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THE EXISTENCE OF PERIODIC SOLUTIONS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS CAUSED BY IMPULSES EFFECTS

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Abstract. The existence for solutions of the periodic boundary value problems concerning the second order impulsive functional differential equation

$$\begin{cases} x''(t)+\alpha x'(t)+\beta x(t)=f(t,x(t),x(\alpha_1(t))\cdots,x(\alpha_n(t))), & a.e. \text{ on } [0,T],\\ \Delta x(t_k)=I_k(x(t_k),x'(t_k)), & k=1,\cdots,m,\\ \Delta x'(t_k)=J_k(x(t_k),x'(t_k)), & k=1,\cdots,m, \end{cases}$$

and the boundary conditions $x(0)=x(T), x'(0)=x'(T)$ at resonance case are established. The method is based upon the theory of coincidence due to Mawhin, which shows that the impulse infects cause the existence of solutions. Related examples are mentioned to support the results of this paper.

1. Introduction

The motivations of this paper are as follows. First, there has been a large number of papers concerning with the solvability of the following periodic boundary value

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problems (PBVP for short)for second-order ordinary differential equations

$$(1.1)\qquad \begin{cases} x''(t) = f(t, x(t)),\ t \in (0, 2\pi), \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi). \end{cases}$$

For example, in [1], it was proved that if f satisfies the non-resonance condition

$$-(N+1)^2 + \epsilon \leq f_u(t, u) \leq -\epsilon - N^2,$$

then PBVP (1.1) has a unique solution, where N is a nonnegative integer and ϵ is a positive constant. The related researches on PBVP(1.1) can be seen in [10, 11, 12, 16, 17, 18] and the references therein.

Second, in [2], Nieto and Rodriguez-Lopez gave a Green′s function to express the unique solution for the following second-order functional differential equation with periodic boundary conditions and functional dependence given by a piecewise constant function

$$(1.2)\qquad \begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t),\ t \in (0, T), \\ x(0) = x(T),\ x'(0) = x'(T). \end{cases}$$

Using upper and lower solution method, they presented sufficient conditions to assure the existence of solutions of PBVP(1.2). The authors in [4] and [5] also studied the solvability of above problem by the similar methods.

In [3, 6], the authors studied the following PBVP

$$(1.3)\qquad \begin{cases} x''(t) + f(t, x(t), x(\theta(t))) = 0,\ t \in (0, T), \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases}$$

where θ is defined by

$$\theta(t) = \begin{cases} t - r, & t \geq r, \\ 0, & t < r. \end{cases}$$

Sufficient conditions for the existence of solutions of PBVP(1.3) were given by using upper and lower solution method. We note that in PBVP(1.2) or PBVP(1.3), the functions $[t]$ or $\theta(t)$ is not differentiable on [0,1]. It is interesting to establish existence results for solutions of PBVP(1.3) when $\theta(t)$ is differentiable.

Third, the periodic boundary value problems for second order impulsive ordinary differential equations (IPBVP for short) were studied in papers [7, 8, 9, 14, 15, 20–25] and the references therein. In [9], Rachunkova and Tvrdy studied the existence of solutions of the following nonlinear IPBVP

$$(1.4)\qquad \begin{cases} x''(t)=f(t,x(t),x'(t)),\ t\in(0,T),\\ x(T_i^+)=J_i(x(t_i)),\ \ i=1,\cdots,m,\\ x'(t_i^+)=M_i(x'(t_i)),\ \ i=1,\cdots,m,\\ x(0)=x(T),\ x'(0)=x'(T) \end{cases}$$

under the existence of lower and upper solutions.

In [14, 20, 21], Chen and Sun, Liang and Shen, Wang and Chen, respectively, studied the existence of solutions of the IPBVP

$$(1.5)\qquad \begin{cases} x''(t)+f(t,x(t),x(\theta(t)))=0,\ t\in(0,T),\\ \Delta x(t_k)=I_k(x(t_k)),\ \ k=1,\cdots,p,\\ \Delta x'(t_k)=I_k^*(x(t_k)),\ \ k=1,\cdots,p,\\ x(0)=x(T)+k_1,\ \ x'(0)=\lambda x'(T)+k_2, \end{cases}$$

which contains IPBVP

$$(1.6)\qquad \begin{cases} x''(t)+f(t,x(t),x(\theta(t)))=0,\ t\in(0,T),\\ x(0)=x(T),\ x'(0)=x'(T),\\ \Delta x^{(i)}(t_k)=I_{i,k}(x(t_k)),\ \ k=1,\cdots,p,\ \ i=0,1 \end{cases}$$

as special case. In (1.5), that $\theta:[0,T]\to[0,T]$ is continuous is supposed and in (1.6) that $0\le\theta(t)\le t$ is supposed. IPBVP(6) was studied in [15]. The methods used in [14, 15] are lower and upper solutions methods and monotone iterative technique. Similar problems were studied in papers [D-H, L-G].

In recent papers [22, 23, 24, 25], the authors studied the existence of solutions of different classes of IPBVPs, but their proofs are based upon the methods of upper and lower solutions and the monotone iterative technique. In [27], by using Schaeffer′s fixed-point theorem and monotone iterative technique, some existence results for IPBVPs were obtained.

Consider the following periodic boundary value problem for the impulsive functional Duffing equation:

$$(1.7)\quad \begin{cases} x''(t)+\alpha x'(t)+\beta x(t)=f(t,x(t),x(\alpha_1(t))\cdots,x(\alpha_n(t))), & a.e. \text{ on } [0,T],\\ \Delta x(t_k)=I_k(x(t_k),x'(t_k)), \quad k=1,\cdots,m,\\ \Delta x'(t_k)=J_k(x(t_k),x'(t_k)), \quad k=1,\cdots,m,\\ x(0)=x(T), \quad x'(0)=x'(T), \end{cases}$$

where $\alpha,\beta\in R$, $T>0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$ are fixed, $f:[0,T]\times R^{n+1}\to R$ is an impulsive Carathéodory function, $I_i,J_i:R\times R\to R$ are continuous, $\Delta y(t)$ stands for $\Delta y(t)=\lim_{t\to t^+}y(t)-\lim_{t\to t^-}y(t)$, $\alpha_i:[0,T]\to[0,T]$ with $\alpha_i\in C^1[0,T]$ and its inverse function $\beta_i\in C^0[0,T](i=1,\cdots,n)$.

