} |S_{j+k,\rho} f(x)|^{q_0} \sup_{i\in\mathbb{Z}} \left||\sigma_i| * u(x)\right| dx \\
\leq \;& \|\sigma^*(u)\|_{L^{(\frac{p_0}{q_0})'}} \left\{ \int_{\mathbb{R}^n} \left[\sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} |S_{j+k,\rho} f(x)|^{q_0} \right]^{p_0/q_0} dx \right\}^{q_0/p_0} \\
= \;& \|\sigma^*(u)\|_{L^r} \left\| \left(\sum_{k\in\mathbb{Z}} (2^{k+j}\rho)^{-s_0 q_0} |S_{j+k,\rho} f|^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}}^{q_0} \leq C\|u\|_{L^r} \|f\|_{\dot{F}_{p_0}^{s_0,q_0}}^{q_0} .
\end{aligned}$$

Therefore,

$$\|T_j f\|_{\dot{F}_{p_0}^{s_0,q_0}} \leq C\|f\|_{\dot{F}_{p_0}^{s_0,q_0}} , \tag{2.4}$$

where C is a constant independent of j and ρ. By interpolation between (2.3) and (2.4)(see [9]), noting (2.2), we get

$$\|Tf\|_{\dot{F}_p^{s,q}} = \sum_{j\in\mathbb{Z}} \|T_j f\|_{\dot{F}_p^{s,q}} \leq C \sum_{j\in\mathbb{Z}} 2^{-\theta\alpha|j|} \|f\|_{\dot{F}_p^{s,q}} = C\|f\|_{\dot{F}_p^{s,q}} ,$$

where $1 < q \leq p < \infty$, $\;1/q - 1/2 < r(1/q - 1/p) < 1/q$.

Setting $1/p'+1/p=1$ for each real number $p>1$. If $1<p\le q<\infty$, $1/2-1/p<r(1/p-1/q)<1-1/q$, then $1<q'\le p'<\infty$ and $0<2(r'/p'-1/q')<r'-1$. By duality, we can obtain $\|Tf\|_{\dot F_p^{s,q}}\le C\|f\|_{\dot F_p^{s,q}}$ for all $s\in\mathbb{R}$.

If $p=q=2$ and $s\in\mathbb{R}$, taking $p_0=q_0=4, s_0=s/2$, then p_0,q_0 satisfy the condition (a). Similarly, taking $p_1=q_1=4/3, s_1=s/2$, then p_1,q_1 satisfy the condition (b). Thus

$$\|Tf\|_{\dot F_{p_0}^{s_0,q_0}}\le C\|f\|_{\dot F_{p_0}^{s_0,q_0}}\ ,\quad \|Tf\|_{\dot F_{p_1}^{s_1,q_1}}\le C\|f\|_{\dot F_{p_1}^{s_1,q_1}}\ .$$

By interpolation between the two inequalities above, we have that T is a bounded operator on $\dot F_p^{s,q}(\mathbb{R}^n)$. This completes the proof.

Remark 2.1. When $q=2$ and $s=0$, T is bounded on $L^p(\mathbb{R}^n)$ where $|1/p-1/2|<1/(2r)$. This is Theorem B in [6] when $a_k=2^k\rho$.

Remark 2.2. If (ii) holds for all $1<r<\infty$, then T is bounded on $\dot F_p^{s,q}(\mathbb{R}^n)$ for all $1<p,q<\infty$, $s\in\mathbb{R}$.

Theorem 2.2. *Let $\alpha,\rho>0$ and $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a sequence of measures on $\mathbb{R}^n$ satisfying (i) and (ii) of Theorem 2.1 and*

(iii) $|\widehat{\sigma}_k(\xi)|\le C min\{2^{k+1}\rho|\xi|,\ [log^+(2^k\rho|\xi|)]^{-(1+\alpha)}\}$ for all $\xi\in\mathbb{R}^n$ and $k\in\mathbb{Z}$.

Suppose $s\in\mathbb{R}$ and $1<p,q<\infty$ satisfy

(a) $q\le p$, $1/q-1/2<r(1/q-1/p)<1/q-1/(2+2\alpha)$; or

(b) $p\le q$, $1/2-1/q<r(1/p-1/q)<(1+2\alpha)/(2+2\alpha)-1/q$; or

(c) $p=q=2$.

