Jordan Journal of Mathematics and Statistics (JJMS) 4 (2), 2011, pp. 79-89

Edge-Maximal C_{2k+1}-vertex disjoint Free Graphs *

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ABSTRACT: Let $k \geq 1$ be a positive integer and $\mathcal{G}(n;V_{2k+1})$ the class of graphs on n vertices containing no $2k+1$ vertex disjoint cycles. Let $f(n;V_{2k+1}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n;V_{2k+1})\}$. In this paper we determine $f(n;V_{2k+1})$ and characterise the edge maximal members in $\mathcal{G}(n;V_{2k+1})$ for $k = 1$ and 2.

1. INTRODUCTION

First, we recall some notation and terminology. For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\varepsilon(G)$, respectively. The cycle on n vertices is denoted by C_n. Let G be a graph and $u \in V(G)$. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbour set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u; we write $d_H(u) = |N_H(u)|$.

Let G_1 and G_2 be graphs. The union $G_1 \cup G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Two graphs G_1 and G_2 are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \phi$; G_1 and G_2 are edge disjoint if

2000 **Mathematics Subject Classification:** Primary 05C38 ; secondary 05 C35

Keywords: Extremal graph; Cycle; Vertex disjoint.

Received on: April 15, 2010 **Accepted on**: June 14, 2011

$E(G_1) \cap E(G_2) = \phi$. The intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is defined similarly, but in this case we need to assume $V(G_1) \cap V(G_2) \neq \phi$. The join $G \vee H$ of two vertex disjoint graphs *G* and *H* is the graph obtained from *G* + *H* by joining each vertex of *G* to each vertex of *H*. For vertex disjoint subgraphs H_1 and H_2 of *G*, we let

$$E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$$

and

$$\varepsilon(H_1, H_2) = |E(H_1, H_2)|.$$

For a proper subgraph *H* of *G* we write *G*[*V*(*H*)] and *G* – *V*(*H*) simply as *G*[*H*] and $G - H$ respectively.

In this paper we consider the Turán-type extremal problem [6] with the odd vertex disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, we only consider non-bipartite graphs. For a positive integer *n* and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on *n* vertices, and $f(n; \mathcal{F}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n; \mathcal{F})\}$. An important problem in extremal graph theory is that of determining the values of the function $f(n; \mathcal{F})$ [6]. Further, characterize the extremal graphs $\mathcal{G}(n; \mathcal{F})$ of where $f(n; \mathcal{F})$ is attained. This problem has been studied by a number of authors [3, 4, 7, 8, 9, 11]. Jia [10] proved that $f(n; C_5) = \left\lfloor \frac{1}{4}(n-2)^2 \right\rfloor + 3$ for $n \geq 9$, and he characterizes the extremal graphs as well. Jia [10] conjectured that $f(n; C_{2k+1}) \leq \left\lfloor \frac{1}{4}(n-2)^2 \right\rfloor + 3$ for $n \geq 4k+2$. Recently, Bataineh [1] confirm positively the above conjecture for $n > 36k$. Most recently, Bataineh and Jaradat [2] proved that for large *n*, $\varepsilon(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1$ where *G* is a graph that contains no *r* edge disjoint copies of C_{2k+1}.
Let $\mathcal{G}(n; V_{2k+1})$ denote the class of graphs on *n* vertices containing no vertex disjoint cycles of length (2*k*+1). Let $f(n; V_{2k+1}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n; V_{2k+1})\}$. In this

paper we determine $f(n;V_{2k+1})$ and characterise the edge maximal members in $\mathcal{G}(n;V_{2k+1})$ for $k = 1$ and 2.

Now, we state a number of results, which we use to prove our main results.

Lemma 1.1 (Bondy & Murty) Let G be a graph on n vertices. If $\varepsilon(G) > \frac{n^2}{4}$, then G contains a cycle of length r for each r, where $3 \le r \le \left\lfloor \frac{n+3}{2} \right\rfloor$.

Theorem 1.1 (Brandt) A non-bipartite graph G of order n and more than $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ edges. Then G contains all cycles of length between 3 and the length of the longest cycle.

Theorem 1.2 (Jia) Let $G \in \mathcal{G}(n; C_5)$, $n \ge 9$. Then

$$f(n;C_5) \le \left\lfloor \frac{1}{4}(n-2)^2 \right\rfloor + 3.$$

Furthermore, equality holds if and only if $G \in \mathcal{G}_5^*(n)$ for $n \ge 10$ where $\mathcal{G}_5^*(n)$ denote the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\left\lfloor \frac{1}{2}(n-2) \right\rfloor, \left\lceil \frac{1}{2}(n-2) \right\rceil}$.