In recent paper [28], the authors investigated the solvability of IPBVP (1.7) under the assumption that system

$$(1.8)\quad \begin{cases} x''(t)+\alpha x'(t)+\beta x(t)=0, & a.e. \text{ on } [0,T],\\ \Delta x(t_k)=0, \quad k=1,\cdots,m,\\ \Delta x'(t_k)=0, \quad k=1,\cdots,m,\\ x(0)=x(T), \quad x'(0)=x'(T) \end{cases}$$

has unique trivial solution $x(t)\equiv 0$, which is called non-resonance case.

In this paper, we investigate the solvability of IPBVP (1.7) under the assumption that system (1.8) has nontrivial solution, which is called resonance case. This is done by applying the well known coincidence degree theory and inequality techniques. Since we do not rely on the existence of Lipschitzian condition and the existence of upper and lower solutions and the method used in this paper is based upon the coincidence degree theory of Mawhin, our methods are different from known ones used [1, 2, 11, 14, 15, 16, 20, 21, 22, 23, 24, 25, 27], and also different from those ones in [19] since the autonomous curvature bound sets are used there.

The remainder of the paper is as follows: In Section 2, we present the main results and the examples to illustrate the main results. In Section 3, we prove the main results.

2. Main Results and Examples

In this section, the main results are presented, as well as the examples are given to illustrate efficiency of the main theorems.

Suppose $u : J = [0,T] \to R$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. For $k = 0, \cdots, m$, define the function $u_k : (t_k, t_{k+1}) \to R$ by $u_k(t) = u(t)$. Choose

$$X = \left\{ u : J \to R \;\middle|\; \begin{array}{l} u_k \in C^0(t_k, t_{k+1}), k = 0, \cdots, m, \\ \text{there exist the limits} \\ \lim_{t \to t_k^-} u(t) = u(t_k), \\ \lim_{t \to t_k^+} u(t), \\ \lim_{t \to 0^+} u(t) = u(0), \\ \lim_{t \to T^-} u(t) = u(T) \end{array} \right\}$$

with the norm

$$||u||_X = \sup_{t\in[0,T]} |u(t)|$$

for $u \in X$. Choose

$$Y = X \times R^m \times R^m$$

with the norm

$$||y||_Y = \max\left\{\sup_{t\in[0,T]} |u(t)|, \quad \max_{1\leq k\leq m} \{|x_k|\}, \quad \max_{1\leq k\leq m} |y_k|\right\}$$

for $y = \{u, x_1, \cdots, x_m, y_1, \cdots, y_m\} \in Y$. Then X and Y real Banach spaces.

Lemma 2.1. *(see [28]) A function $F : [0,1] \times R^{n+1} \to R$ is called an impulsive Carathéodory function if*

- $*$ $F(\bullet, u_0, u_1, \cdots, u_n) \in X$ *for each* $u = (u_0, \cdots, u_n) \in R^{n+1}$*;*
- $*$ $F(t, \bullet, \cdots, \bullet)$ *is continuous for* $t \neq t_k (k = 1, \cdots, m)$*.*

Lemma 2.2. *By a solution of IPBVP(1.7) we mean a function $x : [0,T] \to R$ satisfying the following conditions:*

- $x \in X$ *is differentiable in* (t_k, t_{k+1}) $(k = 0, 1, \cdots, m)$*, there exist the limits* $\lim_{t\to t_k^+} x'(t)$*,* $\lim_{t\to t_k^-} x'(t) = x'(t_k)$ $(k = 0, 1, \cdots, m)$ *and* $\lim_{t\to 0^+} x'(t) = x'(0)$ *and* $\lim_{t\to T^-} x'(t) = x'(T)$*;*
- $x' \in X$ *is differentiable in*(t_k, t_{k+1}) $(k = 0, 1, \cdots, m)$*, there exist the limits* $\lim_{t\to t_k^+} x''(t)$*,* $\lim_{t\to t_k^-} x''(t) = x''(t_k)$ $(k = 0, 1, \cdots, m)$ *and* $\lim_{t\to 0^+} x''(t) = x''(0)$ *and* $\lim_{t\to T^-} x''(t) = x''(T)$*;*
- $x'' \in X$*;*
- *The equations in (1.7) are satisfied.*

We set the following assumptions which should be used in the main results.

(A_1). $I_k(x,y)(2x+I_k(x,y)) \le 0$ for all $x, y \in R$ and $k = 1, \cdots, m$.

(A_2). $xI_k(x,y) \ge 0$ for all $x, y \in R$ and $k = 1, \cdots, m$.

(A_3). $I_k(x,y)J_k(x,y) \le 0$ and $yI_k(x,y) + xJ_k(x,y) + I_k(x,y)J_k(x,y) \ge 0$ for all $x, y \in R$ and $k = 1, \cdots, m$.

(A_4). There exist constants $\theta_k \ge 0$ such that $|I_k(x,y)| \le \theta_k |x|$ for all $x, y \in R$ with $\sum_{k=1}^m \theta_k < 1$.

(C_1). There exist impulsive Carathéodory functions $h : [0,T] \times R^n \to R$, $g_i : [0,T] \times R \to R$, $r \in X$ and constants $q \ge 1$ and $\theta > 0$ such that

$$f(t, x_0, \cdots, x_n) = h(t, x_0, \cdots, x_n) + \sum_{i=0}^n g_i(t, x_i) + r(t)$$

and

$$h(t, x_0, \cdots, x_n)x_0 \ge \theta |x_0|^{q+1}$$

hold for all $(t, x_0, \cdots, x_n) \in [0,T] \times R^{n+1}$ and

$$\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|g_i(t,x)|}{|x|^q} = r_i \in [0, +\infty)$$

for $i = 0, \cdots, n$.

(E_1). There exist a constant $M_0 > 0$ such that

$$c\left(-\beta T c + \int_0^T f(t, c, c, \cdots, c)dt + \sum_{k=1}^m J_k(c, 0) + \alpha \sum_{k=1}^m I_k(c, 0)\right) > 0$$

for all $|c| > M_0$ or

$$c\left(-\beta Tc + \int_0^T f(t,c,c,\cdots,c)dt + \sum_{k=1}^m J_k(c,0) + \alpha\sum_{k=1}^m I_k(c,0)\right) < 0$$

for all $|c| > M_0$.