Then operator T, defined by (2.1) is bounded on $\dot F_p^{s,q}(\mathbb{R}^n)$, and $\|Tf\|_{\dot F_p^{s,q}}\le C\|f\|_{\dot F_p^{s,q}}$, where C is independent of ρ.

Proof. As in the proof of Theorem 2.1, decompose T as follows

$$Tf = \sum_{k\in\mathbb{Z}} \sigma_k * \left(\sum_{j\in\mathbb{Z}} S_{j+k,\rho} S_{j+k,\rho} f\right) = \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} S_{j+k,\rho}(\sigma_k * S_{j+k,\rho} f)\right) := \sum_{j\in\mathbb{Z}} T_j f \ .$$

Let $\Gamma_j = \{\xi \in \mathbb{R}^n : 2^{-j-1}\rho^{-1} \le |\xi| \le 2^{-j+1}\rho^{-1}\}$. By Plancherel's theorem,

$$\|T_j f\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) d\xi \ .$$

For $j \ge 2$ and $k \in \mathbb{Z}$, we get

$$|\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) \le C \left(2^{k+1}\rho|\xi|\right)^2 \chi_{\Gamma_{j+k}}(\xi) \le C \left(2^{k+1}/2^{j+k-1}\right)^2 \le C2^{-2(j-2)} \ .$$

Then, $\|T_j f\|_{L^2} \le C2^{-(j-2)}\|f\|_{L^2}$. For $j \le -1$ and $k \in \mathbb{Z}$, we get

$$\begin{aligned} |\widehat{\sigma}_k(\xi)|^2 \chi_{\Gamma_{j+k}}(\xi) &\le C \left[\log^+(2^k\rho|\xi|)\right]^{-2(1+\alpha)} \chi_{\Gamma_{j+k}}(\xi) \\ &\le C \left[\log^+(2^{-j-1})\right]^{-2(1+\alpha)} \le C|j|^{-2(1+\alpha)} \ . \end{aligned}$$

Then, $\|T_j f\|_{L^2} \le C|j|^{-(1+\alpha)}\|f\|_{L^2}$. As the inequality $|\widehat{\sigma}_k(\xi)| \le \|\sigma_k\| \le 1$, for $j = 0, 1$, we have $\|T_j f\|_{L^2} \le C\|f\|_{L^2}$. Consequently, $\|T_j f\|_{L^2} \le C(1+|j|)^{-(1+\alpha)}\|f\|_{L^2}$ for all $j \in \mathbb{Z}$. Therefore, there is a constant C independent of ρ such that

$$\|T_j f\|_{\dot{F}_2^{0,2}} \le C(1+|j|)^{-(1+\alpha)}\|f\|_{\dot{F}_2^{0,2}} \ . \tag{2.5}$$

If $1 < q \le p < \infty$ and $1/q - 1/2 < r(1/q - 1/p) < 1/q - 1/(2+2\alpha)$, then

$$(r'-1)/(1+\alpha) < 2(r'/p - 1/q) < r' - 1 \ .$$

We choose $1 < q_0 \le p_0 < \infty$ such that

$$\frac{1}{1+\alpha} < \theta = \frac{2(\frac{r'}{p} - \frac{1}{q})}{r'-1} < 1 \ , \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0} \ , \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_0} \ .$$

Hence,

$$\frac{p_0}{q_0} = \frac{1/q - \theta/2}{1/p - \theta/2} = r' > 1 \ .$$

Using similar arguments as in proof of Theorem 2.1, we have

$$\|T_j f\|_{\dot{F}^{s_0,q_0}_{p_0}} \le C\|f\|_{\dot{F}^{s_0,q_0}_{p_0}} \ , \tag{2.6}$$

where C is a constant independent of j and ρ. By interpolation between (2.5) and (2.6)(see [9]), noting $\theta(1+\alpha) > 1$, we get

$$\|Tf\|_{\dot{F}^{s,q}_p} = \sum_{j\in\mathbb{Z}} \|T_j f\|_{\dot{F}^{s,q}_p} \le C \sum_{j\in\mathbb{Z}} (1+|j|)^{-\theta(1+\alpha)} \|f\|_{\dot{F}^{s,q}_p} = C\|f\|_{\dot{F}^{s,q}_p} \ .$$

If $p < q$, $1/2 - 1/q < r(1/p - 1/q) < (1+2\alpha)/(2+2\alpha) - 1/q$, then

$$1 < q' < p' < \infty, \quad 1/q - 1/2 < r(1/q - 1/p) < 1/q - 1/(2+2\alpha)$$

which satisfying (a). By duality, we obtain the same result.