2. Edge-Maximal C_3-vertex disjoint Free Graphs

Let $\mathcal{G}(n;V_3)$ denote the class of graphs on n vertices containing no vertex disjoint cycles of length 3. Let

$$f(n;V_3) = \max\{\varepsilon(G) : G \in \mathcal{G}(n;V_3)\}.$$

In this section we determine $f(n;V_3)$ and characterise the edge maximal members in $\mathcal{G}(n;V_3)$. We begin with the following construction. Let $\Omega(n) = K_{1, \left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil}$.

Observe that $\Omega(n) \subseteq \mathcal{G}(n;V_3)$ and the graph $\Omega(n)$ contains $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1$ edges. Thus, we have established that

$$f\,(n;V_3) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1\,.$$

In the following theorem we establish that equality holds and we determine edge maximum members in $\mathcal{G}(n;V_3)$.

Theorem 2.1 Let $G \in \mathcal{G}(n;V_3)$**.** For $n \geq 10$**,**

$$f\,(n;V_3) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1\,.$$

Furthermore, equality holds if and only if $G = \Omega(n)$.

Proof: Let $G \in \mathcal{G}(n;V_3)$. Suppose G contains a K_5 as a subgraph.
Let $x \in V\,(G - K_5)$, if x is adjacent to K_5 by two edges, then G would have two vertex disjoint cycles of length 3. Thus,

$$\varepsilon(G - K_5, K_5) \leq n - 5.$$

Further, observe that $G - K_5$ cannot have cycles of length 3 as otherwise G would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have

$$\varepsilon(G - K_5) \leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor.$$

Now,

$$\begin{aligned}
\varepsilon(G) &= \varepsilon(G - K_5, K_5) + \varepsilon(G - K_5) + \varepsilon(K_5) \\
&\leq n - 5 + \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 10 \\
&< \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1
\end{aligned}$$

for $n \geq 10$. So, we need to consider the second case when G contains no K_5. Suppose G contains a K_4 as a subgraph. Now, define $A = \{x \in G - K_4 : \varepsilon(x, K_4) = 3\}$. If $|A| \leq 1$ then, we have

$$\varepsilon(G - K_4, K_4) \leq 2(n-4) + 1.$$

Observe that $G - K_4$ contains no cycles of length 3 as otherwise G would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have

$$\varepsilon(G - K_4) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor.$$

Now,

$$\begin{aligned}\varepsilon(G) &= \varepsilon(G - K_4, K_4) + \varepsilon(G - K_4) + \varepsilon(K_4) \\ &\leq 2(n-4) + 1 + \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 6 \\ &< \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.\end{aligned}$$

for $n \geq 10$. So, we need to consider the case when $|A| \geq 2$. Let v and w be two vertices in A. Let $T = G[v, w, K_4]$ and $G_1 = G - T$. Let $g \in V(G_1)$, if g is adjacent to T by 4 edges, then G would have two vertex disjoint cycles of length 3. Thus, we have $\varepsilon(G_1, T) \leq 3(n-6)$. Observe that G_1 cannot have cycles of length 3 as otherwise G would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have

$$\varepsilon(G_1) \leq \left\lfloor \frac{(n-6)^2}{4} \right\rfloor.$$

Now,

$$\begin{aligned}\varepsilon(G) &= \varepsilon(G_1, T) + \varepsilon(G_1) + \varepsilon(T)\\ &\le 3(n-6) + \left\lfloor \frac{(n-6)^2}{4} \right\rfloor + 10\\ &< \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.\end{aligned}$$

for $n \ge 10$. So, we need to consider the case when G contains no K_4 as a subgraph. Suppose G contains a K_3 as a subgraph. Let $T = G[K_3]$ and $G_1 = G - T$. Let $g \in G_1$, if g is adjacent to T by more than 2 edges, then G would have K_4 as a subgraph. Thus, we have $\varepsilon(G_1, T) \le 2(n-3)$. Observe that G_1 cannot have cycles of length 3 as otherwise G would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have

$$\varepsilon(G_1) \le \left\lfloor \frac{(n-3)^2}{4} \right\rfloor.$$

Now,

$$\begin{aligned}\varepsilon(G) &= \varepsilon(G_1, T) + \varepsilon(G_1) + \varepsilon(T)\\ &\le 2(n-3) + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 3\\ &= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.\end{aligned}$$

So, we need to consider the case when G contains no cycles of length 3. By Lemma 1.1, we have

$$\begin{aligned}\varepsilon(G) &\le \left\lfloor \frac{n^2}{4} \right\rfloor\\ &< \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.\end{aligned}$$

This completes the proof.