Theorem 2.1. *Suppose that $\alpha \geq 0$, (E_1), (A_2), (A_3), (A_4) and (C_1) hold. Then IPBVP(1.7) has at least one solution if*

$$\theta > \begin{cases} r_0 + \sum_{k=1}^n r_k||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta \leq 0, \\ \beta + r_0 + \sum_{k=1}^n r_k||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta > 0, q = 1, \\ r_0 + \sum_{k=1}^n r_k||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta > 0, q > 1. \end{cases} \tag{2.1}$$

Theorem 2.2. *Suppose that $\alpha \leq 0$, (E_1), (A_1), (A_3), (A_4) and (C_1) hold. Then IPBVP(1.7) has at least one solution if (2.1) holds.*

Now, we give examples to illustrate the main results.

Example 2.1. *Consider the following IPBVP*

$$\begin{cases} x''(t) + \alpha x'(t) = \sum_{k=1}^{2n+1} \gamma_k x^k(t) + r(t), \\ \quad t \in [0,T],\ t \neq t_k,\ k = 1,\cdots,m, \\ \Delta x(t_k) = b_k x(t_k),\ k = 1,\cdots,m, \\ \Delta x'(t_k) = a_k x'(t_k),\ k = 1,\cdots,m, \\ x(0) = x(T),\ \ x'(0) = x'(T), \end{cases} \tag{2.2}$$

where

- *m, n are positive integers;*
- *$0 < t_1 < \cdots < t_m < T$, $\alpha \geq 0$, $T > 0$;*
- *$b_k \geq 0$ and $a_k \in R$ for all $k = 1,\cdots,m$ with $a_k + b_k \geq 0$ and $a_k + b_k + a_k b_k = 0$*

for all $k = 1,\cdots,m$ and $\sum_{k=1}^m |b_k| < 1$;

- $\gamma_{2n+1} > 0$, $\gamma_i \in R$ *for* $i = 1, 2, \cdots, 2n$;
- r *is continuous on* $[0, T]$.

Corresponding to IBVP(1.7), we have

$$f(t, x_0) = \sum_{k=1}^{2n+1} \gamma_k x_0^k + r(t),\ I_k(x, y) = b_k x,\ J_k(x, y) = a_k y.$$

Choose $h(t, x_0) = \gamma_{2n+1} x_0^{2n+1}$ and $g_0(t, x_0) = \sum_{k=0}^{2n} \gamma_k x_0^k$. One sees that $f(t, x_0) = h(t, x_0) + g_0(t, x_0) + r(t)$. Then we see that (C_1) holds with $\theta = \gamma_{2n+1}$, $q = 2n + 1$, and $r_0 = 0$

It is easy to see that

(i). $b_k \geq 0$ implies that (A_2) holds.

(ii). $a_k b_k \leq 0$ and $a_k + b_k + a_k b_k = 0$ implies that (A_3) holds.

(iii). $\Delta x(t_k) = b_k x(t_k),\ k = 1, \cdots, m$ and $\sum_{k=1}^m |b_k| < 1$ imply that (A_4) holds.

(iv). One finds that

$$\begin{aligned} & c\left(\int_0^T f(s, c) + \sum_{k=1}^m J_k(c, 0) + \alpha \sum_{k=1}^m I_k(c, 0)\right) \\ = \ & c\left(\int_0^T \left(\sum_{k=1}^{2n+1} \gamma_k c^k + r(s)\right) ds + c\left(\sum_{k=1}^m a_k + \alpha \sum_{k=1}^m b_k\right)\right) \\ = \ & \gamma_{2n+1} T c^{2n+2} + cT \sum_{k=1}^{2n} \gamma_k c^k + c\int_0^T r(s) ds + c\sum_{k=1}^m (a_k + \alpha b_k). \end{aligned}$$

Since $\gamma_{2n+1} > 0$, we find that (E_1) holds.

It follows from Theorem 2.1 that IPBVP(2.2) has at least one solution.

Example 2.2. *Consider the following IPBVP*

$$(2.3)\qquad \begin{cases} x''(t)+\alpha x'(t)+\beta x(t)=\sum_{k=1}^{2n+1}c_kx^k(t)+\sum_{k=1}^{l}d_kx^{2n+1}\left(\frac{1}{k+1}t\right)+r(t),\\ \quad t\in[0,T],\ t\neq t_k,\ k=1,\cdots,m,\\ \Delta x(t_k)=b_kx(t_k),\ k=1,\cdots,m,\\ \Delta x'(t_k)=a_kx'(t_k),\ k=1,\cdots,m,\\ x(0)=x(T),\ \ x'(0)=x'(T), \end{cases}$$

where

- $\alpha\le 0,\ \beta<0,\ T>0,\ m,n,l$ *are positive integers;*
- $0<t_1<\cdots<t_m<T$*;*
- $c_{2n+1}>0,\ c_k,d_k\in R,\ n$ *is a positive integer;*
- $a_k,b_k\in R$*;*
- r *is continuous.*

It is easy to see that

(i). if $b_k(2+b_k)\le 0$, then (A_1) holds.

(ii). if $a_kb_k\le 0$ and $a_k+b_k+a_kb_k=0$, then (A_3) holds.

(iii). if $\sum_{k=1}^{m}|b_k|<1$, then (A_4) holds.

(iv). corresponding to IPBVP(1.7), we have

$$f(t,x_0,x_1,\cdots,x_l)=c_{2n+1}x_0^{2n+1}+\sum_{k=1}^{2n}c_kx_0^k+\sum_{k=1}^{l}d_kx_k^{2n+1}+r(t),$$

choose $h(t,x)=c_{2n+1}x^{2n+1}$, $g_0(t,x)=\sum_{k=1}^{2n}c_kx^k$ and $g_k(t,x)=d_kx^{2n+1}$ for $k=1,\cdots,l$. If $c_{2n+1}>0$, then (C_1) holds with $\theta=c_{2n+1},q=2n+1,r_0=0,r_k=d_k$.