If $p = q = 2$ and $s \in \mathbb{R}$, taking $p_0 = q_0 = 2+\alpha, s_0 = s/2$, then p_0, q_0 satisfy the condition (a). Similarly, taking $p_1 = q_1 = (2+\alpha)/(1+\alpha), s_1 = s/2$, then p_1, q_1 satisfy the condition (b). Thus

$$\|Tf\|_{\dot{F}^{s_0,q_0}_{p_0}} \le C\|f\|_{\dot{F}^{s_0,q_0}_{p_0}} \ , \quad \|Tf\|_{\dot{F}^{s_1,q_1}_{p_1}} \le C\|f\|_{\dot{F}^{s_1,q_1}_{p_1}} \ .$$

By interpolation between the two inequalities above, we have T is bounded on $\dot{F}^{s,q}_p(\mathbb{R}^n)$. This completes the proof.

Remark 2.3. Let $q = 2$ and $s = 0$. If (ii) holds for all $1 < r < \infty$, then T is bounded on $L^p(\mathbb{R}^n)$ where $(2+2\alpha)/(1+2\alpha) < p < 2+2\alpha$. This is the corresponding result in [1] when $a_k = 2^k\rho$.

3. Applications to Singular Integral Operators with Rough Kernels

In this section, we apply the main theorems obtained in section 2 to prove the boundedness for singular integral operators with rough kernels on homogeneous Triebel-Lizorkin spaces. Let $T_{\Omega,h}$ be the singular integral operator defined by (1.1) and

$h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$, where $\Delta_\gamma(\mathbb{R}_+),\ \gamma > 1$ is defined by

$$\Delta_\gamma(\mathbb{R}_+) = \left\{ h : \sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^\gamma dt < \infty \right\} .$$

It is easy to see that $L^\infty \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\gamma(\mathbb{R}_+)$ for any $1 < \gamma < \beta$ and the inclusions are proper. We will use Theorem 2.1 to show $T_{\Omega,h}$ is bounded on $\dot{F}_p^{s,q}$,which improves the corresponding results in [6] and [8].

Now in order to prove Theorem 4.1, Let us recall the following Lemmas, which are proved by Duoandikoetxea and Van der Corput, respectively.

Lemma 3.1 ([6])**.** *Let $\alpha > 0$ and $\{\mu_k\}_{k\in\mathbb{Z}}$ be a sequence of measures on $\mathbb{R}^n$ such that for all $k \in \mathbb{Z},\ \mu_k \geq 0$ and*

$$|\widehat{\mu}_k(\xi) - 1| \leq C|2^k\xi| , \qquad |\widehat{\mu}_k(\xi)| \leq C|2^k\xi|^{-\alpha} .$$

Then the maximal operator $\mu^*(f)(x) = \sup_{k\in\mathbb{Z}} |\mu_k * f(x)|$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Lemma 3.2 ([11])**.** *Let $u(t)$ be a real-valued function, which satisfies $u'(t)$ is monotonic on (a, b) and $u'(t) \geq 1$ for all $t \in (a, b)$. If $\psi(t)$ is a function on (a, b) with integrable derivative, then*

$$\left| \int_a^b e^{i\lambda u(t)} \psi(t) dt \right| \leq C\lambda^{-1} \left[|\psi(b)| + \int_a^b |\psi'(s)| ds \right] .$$

Theorem 3.1. *Let $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$. Let $\Omega \in L^{q_0}(S^{n-1})$ be a homogeneous function of degree zero and satisfy (1.2), where $\max\{0, 1 - 2/\gamma'\} < 1/q_0 < 1$. Then for all $1 < p, q < \infty,\ s \in \mathbb{R}$,*

$$\|T_{\Omega,h} f\|_{\dot{F}_p^{s,q}} \leq C\|\Omega\|_{L^{q_0}(S^{n-1})} \|f\|_{\dot{F}_p^{s,q}} .$$

