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ON SOME PROPERTIES FOR NEW GENERALIZED DERIVATIVE OPERATOR

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Abstract. Motivated by many well-known differential operators, we introduce a new generalized derivative operator and study its characterization properties. In addition, we determine conditions under which the partial sums of this operator of bounded turning are also of bounded turning.

1. Introduction

The theory of univalent function is a beautiful subject as we can see in recent years, many new articles are written in this area. This field which is often associated with geometry and analysis has raised the interest of many since the beginning of 20th century to recent times. The name univalent functions or schlicht (the German word for simple) functions is given to functions defined on the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ that are characterized by the fact that such a function provides one-to-one mapping onto its image. Geometrically, the function f is univalent if $f(z_1) = f(z_2)$ implies $z_1 = z_2$ in $\mathbb{U}$ and is locally univalent at $z_0 \in \mathbb{U}$ if it is univalent in some neighborhood of z_0. The Koebe function

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$k(z) = z/(1-z)^2$ is a univalent function. In fact, the Koebe function and its rotations $e^{-i\gamma}k(e^{i\gamma}z), \gamma \in R$ are the only extremal functions for various problems. The famous findings of Bieberbach conclude the claims given. In brief, Bieberbach [1] proved that if f is normalized by $f(0) = 0$ and $f^{'}(0) = 1$ then the second coefficient $|a_2| \leq 2$ with equality if and only if f is a rotation of the Koebe function. He also conjectured that$|a_n| \leq n, (n = 2, 3,)$ which is generally valid and this was proved by de Branges in 1985. Now operators of normalized analytic functions become very popular, namely for differential and integral. Many articles discuss on operators and new generalizations of various authors. To our best knowledge, perhaps Ruscheweyh [4] was the first in 1975 to introduce the differential operator and followed by Salagean [5] in 1985. These two operators were quite a while being used to study different properties and problems involving subclasses of univalent functions. In 2004, Al-Oboudi [7] generalized Salagean operator and followed by Shaqsi and Darus ([6],[8]) generalized both differential of Ruscheweyh and Salagean in 2008. Many authors started to introduce new operator in their own style based on the Salagean and Ruscheweyh operators. For example see ([10], [11]).

We use these operators to find another type of differential operator and obtain certain conditions on bounded turning. In addition, we obtain the Cesáro means for the operator defined. This type of problems can be seen in various work for example (see [11],[12],[13]).

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$.

Let A denote the subclass of H consisting of functions normalized by
$f(0) = 0, f'(0) = 1$.

Let the functions f given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathbb{U}).$$

Then the Hadamard product (convolution) of f and g, defined by :

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U}).$$

Now, $(c)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by
$(c)_k$

$$= \begin{cases} 1 \qquad for \quad k = 0, \\ c(c+1)(c+2)..(c+k-1) \qquad for \quad k \in \mathbb{N} = \{1, 2, 3, ...\}, c \in \mathbb{C} - \{0\}. \end{cases}$$

In order to derive our new generalized derivative operator, we define the analytic function

$$\varphi^m(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1) + l)^{m-1}}{(1+l)^{m-1}(1 + \lambda_2(k-1))^m} z^k, \tag{1.2}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\lambda_2, \lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$.

Now, we introduce the new generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ as the following:

Definition 1.1. For $f \in A$ the operator $I^m(\lambda_1, \lambda_2, l, n)$ is defined by
$I^m(\lambda_1, \lambda_2, l, n) : A \to A$

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = \varphi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z) \qquad (z \in \mathbb{U}), \tag{1.3}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [4], and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, (n \in \mathbb{N}_0, z \in \mathbb{U}),$$

where $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If f is given by (1.1), then we easily find from the equality (1.3) that

$$I^m(\lambda_1, \lambda_2, l, n) f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1) + l)^{m-1}}{(1+l)^{m-1}(1 + \lambda_2(k-1))^m} c(n,k) a_k z^k,$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes:

• the Ruscheweyh derivative operator [4] in the cases:

$$I^1(\lambda_1, 0, l, n) \equiv I^1(\lambda_1, 0, 0, n) \equiv I^1(0, 0, l, n) \equiv I^0(0, \lambda_2, 0, n)$$

$$\equiv I^0(0, 0, 0, n) \equiv I^{m+1}(0, 0, l, n) \equiv I^{m+1}(0, 0, 0, n) \equiv R^n,$$