(v). One sees that

$$\begin{aligned}
& c\left(-\beta T c+\int_0^T f(s,c,\cdots,c)+\sum_{k=1}^m J_k(c,0)+\alpha\sum_{k=1}^m I_k(c,0)\right)\\
= & \ c\left(-\beta T c+\int_0^T\left(\sum_{k=1}^{2n+1} c_k c^k+\sum_{k=1}^l d_k c^{2n+1}+r(s)\right)ds+c\left(\sum_{k=1}^m a_k+\alpha\sum_{k=1}^m b_k\right)\right)\\
= & \left(c_{2n+1}T+T\sum_{k=1}^l d_k\right)c^{2n+2}+c\sum_{k=1}^{2n} c_k c^k+c\int_0^T r(s)ds+c\sum_{k=1}^m(a_k+\alpha b_k)-\beta T c^2.
\end{aligned}$$

Since $c_{2n+1}+\sum_{k=1}^l d_k\neq 0$ and $n\geq 1$, we find that (E_1) holds.

(vi). choose $\alpha_i(t)=\frac{1}{i+1}t$, then $\beta_i(t)=(i+1)t$ and $||\beta_i||=(i+1)T$.

It follows from Theorem 2.2 that IPBVP(2.3) has at least one solution if

$$b_k(2+b_k)\leq 0,\ \ a_kb_k\leq 0,\ \ a_k+b_k+a_kb_k=0,\ \ k=1,\cdots,m,$$

and

$$\sum_{k=1}^m |b_k|<1,\ \ c_{2n+1}>0,\ \ \sum_{k=1}^l d_k+c_{2n+1}\neq 0,$$

and

$$T^{(2n+1)/(2n+2)}\sum_{k=1}^l d_k(k+1)^{(2n+1)/(2n+2)}<c_{2n+1}.$$

Remark 1. It is easy to see that $f(t,x_0)$ in IPBVP(2.2) and $f(t,x_0,x_1,\cdots,x_l)$ in IPBVP(2.3) do not satisfy either the Lipschitzian condition, or left Lipschitzian condition, or right Lipschitzian condition. Hence Example 2.1 and Example 2.2 can not be solved by known theorems in [7-9, 14, 15, 20-25].

3. Proofs of the Main Results

Let X and Y be real Banach spaces, $L:\ D(L)(\subset X)\to Y$ be a Fredholm operator of index zero, $P:\ X\to X$, $Q:\ Y\to Y$ be projectors such that

$$\text{Im } P=\text{Ker } L,\ \text{Ker } Q=\text{Im } L,\ \ X=\text{Ker } L\oplus\text{Ker } P,\ \ Y=\text{Im } L\oplus\text{Im } Q.$$

It follows that

$$L|_{D(L)\cap \text{Ker } P}: \ D(L)\cap \text{Ker } P \to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega}\neq\emptyset$, the map $N: \ X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \ \overline{\Omega}\to X$ is compact.

We need the following fixed point theorems, one may see the text book [13].

Lemma 3.1. *([13], Theorem IV) Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*

(i). *$Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(D(L)\setminus KerL)\cap\partial\Omega]\times(0,1)$;*

(ii). *$Nx\notin ImL$ for every $x\in KerL\cap\partial\Omega$;*

(iii). *$deg(\wedge QN\big|_{KerL}, \ \Omega\cap KerL,0)\neq 0$, where $\wedge: \ KerL\to Y/ImL$ is an isomorphism.*

Then the equation $Lx=Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

Now, we define the linear operator $L: D(L)(\subseteq X)\to Y$ by

$$Lx(t)=\begin{pmatrix} x''(t)+\alpha x'(t)\\ \Delta x(t_1)\\ \vdots\\ \Delta x(t_m)\\ \Delta x'(t_1)\\ \vdots\\ \Delta x'(t_m)\end{pmatrix} \text{ for } x\in D(L)$$

where

$$D(L)=\left\{u:[0,T]\to R\;\middle|\;\begin{array}{l} u\in X \text{ is differentiable in } (t_k,t_{k+1})(k=0,1,\cdots,m),\\ \text{ there exist the limits } \lim_{t\to t_k^+} x'(t),\\ \lim_{t\to t_k^-} x'(t)=x'(t_k)(k=0,1,\cdots,m),\\ \lim_{t\to 0^+} x'(t)=x'(0), \lim_{t\to T^-} x'(t)=x'(T),\\ x'\in X \text{ there exist the limits } \lim_{t\to t_k^+} x''(t),\\ \lim_{t\to t_k^-} x''(t)=x''(t_k)(k=0,1,\cdots,m),\\ \lim_{t\to 0^+} x''(t)=x''(0), \lim_{t\to T^-} x''(t)=x''(T),\\ x''\in X \end{array}\right\}$$

and the nonlinear operator $N:X\to Y$ by

$$Nx(t)=\begin{pmatrix} -\beta x(t)+f(t,x(t),x(\alpha_1(t)),\cdots,x(\alpha_n(t)))\\ I_1(x(t_1),x'(t_1))\\ \cdot\\ \cdot\\ \cdot\\ I_m(x(t_m),x'(t_m))\\ J_1(x(t_1),x'(t_1))\\ \cdot\\ \cdot\\ \cdot\\ J_m(x(t_m),x'(t_m)) \end{pmatrix} \quad \text{for } x\in X.$$

Lemma 3.2. *The following results hold:*

(i). $KerL=\{x(t)=c,\ t\in[0,T],\ c\in R\}$.

(ii). *it holds that*

$$ImL = \left\{ (y(t), a_1, \cdots, a_m, b_1, \cdots, b_m) \in Y, \int_0^T y(s)ds + \sum_{k=1}^{m} b_k + \alpha \sum_{k=1}^{m} a_k = 0 \right\}.$$

(iii). *L is a Fredholm operator of index zero.*

(iv). *There exist projectors $P: X \to X$ and $Q: Y \to Y$ such that $KerL = ImP$, $KerQ = ImL$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$.*

(v). *$x \in D(L)$ is a solution of IPBVP(1.7) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.*