Proof. As $\Delta_\gamma(\mathbb{R}_+) \subset \Delta_2(\mathbb{R}_+)$, where $\gamma > 2$. Define a measure $1 < \gamma \leq 2$ and $1 - 2/\gamma' < 1/q_0 < 1$. Define measure σ_k on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f d\sigma_k = \frac{1}{\|\Omega\|_{L^{q_0}}} \int_{2^k \leq |y| < 2^{k+1}} f(y) h(|y|) \frac{\Omega(y)}{|y|^n} dy \ .$$

Then

$$T_{\Omega,h} f = \|\Omega\|_{L^{q_0}} \sum_{k \in \mathbb{Z}} \sigma_k * f \ .$$

It follows from Hölder inequality that

$$\begin{aligned} \|\sigma_k\| &\leq \frac{1}{\|\Omega\|_{L^{q_0}}} \int_{S^{n-1}} |\Omega(y)| \left(\int_{2^k}^{2^{k+1}} \frac{|h(t)|}{t} dt \right) d\sigma(y) \\ &\leq C \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} |h(t)|^\gamma dt \right)^{1/\gamma} \leq C \ . \end{aligned}$$

Write

$$\begin{aligned} |\widehat{\sigma}_k(\xi)| &= \frac{1}{\|\Omega\|_{L^{q_0}}} \left| \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} e^{-2\pi i t \xi \cdot y} \Omega(y) d\sigma(y) \frac{h(t)}{t} dt \right| \\ &\leq \frac{C}{\|\Omega\|_{L^{q_0}}} \left(\int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-2\pi i t \xi \cdot y} \Omega(y) d\sigma(y) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \\ &\leq \frac{C}{\|\Omega\|_{L^{q_0}}} \left(\int_1^2 \left| \int_{S^{n-1}} e^{-2\pi i 2^k t \xi \cdot y} \Omega(y) d\sigma(y) \right|^{\gamma'} dt \right)^{1/\gamma'} \ . \end{aligned} \tag{3.1}$$

Setting $H_\xi(\Omega)(t) = \chi_{[1,2]}(t) \int_{S^{n-1}} e^{-2\pi i 2^k t \xi \cdot y} \Omega(y) d\sigma(y)$, we have

$$\|H_\xi(g)\|_{L^\infty(\mathbb{R})} \leq C \|g\|_{L^1(S^{n-1})} \ . \tag{3.2}$$

Since

$$|H_\xi(g)|^2 = \int_{S^{n-1}} \int_{S^{n-1}} e^{-2\pi i 2^k t \xi \cdot (y-z)} g(y) \overline{g(z)} d\sigma(y) d\sigma(z) \ , \tag{3.3}$$

it follows from Lemma 3.2 that

$$\left|\int_1^2 e^{-2\pi i 2^k t\xi\cdot(y-z)}\frac{dt}{t}\right| \le C2^{-k}\left|\xi\cdot(y-z)\right|^{-1} = C(2^k|\xi|)^{-1}\left|\xi'\cdot(y-z)\right|^{-1} .$$

On the other hand, $\left|\int_1^2 e^{-2\pi i 2^k t\xi\cdot(y-z)}\frac{dt}{t}\right| \le 1$. Since $0 < 1-1/q_0 < 2/\gamma'$, we choose $1 < q_1 < \infty$ such that

$$\frac{1}{q_1} = 1 - \frac{\gamma'}{2}\left(1-\frac{1}{q_0}\right)$$

Thus, we have for all $0 < \alpha < 1/q_1'$,

$$\left|\int_1^2 e^{-2\pi i 2^k t\xi\cdot(y-z)}\frac{dt}{t}\right| \le C(2^k|\xi|)^{-\alpha}\left|\xi'\cdot(y-z)\right|^{-\alpha} . \tag{3.4}$$