•the Salagean derivative operator [5]:

$$I^{m+1}(1, 0, 0, 0) \equiv S^n,$$

• the generalized Ruscheweyh derivative operator [6]:

$$I^2(\lambda_1, 0, 0, n) \equiv R^n_\lambda,$$

•the generalized Salagean derivative operator introduced by Al-Oboudi [7]:

$$I^{m+1}(\lambda_1, 0, 0, 0) \equiv S^n_\beta,$$

•the generalized Al-Shaqsi and Darus derivative operator[8]:

$$I^{m+1}(\lambda_1, 0, 0, n) \equiv D^n_{\lambda,\beta},$$

• the Al-Abbadi and Darus generalized derivative operator [9]:

$$I^m(\lambda_1, \lambda_2, 0, n) \equiv \mu^{n,m}_{\lambda_1,\lambda_2},$$

and finally

• the Catas drivative operator [10]:

$$I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda, \beta, l).$$

Using simple computation one obtains the next result.

$$(l+1)I^{m+1}(\lambda_1, \lambda_2, l, n)f(z) = (1+l-\lambda_1)[I^m(\lambda_1, \lambda_2, l, n) * \varphi^1(\lambda_1, \lambda_2, l)(z)]f(z)+$$

$$\lambda_1 z[(I^m(\lambda_1, \lambda_2, l, n) * \varphi^1(\lambda_1, \lambda_2, l)(z)]'. \tag{1.4}$$

Where $(z \in \mathbb{U})$ and $\varphi^1(\lambda_1, \lambda_2, l)(z)$ analytic function and from (1.2) given by

$$\varphi^1(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k.$$

For $0 \le \beta < 1, \alpha > 0$, let $\Omega(\beta)$ denote the class of functions f of the form (1.1) so that $\mathrm{Re}\{(\frac{f(z)}{z})^\alpha\} > \beta$ in $\mathbb{U}$. The functions in $\Omega(\beta)$ are called functions of bounded turning(cf. [3],vol II).

By the Nashiro-Warschowski theorem (see, e.g, [3], vol I), the functions in $\Omega(\beta)$ are univalent in $\mathbb{U}$.

The $v-th$ partial sums $F_v(z)$ of the operator (1.2) are given by

$$F_v(z)=z+\sum_{k=2}^{v}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_kz^k,(z\in\mathbb{U}). \tag{1.5}$$

We show that if f of the form (1.1) and belongs to the class $\Omega(\beta)$,
i.e. $\text{Re}\left\{\left(\frac{f(z)}{z}\right)^\alpha\right\}>\beta$ then $F_v(z)$ also belong to the class $\Omega(\beta)$, i.e.

$$\text{Re}\left\{\left(\frac{F_v(z)}{z}\right)^\alpha\right\}>\beta,\quad 0\le\beta<1,\alpha>0,v=1,2,\cdots.$$

For this purpose,to prove our results we will need the following three lemmas.

Lemma 1.1. [2] *For $z\in\mathbb{U}$ we have*

$$Re\left\{\sum_{k=1}^{j}\frac{z^k}{k+2}\right\}\ge-\frac{1}{3}.$$

Lemma 1.2. [3] *Let $P(z)$ be analytic in $\mathbb{U}$, such that $P(0)=1$ and let $Re(P(z))>1/2$ in $\mathbb{U}$. For the function Qanalytic in $\mathbb{U}$, the convolution function $P*Q$ takes values in the convex hull of the image on $\mathbb{U}$ under Q.*

2. Main Results

By making use Lemma 1.1 and Lemma 1.2, we illustrate the conditions under which the $v-th$ partial sums (1.5) of the functions in $\Omega(\beta)$ of bounded turning are also of bounded turning.

Theorem 2.1. *Let the function $f(z)=z+a_2z^2+\cdots$ belongs to the class $\Omega(\beta)$. If $\frac{1}{4}<\beta<1,\alpha>0$, then $F_v(z)\in\Omega(\frac{3-(1-\beta)^\alpha}{3})$.*

Proof. Let f be of the form (1.1) and belong to $\Omega(\beta)$ for $\frac{1}{4} < \beta < 1$.