Proof. The proofs of (i), (ii), (iii) and (v) are standard. We omit the details. To show (iv), we present the projectors $P : X \to X$ and $Q : Y \to Y$, the isomorphism $\wedge : \mathrm{Ker}L \to Y/\mathrm{Im}L$ and the generalized inverse $K_p : \mathrm{Im}L \to D(L) \cap \mathrm{Im}P$:

$$Px(t) = x'(0) + \alpha x(0) \text{ for } x \in X,$$

$$\begin{aligned}
&Q(y(t), a_1, \cdots, a_m, b_1, \cdots, b_m) \\
&= \left(\frac{1}{T} \left(\int_0^T y(s)ds + \sum_{k=1}^{m} b_k + \alpha \sum_{k=1}^{m} a_k \right), \ \ 0, \cdots, 0 \right), \\
&\wedge(c) = (c, 0, \cdots, 0), \ c \in R,
\end{aligned}$$

$$\begin{aligned}
&K_p(y(t), a_1, \cdots, a_m, b_1, \cdots, b_m) \\
&= e^{-\alpha t} \left[\sum_{0<t_k<t} a_k e^{\alpha t_k} + \int_0^t e^{\alpha s} \sum_{0<t_k<s} (b_k + \alpha a_k) ds \right. \\
&+ \int_0^t \int_0^s y(u) du e^{\alpha s} ds + \left(\int_0^t e^{\alpha s} ds \right) \left(\int_0^T e^{\alpha s} ds \right)^{-1} \times \\
&\left. \left(\sum_{k=1}^{m} a_k e^{\alpha t_k} + \int_0^T e^{\alpha s} \sum_{0<t_k<s} (b_k + \alpha a_k) ds + \int_0^T \int_0^s y(u) du e^{\alpha s} ds \right) \right].
\end{aligned}$$

Lemma 3.3. *Let*

$$\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \exists \lambda \in (0,1)\}.$$

Suppose that $(A_2), (A_3), (A_4)$ *and* (C_1) *hold. Then* Ω_1 *is bounded if*

$$\theta > \begin{cases} r_0 + \sum_{k=1}^n r_k ||\beta_k'||^{q/(q+1)} \text{ for } \beta \le 0 \text{ or } q > 1, \\ \beta + r_0 + \sum_{k=1}^n r_k ||\beta_k'||^{q/(q+1)} \text{ for } \beta > 0 \text{ and } q = 1. \end{cases}$$

Proof. Suppose $x \in \Omega_1$, then

$$\begin{cases} x''(t) + \alpha x'(t) = -\lambda\beta x(t) + \lambda f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t))), \\ \Delta x(t_k) = \lambda I_k(x(t_k), x'(t_k)),\ k = 1, \cdots, m, \\ \Delta x'(t_k) = \lambda J_k(x(t_k), x'(t_k)),\ k = 1, \cdots, m, \\ x(0) = x(T),\ \ x'(0) = x'(T). \end{cases} \tag{3.1}$$

Step 1. Prove that there exists a constant $M_1 > 0$ so that $\int_0^T |x(s)|^{q+1} ds \le M_1$ for each $x \in \Omega_1$.

Multiplying both sides of the first equation of (3.1) by $x(t)$, integrating it from 0 to T, we get from (C_1) that

$$\begin{aligned} & x'(T)x(T) - x'(0)x(0) - \sum_{k=1}^m [x'(t_k^+)x(t_k^+) - x'(t_k)x(t_k)] - \int_0^T [x'(s)]^2 ds \\ & + \frac{\alpha}{2}[(x(T))^2 - (x(0))^2] - \frac{\alpha}{2}\sum_{k=1}^m \left[\left(x(t_k^+)\right)^2 - (x(t_k))^2\right] \\ = \ & -\lambda\beta \int_0^T |x(t)|^2 dt + \lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\ = \ & -\lambda\beta \int_0^T |x(t)|^2 dt + \lambda \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \right. \\ & \left. + \int_0^T g_0(s, x(s))x(s)ds + \sum_{i=1}^n \int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T r(s)x(s)ds \right). \end{aligned}$$

It follows from (A_2) that

$$\begin{aligned}
&\sum_{k=1}^{m}\left[\left(x(t_k^+)\right)^2-(x(t_k))^2\right]=\sum_{k=1}^{m}\left(x(t_k^+)-x(t_k)\right)\left(x(t_k^+)+x(t_k)\right)\\
=\;&\sum_{k=1}^{m}\Delta x(t_k)\left(2x(t_k)+\Delta x(t_k)\right)\\
=\;&\lambda\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))\left(2x(t_k)+\lambda I_k(x(t_k),x'(t_k))\right)\\
=\;&2\lambda\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))x(t_k)+\lambda^2\sum_{k=1}^{m}[I_k(x(t_k),x'(t_k))]^2\\
\geq\;&2\lambda\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))x(t_k)\geq 0.
\end{aligned}$$

On the other hand, (A_3) implies that

$$\begin{aligned}
&\sum_{k=1}^{m}\left(x'(t_k^+)x(t_k^+)-x'(t_k)x(t_k)\right)\\
=\;&\sum_{k=1}^{m}\Big(x'(t_k^+)(x(t_k^+)-x(t_k))+(x'(t_k^+)-x'(t_k))x(t_k)\Big)\\
=\;&\lambda\sum_{k=1}^{m}\Big(x'(t_k)I_k(x(t_k),x'(t_k))+x(t_k)J_k(x(t_k),x'(t_k))\Big)\\
&+\lambda^2\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))J_k(x(t_k),x'(t_k))\\
\geq\;&\lambda\sum_{k=1}^{m}\Big(x'(t_k)I_k(x(t_k),x'(t_k))+x(t_k)J_k(x(t_k),x'(t_k))\\
&+I_k(x(t_k),x'(t_k))J_k(x(t_k),x'(t_k))\Big)\\
\geq\;&0.
\end{aligned}$$

We get

$$\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds$$
$$+\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\le\beta\int_0^T|x(t)|^2dt.$$

It follows from (C_1) that

$$\begin{aligned}\theta\int_0^T|x(s)|^{q+1}ds &\le -\int_0^T g_0(s,x(s))x(s)ds-\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s))x(s)ds\\ &\quad -\int_0^T r(s)x(s)ds+\beta\int_0^T|x(t)|^2dt\\ &\le \int_0^T|g_0(s,x(s))||x(s)|ds+\sum_{i=1}^n\int_0^T|g_i(s,x(\alpha_i(s))||x(s)|ds\\ &\quad +\int_0^T|r(s)||x(s)|ds+\sigma(\beta)\int_0^T|x(t)|^2dt,\end{aligned}$$

where $\sigma(\beta)=0$ if $\beta\le 0$ and $\sigma(\beta)=\beta$ if $\beta>0$. Let $\epsilon>0$ satisfy that

$$\theta>\begin{cases}(r_0+\epsilon)+\sum_{k=1}^n(r_k+\epsilon)||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta\le 0,\\ \beta+(r_0+\epsilon)+\sum_{k=1}^n(r_k+\epsilon)||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta>0 \text{ and } q=1,\\ (r_0+\epsilon)+\sum_{k=1}^n(r_k+\epsilon)||\beta_k'||^{\frac{q}{q+1}} \text{ for } \beta>0 \text{ and } q>1.\end{cases}\tag{3.2}$$