Therefore, it follows from (3.3) and (3.4) that

$$\begin{aligned}
\|H_\xi(g)\|_{L^2(\mathbb{R})} &\le C\left(\int_1^2 |H_\xi(g)(t)|^2\frac{dt}{t}\right)^{1/2}\\
&= C\left(\int_1^2\int_{S^{n-1}}\int_{S^{n-1}} e^{-2\pi i 2^k t\xi\cdot(y-z)}g(y)\overline{g(z)}d\sigma(y)d\sigma(z)\frac{dt}{t}\right)^{1/2}\\
&= C\left(\int_{S^{n-1}}\int_{S^{n-1}}\left|\int_1^2 e^{-2\pi i 2^k t\xi\cdot(y-z)}\frac{dt}{t}\right| g(y)\overline{g(z)}d\sigma(y)d\sigma(z)\right)^{1/2}\\
&\le C\left(2^k|\xi|\right)^{-\alpha/2}\left(\int_{S^{n-1}}\int_{S^{n-1}}\frac{g(y)\overline{g(z)}}{\left|\xi'\cdot(y-z)\right|^{\alpha}}d\sigma(y)d\sigma(z)\right)^{1/2}\\
&\le C\left(2^k|\xi|\right)^{-\alpha/2}\|g\|_{L^{q_1}(S^{n-1})}\left(\int_{S^{n-1}}\int_{S^{n-1}}\frac{1}{\left|\xi'\cdot(y-z)\right|^{\alpha q_1'}}d\sigma(y)d\sigma(z)\right)^{1/2q_1'} .
\end{aligned}$$

Hence

$$\|H_\xi(g)\|_{L^2(\mathbb{R})} \le C\left(2^k|\xi|\right)^{-\alpha/2}\|g\|_{L^{q_1}(S^{n-1})} . \tag{3.5}$$

Taking $\theta = 2/\gamma'$, then $1/q_0 = \theta/q_1 + 1 - \theta$. By interpolation between (3.2) and (3.5), we have

$$\|H_\xi(g)\|_{L^{\gamma'}(\mathbb{R})} \le C\left(2^k|\xi|\right)^{-\alpha/\gamma'}\|g\|_{L^{q_0}(S^{n-1})} .$$

It follows from (3.1) that $|\widehat{\sigma}_k(\xi)| \le C\left(2^k|\xi|\right)^{-\alpha/\gamma'}$. Moreover, since $\int_{S^{n-1}} \Omega(y)d\sigma(y) = 0$, we have

$$\begin{aligned} |\widehat{\sigma}_k(\xi)| &= \frac{1}{\|\Omega\|_{L^{q_0}}} \int_1^2 \int_{S^{n-1}} \left| e^{-2\pi i 2^k t y\cdot\xi} - 1 \right| \ |\Omega(y)| d\sigma(y) \frac{|h(t)|}{t} dt \\ &\le C 2^k |\xi| \frac{1}{\|\Omega\|_{L^{q_0}}} \int_{S^{n-1}} |y\cdot\xi'| \ |\Omega(y)| d\sigma(y) \left(\sup_R \frac{1}{R} \int_0^R |h(t)| dt \right)^{1/\gamma} \\ &\le C 2^k |\xi| \ . \end{aligned}$$

Let $\mu_k = |\sigma_k|$, note that $\widehat{\mu}_k(0) = \|\sigma_k\| \le C < \infty$, then the similar arguments as above yield:

$$|\widehat{\mu}_k(\xi)| \le C(2^k|\xi|)^{-\alpha/\gamma'} \ , \qquad |\widehat{\mu}_k(\xi) - \widehat{\mu}_k(0)| \le C 2^k |\xi| \ .$$

By Lemma 3.1, $\sigma^*(f)(x) = \sup_{k\in\mathbb{Z}} |\ |\sigma_k| * f(x)| = \mu^*(f)(x)$ is a bounded operator on $L^r(\mathbb{R}^n)$ for all $1 < r < \infty$. It follows from Theorem 2.1 and Remark 2.2 that for all $1 < p, q < \infty, \ s \in \mathbb{R}$,

$$\|T_{\Omega,h} f\|_{\dot{F}_p^{s,q}} = \|\Omega\|_{L^{q_0}} \left\| \sum_{k\in\mathbb{Z}} \sigma_k * f \right\|_{\dot{F}_p^{s,q}} \le C \|\Omega\|_{L^{q_0}} \|f\|_{\dot{F}_p^{s,q}} \ .$$

This completes the proof.