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case where G have a K_3 as a subgraph, $G-K_3$ is a complete a bipartite graph $K_{\left\lfloor\frac{n-3}{2}\right\rfloor,\left\lceil\frac{n-3}{2}\right\rceil}$ and $\varepsilon(K_3,\ G-K_3)=2(n-3)$. This gives rise to the graph $\Omega(n)=K_{1,\left\lfloor\frac{n-1}{2}\right\rfloor,\left\lceil\frac{n-1}{2}\right\rceil}$.

In the following section we determine edge maximum members in $\mathcal{G}(n;V_5)$.

3. Edge-Maximal C5-vertex disjoint Free Graphs

Let $k\geq 2$ be a positive integer. Let $\mathcal{G}(n;V_{2k+1})$ denote the class of graphs on n vertices containing no 2k+1 vertex disjoint cycles. Let
$f\,(n;V_{2k+1})=\max\{\varepsilon(G):G\in\mathcal{G}(n;V_{2k+1})\}$.
In this section, we determine $f\,(n;V_5)$ and characterise the edge maximal members in $\mathcal{G}(n;V_5)$. Let $\Omega(n)=K_{1,\left\lfloor\frac{n-1}{2}\right\rfloor,\left\lceil\frac{n-1}{2}\right\rceil}$. Observe that $\Omega(n)\subseteq\mathcal{G}(n;V_5)$ and the graph $\Omega(n)$ contains $\left\lfloor\frac{(n-1)^2}{4}\right\rfloor+n-1$ edges. Thus, we have established that

$$f\,(n;V_5)\geq\left\lfloor\frac{(n-1)^2}{4}\right\rfloor+n-1.$$

In this section, we prove that equality holds. In the following theorem we determine edge maximum members in $\mathcal{G}(n;V_5)$.

Theorem 3.1 Let $G\in\mathcal{G}(n;V_5)$. If $\delta(G)\geq 40$, then

$$f\,(n;V_5)\leq\left\lfloor\frac{(n-1)^2}{4}\right\rfloor+n-1.$$

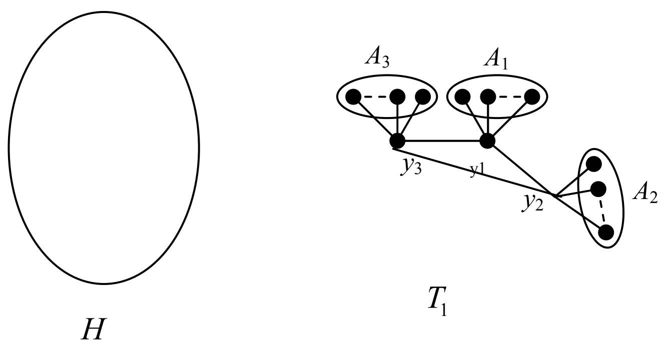
Furthermore, equality holds if and only if $G=\Omega(n)$.

Proof: Suppose G contains no two vertex-disjoint cycles of length 3. Then by the Theorem 2.1, we have

$$\varepsilon(G) \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.$$

So, we need to consider the case when G has at least two vertex-disjoint cycles of length 3. Let $C_3 = x_1, x_2, x_3, x_1$ and $C_3' = y_1, y_2, y_3, y_1$ be two vertex-disjoint cycles of length 3. We consider two cases:

Case 1: C_3 form a cycle of length 5 in $G - C_3'$ or C_3' form a cycle of length 5 in $G - C_3$. Without loss of generality, assume C_3 form a cycle of length 5 in $G - C_3'$. Let $C_5 = z_1, z_2, z_3, z_4, z_5, z_1$ be the cycle of length 5 in $G - C_3'$. Define $H = (G - C_3') - C_5$. Note that the vertices in G have degree more than or equal to 40 in G. So, for j = 1, 2, 3, let A_j be a set that consist of 4 neighbours of y_j in H, selected so that $A_l \cap A_j = \phi$, for $l \ne j$. Let $T_1 = G\left[\bigcup_{j=1}^{3} y_j, \bigcup_{j=1}^{3} A_j\right]$. The situation as shown below:



Let $u \in V(H)$. If u is adjacent to a vertex in A_j, for j = 1, 2, 3, then u can not be adjacent to any vertex in $A_{j+2} \cup A_{j-2}$, and to x_{j+1} and x_{j-1} . Thus, $\varepsilon(\{u\}, T_1) \le 5$. Consequently, $\varepsilon(H, T_1) \le 5\,(n-20)$. Further, $\varepsilon(T_1, C_5) \le 25$

By Theorem 1.2, we have $\varepsilon(T_1) \le \left\lfloor \frac{13^2}{4} \right\rfloor + 3$. Note that, $\varepsilon(H, C_5) + \varepsilon(C_5) \le 5(n-20) + 10$. Now,

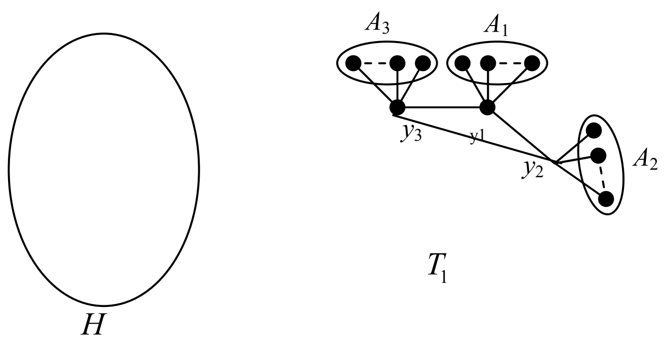
$$\varepsilon(G) = \varepsilon(H) + \varepsilon(H, T_1) + \varepsilon(T_1) + \varepsilon(H, C_5) + \varepsilon(C_5) + \varepsilon(T_1, C_5)$$

$$\le \left\lfloor \frac{(n-20)^2}{4} \right\rfloor + 5(n-20) + \left\lfloor \frac{13^2}{4} \right\rfloor + 3 + 5(n-20) + 10 + 25 \quad \text{(Lemma 1.1)}$$

$$\le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n.$$

Case 2: C_3 does not form a cycle of length 5 in $G - C_3'$ and C_3' does not form a cycle of length 5 in $G - C_3$.

Define $H = (G - C_3') - C_3$. Note that the vertices in *G* have degree more than or equal to 40 in *G*. So, for *j* = 1, 2, 3, let A_j be a set that consist of 4 neighbours of y_j in *H*, selected so that $A_l \cap A_j = \phi$, for $l \ne j$. Let $T_1 = G\left[\bigcup_{j=1}^{3} y_j, \bigcup_{j=1}^{3} A_j\right]$. The situation is shown below:



Let $u \in V(H)$. If *u* is adjacent to a vertex in A_j, for *j* = 1, 2, 3, then *u* can not be adjacent to any vertex in $A_{j+2} \cup A_{j-2}$, and to x_{j+1} and x_{j-1}.

Thus, $\varepsilon(\{u\},T_1) \le 5$. Consequently, $\varepsilon(H,T_1) \le 5\,(n-18)$. Further, $\varepsilon(T_1,\, C_3) \le 15$.

By Theorem 1.2, we have $\varepsilon(T_1) \le \left\lfloor \frac{13^2}{4} \right\rfloor + 3$. Note that, $\varepsilon(H,\, C_3) + \varepsilon(C_3)$ $\le 3(n-20)+3$. Now,

$$\begin{aligned}\varepsilon(G) &= \varepsilon(H) + \varepsilon(H,T_1) + \varepsilon(T_1) + \varepsilon(H,C_3) + \varepsilon(C_3) + \varepsilon(T_1,C_3) \\ &\le \left\lfloor \frac{(n-18)^2}{4} \right\rfloor + 5\,(n-18) + \left\lfloor \frac{13^2}{4} \right\rfloor + 3 + 3(n-18) + 3 + 15. \qquad \text{(Lemma 1.1)} \\ &< \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n.\end{aligned}$$

This completes the proof.

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case where G have a K_3 as a subgraph, $G - K_3$ is a complete a bipartite graph $K_{\left\lfloor \frac{n-3}{2} \right\rfloor, \left\lceil \frac{n-3}{2} \right\rceil}$ and $\varepsilon(K_3,\; G-K_3) = 2(n-3)$. This gives rise to the graph $\Omega(n) = K_{1,\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil}$.

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