For such $\epsilon>0$, there is $\delta>0$ so that for every $i=0,1,\cdots,n$,

$$|g_i(t,x)|<(r_i+\epsilon)|x|^q \text{ for } a.e.t\in[0,T] \text{ and all } x \text{ such that } |x|>\delta.\tag{3.3}$$

Denote, for $i = 1, \cdots, n$,

$$\begin{aligned}
\Delta_{1,i} &= \{t: \ t \in [0,T], \ |x(\alpha_i(t))| \le \delta\},\\
\Delta_{2,i} &= \{t: \ t \in [0,T], \ |x(\alpha_i(t))| > \delta\},\\
g_{\delta,i} &= \max_{t\in[0,T],|x|\le\delta} |g_i(t,x)|,\\
\Delta_1 &= \{t \in [0,T], |x(t)| \le \delta\},\\
\Delta_2 &= \{t \in [0,T], |x(t)| > \delta\},\\
\delta' &= \max\{g_{\delta,k} : k = 0, \cdots, n\}.
\end{aligned}$$

Using Holder's inequality, we get

$$\begin{aligned}
&\theta \int_0^T |x(s)|^{q+1} ds\\
= \ & \int_{\Delta_1} |g_0(s,x(s))||x(s)|ds + \int_{\Delta_2} |g_0(s,x(s))||x(s)|ds\\
&+ \sum_{i=1}^n \int_{\Delta_{1,i}} |g_i(s,x(\alpha_i(s))||x(s)|ds\\
&+ \sum_{i=1}^n \int_{\Delta_{2,i}} |g_i(s,x(\alpha_i(s))||x(s)|ds + \int_0^T |r(s)||x(s)|ds + \sigma(\beta)\int_0^T |x(t)|^2 dt\\
\le \ & (r_0+\epsilon)\int_0^T |x(s)|^{q+1}ds + \sum_{k=1}^n (r_k+\epsilon)\int_0^T |x(\alpha_k(s))|^q |x(s)|ds\\
&+ \int_0^T |r(s)||x(s)|ds + g_{\delta,0}\int_0^T |x(s)|ds + \sum_{k=1}^n g_{\delta,k}\int_0^T |x(s)|ds\\
&+\sigma(\beta)\int_0^T |x(t)|^2 dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma(\beta)\int_0^T |x(t)|^2dt + (r_0+\epsilon)\int_0^T |x(s)|^{q+1}ds \\
&\quad +\sum_{k=1}^{n}(r_k+\epsilon)\left(\int_0^T |x(\alpha_k(s))|^{q+1}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +(n+1)\delta'\int_0^T |x(s)|ds \\
&= \sigma(\beta)\int_0^T |x(t)|^2dt + (r_0+\epsilon)\int_0^T |x(s)|^{q+1}ds \\
&\quad +\sum_{k=1}^{n}(r_k+\epsilon)\left|\int_{\alpha_k(0)}^{\alpha_k(T)} |x(u)|^{q+1}|\beta_k'(u)|du\right|^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\leq \sigma(\beta)\int_0^T |x(t)|^2dt + (r_0+\epsilon)\int_0^T |x(s)|^{q+1}ds \\
&\quad +\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||^{q/(q+1)}\left(\int_0^T |x(u)|^{1+q}|du\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&= \sigma(\beta)\int_0^T |x(t)|^2dt + \left((r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||^{q/(q+1)}\right)\int_0^T |x(s)|^{q+1}ds \\
&\quad +\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\quad +(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}.
\end{aligned}
$$

If $\beta \leq 0$, we get

$$\begin{aligned}\theta\int_0^T |x(s)|^{q+1}ds \;\leq\; & \left((r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||^{q/(q+1)}\right)\int_0^T |x(s)|^{q+1}ds\\ &+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\ &+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}.\end{aligned}$$

Then (3.1) implies that there exists a constant $M_1 > 0$ such that $\int_0^T |x(s)|^{q+1}ds \leq M_1$.

If $\beta > 0$, one sees from $q \geq 1$ that

$$\begin{aligned}&\theta\int_0^T |x(s)|^{q+1}ds\\ \leq\;& \beta\int_0^T |x(t)|^2dt+\left((r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||^{q/(q+1)}\right)\int_0^T |x(s)|^{q+1}ds\\ &+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\ &+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\ \leq\;& \beta T^{(q-1)/(q+1)}\left(\int_0^T |x(t)|^{q+1}dt\right)^{2/(q+1)}\\ &+\left((r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||^{q/(q+1)}\right)\int_0^T |x(s)|^{q+1}ds\\ &+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\ &+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}.\end{aligned}$$

Then (3.1) implies that there exists a constant $M_1 > 0$ such that $\int_0^T |x(s)|^{q+1}ds \leq M_1$.

Step 2. Prove that there exists a constant $M_2 > 0$ such that $||x||_X \leq M_2$ for each $x \in \Omega_1$.