Let $h \equiv 1$. Grafakos and Stefanov in [10] introduced the following condition

$$\sup_{\xi\in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi\cdot y|} \right)^{1+\alpha} d\sigma(y) < \infty \ , \tag{3.6}$$

where $\alpha > 0$ is a fixed constant. They showed that it implies the L^p-boundedness of $T_\Omega = T_{\Omega,1}$ which defined in (1.1), where $(2+\alpha)/(1+\alpha) < p < 2+\alpha$. The range for p was improved later to $(2+2\alpha)/(1+2\alpha) < p < 2+\alpha$ in [7]. Let $F_\alpha(S^{n-1})$ denote the space of all integrable functions Ω on S^{n-1} satisfying (3.6). It should be noted that

Grafakos and Stefanov showed that

$$\bigcap_{\alpha>0} F_\alpha(S^{n-1}) \not\subset H^1(S^{n-1}) \not\subset \bigcup_{\alpha>0} F_\alpha(S^{n-1}) \ ,$$

where H^1 is the Hardy space. We will use Theorem 2.2 to improve the results stated above.

Now, let us recall the following Lemma, which will be used in order to prove Theorem 3.2.

Lemma 3.3 ([11]). *Let $y \in S^{n-1}$ and*

$$(\mathcal{M}_y f)(x) = \sup_{r>0} \frac{1}{r} \left| \int_{|t|<r} f(x-ty)dt \right| \ .$$

For every $1 < p \leq \infty$, there exists a positive constant $C_p = C(p,d)$ which is independent of y such that

$$\|\mathcal{M}_y f\|_{L^p} \leq C_p \|f\|_{L^p} \ .$$

Theorem 3.2. *Let Ω be a homogeneous function of degree zero satisfied (1.2) and (3.6). Then T_Ω is bounded on $\dot{F}_p^{s,q}(\mathbb{R}^n)$ where $s \in \mathbb{R}$ and p, q satisfy*

(a) $(2+2\alpha)/(1+2\alpha) < p < q < \infty$;

(b) $1 < q < p < 2+2\alpha$;

(c) $1 < p = q < \infty$.

Proof. Using similar arguments as in [1]. Define a measure σ_k on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f d\sigma_k = \int_{2^k \leq |y| < 2^{k+1}} f(y) \frac{\Omega(y)}{|y|^n} dy \ .$$

Then $T_\Omega f = \sum_{k\in\mathbb{Z}} \sigma_k * f$. There exist finite numbers: $\{\eta_1, \cdots, \eta_m\}$ in S^{n-1} and $A_i \subset S^{n-1}, 1 \leq i \leq m$ such that $\bigcup_{i=1}^m A_i = S^{n-1}$ and $|y \cdot \eta_i| < 1/10$ for all $y \in A_i$. It follows from (3.6) that

$$\begin{aligned}\|\sigma_k\| &= \int_{S^{n-1}} |\Omega(y)| d\sigma(y) \le \bigcup_{i=1}^{m} \int_{A_i} |\Omega(y)| \left(\log \frac{1}{|\eta_i \cdot y|}\right)^{1+\alpha} d\sigma(y) \\ &\le C \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi \cdot y|}\right)^{1+\alpha} d\sigma(y) \le C \ .\end{aligned}$$

Since

$$\begin{aligned}|\sigma^*(f)(x)| &= \sup_{k\in\mathbb{Z}} \int_{S^{n-1}} |\Omega(y)| \int_{2^k}^{2^{k+1}} |f(x-ty)| t^{-1} dt d\sigma(y) \\ &\le C \int_{S^{n-1}} |\Omega(y)| \sup_{k\in\mathbb{Z}} \frac{1}{2^{k+1}} \int_{|t|<2^{k+1}} |f(x-ty)| \, dt d\sigma(y) \\ &\le C \int_{S^{n-1}} |\Omega(\xi)| \left[\mathcal{M}_y |f|(x)\right] d\sigma(y) \ ,\end{aligned}$$

by Lemma 3.3, we have for all $1 < r < \infty$

$$\|\sigma^*(f)\|_{L^r} \le C \int_{S^{n-1}} \|\mathcal{M}_y(|f|)\|_{L^r} |\Omega(y)| d\sigma(y) \le C \|f\|_{L^r} \ .$$