It follows from Step 1 that there exists $\xi \in [0, T]$ such that $|x(\xi)| \leq (M_1/T)^{1/(q+1)}$. Multiplying both sides of the first equation of (3.1) by $x(t)$, integrating it from 0 to T, we get, using (A_2), (A_3) and (C_1) that

$$\begin{aligned}
& \int_0^T [x'(s)]^2 ds \\
= \;& -\frac{\alpha}{2}\sum_{k=1}^{m}\left[\left(x(t_k^+)\right)^2 - (x(t_k))^2\right] - \sum_{k=1}^{m}[x'(t_k^+)x(t_k^+) - x'(t_k)x(t_k)] \\
& -\lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds + \lambda\beta \int_0^T |x(t)|^2 dt \\
\leq \;& -\lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds + \lambda\beta \int_0^T |x(t)|^2 dt \\
\leq \;& -\lambda \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds \right. \\
& \left. + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T r(s)x(s)ds\right) + \lambda\beta \int_0^T |x(t)|^2 dt \\
\leq \;& -\lambda \int_0^T g_0(s, x(s))x(s)ds - \lambda\sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s))x(s)ds - \lambda\int_0^T r(s)x(s)ds \\
& +\lambda\beta \int_0^T |x(t)|^2 dt \\
\leq \;& \int_0^T |g_0(s, x(s))||x(s)|ds + \sum_{i=1}^{n}\int_0^T |g_i(s, x(\alpha_i(s))||x(s)|ds + \int_0^T |r(s)||x(s)|ds \\
& +\lambda\beta \int_0^T |x(t)|^2 dt \\
\leq \;& \left[\left((r_0 + \epsilon) + \sum_{k=1}^{n}(r_k + \epsilon)||\beta_k'||^{q/(1+q)}\right)\int_0^T |x(s)|^{q+1} ds\right.
\end{aligned}$$

$$+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\Bigg]$$
$$+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}+\sigma(\beta)\int_0^T |x(t)|^2dt.$$

If $\beta \leq 0$, then

$$\begin{aligned}
&\int_0^T [x'(s)]^2ds\\
\leq\ &\left[\left((r_0+\epsilon)+\sum_{k=1}^n (r_k+\epsilon)||\beta_k'||^{q/(1+q)}\right)M_1\right.\\
&\left.+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}M_1^{1/(q+1)}\right]+(n+1)\delta' T^{q/(q+1)}M_1^{1/(q+1)}\\
=:\ &M_2.
\end{aligned}$$

If $\beta > 0$, then

$$\begin{aligned}
&\int_0^T [x'(s)]^2ds\\
\leq\ &\left[\left((r_0+\epsilon)+\sum_{k=1}^n (r_k+\epsilon)||\beta_k'||^{q/(1+q)}\right)M_1+\beta\int_0^T |x(t)|^2dt\right.\\
&\left.+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}M_1^{1/(q+1)}\right]+(n+1)\delta' T^{q/(q+1)}M_1^{1/(q+1)}\\
\leq\ &\left[\left((r_0+\epsilon)+\sum_{k=1}^n (r_k+\epsilon)||\beta_k'||^{q/(1+q)}\right)M_1+\beta T^{(q-1)/(q+1)}M_1^{2/(q+1)}\right.\\
&\left.+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}M_1^{1/(q+1)}\right]+(n+1)\delta' T^{q/(q+1)}M_1^{1/(q+1)}\\
=:\ &M_2.
\end{aligned}$$

Due to (A_4) one sees that

$$
\begin{aligned}
|x(t)| &= \begin{cases} \left|x(\xi) + \lambda \sum_{\xi \le t_k < t} I_k(x(t_k), x'(t_k)) + \int_\xi^t x'(s)ds\right| & \text{if } t \ge \xi, \\ \left|x(\xi) - \lambda \sum_{t \le t_k < \xi} I_k(x(t_k), x'(t_k)) - \int_t^\xi x'(s)ds\right| & \text{if } t < \xi, \end{cases} \\
&\le (M_1/T)^{1/(q+1)} + \sum_{k=1}^{m} \theta_k ||x||_X + \int_0^T |x'(s)|ds \\
&\le (M_1/T)^{1/(q+1)} + \sum_{k=1}^{m} \theta_k ||x||_X + T^{1/2} \left(\int_0^T |x'(s)|^2 ds \right)^{1/2} \\
&\le (M_1/T)^{1/(q+1)} + \sum_{k=1}^{m} \theta_k ||x||_X + T^{1/2} M_2^{1/2}.
\end{aligned}
$$

It follows from (A_4) that

$$
||x||_X \le \frac{1}{1 - \sum_{k=1}^m \theta_k} \left((M_1/T)^{1/(q+1)} + T^{1/2} M_2^{1/2} \right).
$$

It follows that Ω_1 is bounded. This completes the proof of Lemma 3.3.

Lemma 3.4. *Let*

$$
\Omega_2 = \{x \in KerL,\ Nx \in ImL\}.
$$

Suppose that (E_1) holds, then Ω_2 is bounded.

Proof. Suppose $x \in \Omega_2$, then $x(t) = c \in R$ and

$$
Nx(t) = \begin{pmatrix} -\beta c + f(t, c, c, \cdots, c) \\ I_1(c, 0) \\ \cdots \\ I_m(c, 0) \\ J_1(c, 0) \\ \cdots \\ J_m(c, 0) \end{pmatrix},
$$

then $Nx \in \text{Im}L$ implies that

$$-\beta Tc + \int_0^T f(t, c, c, \cdots, c)dt + \sum_{k=1}^{m} J_k(c, 0) + \alpha \sum_{k=1}^{m} I_k(c, 0) = 0.$$

It follows from (E_1) that $|c| \le M_0$.

Lemma 3.5. *If the first case in (E_1) holds, let*

$$\Omega_3 = \{x \in KerL,\ \lambda \wedge x + (1-\lambda)QNx = 0,\ \exists \lambda \in [0,1]\},$$

where $\wedge : KerL \to ImQ$ is the linear isomorphism given by $\wedge(c) = (c, 0, \cdots, 0)$ for all $c \in R$. If the second case in (E_1) holds, let

$$\Omega_3 = \{x \in KerL,\ \lambda \wedge x - (1-\lambda)QNx = 0,\ \exists \lambda \in [0,1]\}.$$

Then Ω_3 is bounded.

Proof. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to +\infty$ as n tends to infinity. Then

$$Nx_n(t) = \begin{pmatrix} -\beta c_n + f(t, c_n, c_n, \cdots, c_n) \\ I_1(c_n, 0) \\ \cdots \\ I_m(c_n, 0) \\ J_1(c_n, 0) \\ \cdots \\ J_m(c_n, 0) \end{pmatrix}.$$

It follows that

$$QNx_n = \left(\frac{-\beta T c_n + \int_0^T f(t, c_n, c_n, \cdots, c_n)dt}{T} + \frac{\sum_{k=1}^{m} J_k(c_n, 0) + \alpha \sum_{k=1}^{m} I_k(c_n, 0)}{T}, 0, \cdots, 0 \right).$$

Hence

$$\begin{aligned} 0 &= \lambda \wedge (c_n) + \frac{1-\lambda}{T}\left(-\beta T c_n + \int_0^T f(t, c_n, c_n, \cdots, c_n)dt \right.\\ &\left. + \sum_{k=1}^{m} J_k(c_n, 0) + \alpha \sum_{k=1}^{m} I_k(c_n, 0)\right). \end{aligned}$$

So

$$\begin{aligned} \lambda c_n^2 &= -\frac{1-\lambda}{T} c_n \left(-\beta T c_n + \int_0^T f(t, c_n, \cdots, c_n)dt \right.\\ &\left. + \sum_{k=1}^{m} J_k(c_n, 0) + \alpha \sum_{k=1}^{m} I_k(c_n, 0)\right). \end{aligned}$$

If $\lambda = 1$, then $c_n = 0$. If $\lambda \in [0, 1)$ and $|c_n| > M_0$, then $\lambda c_n^2 < 0$, a contradiction. Hence $|c_n| \le M_0$. Ω_3 is bounded.

If the second case in (E_1) holds, similar to above discussion, we get Ω_3 is bounded.

Proof of Theorem 2.1. We show that all conditions of Lemma 3.1 are satisfied. Let Ω be a non-empty open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^{3} \overline{\Omega_i}$ centered at zero. since L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(a). $Lx \ne \lambda N x$ for $x \in (D(L) \setminus \text{Ker}L) \cap \partial\Omega$ and $\lambda \in (0, 1)$;

(b). $Nx \notin \text{Im}L$ for $x \in \text{Ker}L \cap \partial\Omega$.

We prove **(c).** $\deg(QN|_{\text{Ker}L},\ \Omega \cap \text{Ker}L, 0) \ne 0$.

In fact, let $H(x, \lambda) = \lambda \wedge x \pm (1-\lambda)QNx$. According the definition of Ω, we know $H(x, \lambda) \ne 0$ for $x \in \partial\Omega \cap \text{Ker}L$ and $\lambda \in (0, 1)$, thus by the homotopy property of degree,

$$\begin{aligned} &\deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) = \deg(H(\cdot, 0), \Omega \cap \text{Ker}L, 0) \\ = \;&\deg(H(\cdot, 1), \Omega \cap \text{Ker}L, 0) = \deg(I, \Omega \cap \text{Ker}L, 0) \ne 0. \end{aligned}$$

Thus by Lemma 3.1, $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$, which is a solution of IPBVP(1.7). The proof is complete.

Lemma 3.6. *Let*

$$\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \exists \lambda \in (0,1)\}.$$

Suppose that $(A_1), (A_3), (A_4)$ *and* (C_1) *hold. Then* Ω_1 *is bounded if*

$$\theta > \begin{cases} r_0 + \sum_{k=1}^n r_k ||\beta_k'||^{q/(q+1)} \ \textit{for}\ \beta \le 0 \ \textit{or}\ q > 1, \\ \beta + r_0 + \sum_{k=1}^n r_k ||\beta_k'||^{q/(q+1)} \ \textit{if}\ \beta > 0 \ \textit{and}\ q = 1. \end{cases}$$

Proof. Suppose $x \in \Omega_1$, we get (3.1). Multiplying both sides of the first equation of (3.1) by $x(t)$, integrating it from 0 to T, we get from (C_1) that

$$\begin{aligned} & x'(T)x(T) - x'(0)x(0) - \sum_{k=1}^{m} [x'(t_k^+)x(t_k^+) - x'(t_k)x(t_k)] - \int_0^T [x'(s)]^2 ds \\ & + \frac{\alpha}{2}[(x(T))^2 - (x(0))^2] - \frac{\alpha}{2} \sum_{k=1}^{m} \left[\left(x(t_k^+)\right)^2 - (x(t_k))^2 \right] \\ = \; & -\lambda\beta \int_0^T |x(t)|^2 dt + \lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\ = \; & -\lambda\beta \int_0^T |x(t)|^2 dt + \lambda \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \right. \\ & + \int_0^T g_0(s, x(s))x(s)ds \\ & \left. + \sum_{i=1}^{n} \int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T r(s)x(s)ds \right). \end{aligned}$$

It follows from (A_1) that

$$\begin{aligned}
&\sum_{k=1}^{m}\left[\left(x(t_k^+)\right)^2-(x(t_k))^2\right]=\sum_{k=1}^{m}\left(x(t_k^+)-x(t_k)\right)\left(x(t_k^+)+x(t_k)\right)\\
=\ &\sum_{k=1}^{m}\Delta x(t_k)\left(2x(t_k)+\Delta x(t_k)\right)\\
=\ &\lambda\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))\left(2x(t_k)+\lambda I_k(x(t_k),x'(t_k))\right)\\
\leq\ &\lambda\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))\left(2x(t_k)+I_k(x(t_k),x'(t_k))\right)\\
\leq\ &0.
\end{aligned}$$

On the other hand, (A_3) implies that

$$\begin{aligned}
&\sum_{k=1}^{m}\left(x'(t_k^+)x(t_k^+)-x'(t_k)x(t_k)\right)\\
=\ &\sum_{k=1}^{m}\Big(\,x'(t_k^+)(x(t_k^+)-x(t_k))+(x'(t_k^+)-x'(t_k))x(t_k)\Big)\\
=\ &\lambda\sum_{k=1}^{m}\Big(\,x'(t_k)I_k(x(t_k),x'(t_k))+x(t_k)J_k(x(t_k),x'(t_k))\Big)\\
&+\lambda^2\sum_{k=1}^{m}I_k(x(t_k),x'(t_k))J_k(x(t_k),x'(t_k))\\
\geq\ &\lambda^2\sum_{k=1}^{m}\Big(\,x'(t_k)I_k(x(t_k),x'(t_k))+x(t_k)J_k(x(t_k),x'(t_k))\\
&\quad+I_k(x(t_k),x'(t_k))J_k(x(t_k),x'(t_k))\Big)\\
\geq\ &0.
\end{aligned}$$

We get

$$\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds$$
$$+\sum_{i=1}^n \int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T r(s)x(s)ds \le \beta \int_0^T |x(t)|^2 dt.$$

The remainder of the proof ia similar to that of Lemma 3.3 and is omitted.

Proof of Theorem 2.2. It is similar to that of Theorem 2.1 and is omitted.

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