Moreover, since $\int_{S^{n-1}} \Omega(y) d\sigma(y) = 0$, we have

$$\begin{aligned}|\widehat{\sigma}_k(\xi)| &= \int_1^2 \int_{S^{n-1}} \left|e^{-2\pi i 2^k t y\cdot\xi} - 1\right| \ |\Omega(y)| d\sigma(y) \frac{dt}{t} \\ &\le C 2^k |\xi| \int_{S^{n-1}} |y \cdot \xi'| \ |\Omega(y)| d\sigma(y) \le C 2^k |\xi| \ .\end{aligned}$$

Write

$$\widehat{\sigma}_k(\xi) = \int_{S^{n-1}} \left(\int_1^2 e^{-2\pi i 2^k t y\cdot\xi} \frac{dt}{t}\right) \Omega(y) d\sigma(y) \ .$$

It follows from Van der Corput's Lemma 3.2 that

$$|I_k(\xi,y)| = \left|\int_1^2 e^{-i 2^k t y\cdot\xi} \frac{dt}{t}\right| \le C \left|2^k (\xi\cdot y)\right|^{-1} = C \left(2^k|\xi| \ |\xi'\cdot y|\right)^{-1} \ .$$

On the other hand, $|I_k(\xi,y)| \le 1$. If $\left(2^k|\xi| \ |\xi'\cdot y|\right)^{-1} \ge 1$, then $\log \frac{1}{|\xi'\cdot y|} \ge \log^+ (2^k|\xi|)$. If $\left(2^k|\xi| \ |\xi'\cdot y|\right)^{-1} < 1$, then $1 \le \frac{1}{|\xi'\cdot y|} < 2^k|\xi|$. Since $t^{-1/(1+\alpha)}\log t$ is a decreasing function when $\log t \geq 1+\alpha$ and is a increasing function when $0 < \log t < 1+\alpha$, we have for $\log \frac{1}{|\xi'\cdot y|} \geq 1+\alpha$,

$$\frac{\log^+ (2^k|\xi|)}{(2^k|\xi|)^{1/(1+\alpha)}} - \frac{\log(1/|\xi'\cdot y|)}{(1/|\xi'\cdot y|)^{1/(1+\alpha)}} \leq 0 < \frac{1+\alpha}{(1/|\xi'\cdot y|)^{1/(1+\alpha)}} ,$$

and for $0 < \log \frac{1}{|\xi'\cdot y|} < 1+\alpha$,

$$\frac{\log^+ (2^k|\xi|)}{(2^k|\xi|)^{1/(1+\alpha)}} \leq \max_{t>1}\{t^{-1/(1+\alpha)}\log t\} = \frac{1+\alpha}{10} \leq \frac{1+\alpha}{(1/|\xi'\cdot y|)^{1/(1+\alpha)}} .$$

Therefore,

$$\left(2^k|\xi| \ |\xi'\cdot y|\right)^{-1/(1+\alpha)} \leq \left[\log^+ (2^k|\xi|)\right]^{-1}\left(1+\alpha+\log\frac{1}{|\xi'\cdot y|}\right) .$$

Thus

$$|I_k(\xi,y)| \leq C\min\left\{1,\ \left(2^k|\xi| \ |\xi'\cdot y|\right)^{-1}\right\} \leq \left[\log^+ (2^k|\xi|)\right]^{-(1+\alpha)}\left(1+\alpha+\log\frac{1}{|\xi'\cdot y|}\right)^{1+\alpha} .$$

Therefore,

$$\begin{aligned} |\widehat{\sigma}_k(\xi)| &\leq C\left[\log^+ (2^k|\xi|)\right]^{-(1+\alpha)} \sup_{x\in S^{n-1}}\int_{S^{n-1}}|\Omega(y)|\left(\log\frac{1}{|x\cdot y|}\right)^{1+\alpha} d\sigma(y) \\ &\leq C\left[\log^+ (2^k|\xi|)\right]^{-(1+\alpha)} . \end{aligned}$$

Moreover, T_Ω is a bounded operator on $\dot{F}_p^{s,q}(\mathbb{R}^n)$, follows from Theorem 2.2.

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