Since $\operatorname{Re}\{(\frac{f(z)}{z})^{\alpha}\} > \beta$, we have

$$\operatorname{Re}\left\{\left(1+\sum_{k=2}^{\infty} a_k z^{k-1}\right)^{\alpha}\right\} > \beta > \frac{1}{2},$$

$$\operatorname{Re}\left\{\left(\frac{1+\sum_{k=2}^{\infty} a_k z^{k-1}}{1-\beta}\right)^{\alpha}\right\} > \operatorname{Re}\left\{\left(1+\sum_{k=2}^{\infty} a_k z^{k-1}\right)^{\alpha}\right\} > \beta > \frac{1}{2},$$

and then

$$\operatorname{Re}\left\{\left(\frac{1+\sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^{k-1}}{1-\beta}\right)^{\alpha}\right\} > \frac{1}{2}.$$

Applying the convolution properties of power series to $\left(\frac{F_v(z)}{z}\right)^{\alpha}$, we may write

$$\left(\frac{F_v(z)}{z}\right)^{\alpha} = \left(1+\sum_{k=2}^{v} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^{k-1}\right)^{\alpha}$$

$$= \left(1+\sum_{k=2}^{v} \frac{1}{(1-\beta)} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^{k-1}\right)^{\alpha}$$

$$* \left(1+\sum_{k=2}^{v} (1-\beta)^{\alpha} z^{k-1}\right) = P(z) * Q(z). \tag{2.1}$$

From Lemma 1.1 for $j = v-1$, we obtain

$$\operatorname{Re}\left\{\sum_{k=1}^{v} \frac{z^k}{k+2}\right\} \geq -\frac{1}{3},$$

$$\operatorname{Re}\left\{\sum_{k=1}^{v} z^{k-1}\right\} \geq \operatorname{Re}\left\{\sum_{k=1}^{v} \frac{z^k}{k+2}\right\} \geq -\frac{1}{3}. \tag{2.2}$$

Applying a simple algebra to inequality (2.2) and $Q(z)$ in (2.1) yields

$$\begin{aligned}\operatorname{Re}\{Q(z)\} &= \operatorname{Re}\left\{1+\sum_{k=2}^{v}(1-\beta)^{\alpha}z^{k-1}\right\}\\ &> 1+((1-\beta)^{\alpha})(\frac{-1}{3})\\ &= \frac{3-(1-\beta)^{\alpha}}{3}.\end{aligned}$$

On the other hand, the power series $P(z)$ in (2.1) yield $\operatorname{Re}(P(z)) > 1/2$. Therefore, by Lemma 1.2,

$$\operatorname{Re}\{P(z)\} = \operatorname{Re}\left\{\left(1+\sum_{k=2}^{v}\frac{1}{(1-\beta)^{\alpha}}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^{m}}c(n,k)a_k z^{k-1}\right)^{\alpha}\right\} > \frac{1}{2},$$

for $(z \in \mathbb{U})$. Thus

$$\operatorname{Re}\left\{\left(\frac{F_v(z)}{z}\right)^{\alpha}\right\} > \frac{3-(1-\beta)^{\alpha}}{3}.$$

This concludes the Main Theorem. □

Next we determine the bounded turning for the Cesáro sums of order v.

From the partial sum

$$s_v(z) = z + \sum_{k=2}^{v} a_k z^k, (z \in \mathbb{U}),$$

with $s_1(z) = z$ we construct the Cesáro means $\sigma_v(z)$ of $f \in A$ by

$$\begin{aligned}
\sigma_v(z,f) &= \frac{1}{v}\sum_{k=1}^{v} s_k(z) \\
&= \frac{1}{v}\left[s_1(z)+...+s_v(z)\right] \\
&= \frac{1}{v}\left[z+(z+a_2z^2)+...+(z+...+a_vz^v)\right] \\
&= \frac{1}{v}\left[vz+(v-1)a_2z^2+...+a_vz^v)\right] \\
&= z+\sum_{k=2}^{v}\left(\frac{v-k+1}{v}\right)a_kz^k \\
&= f(z)*\left[z+\sum_{k=2}^{v}\left(\frac{v-k+1}{v}\right)z^k\right] \\
&= f(z)*g_v(z),
\end{aligned}$$

where

$$g_v(z)=z+\sum_{k=2}^{v}\left(\frac{v-k+1}{v}\right)z^k.$$

Now, we have the following result:

Theorem 2.2. *Let the function* $f(z)=z+a_2z^2+\cdots$ *belongs to the class* $\Omega(\beta)$, *If* $\frac{1}{4}<\beta<1$, *then* $\sigma_v(z,I^m(\lambda_1,\lambda_2,l,n)f(z))\in\Omega(\frac{3v^\alpha-(1-\beta)^\alpha}{3v^\alpha})$.

Proof. $\left(\frac{\sigma_v(z,I^m(\lambda_1,\lambda_2,l,n)f(z))}{z}\right)^\alpha(z)$

$$=\left(1+\sum_{k=2}^{v}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(\frac{v-k+1}{v})a_kz^{k-1}\right)^\alpha$$

$$=\left(1+\sum_{k=2}^{v}\frac{1}{(1-\beta)}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_kz^{k-1}\right)^\alpha$$

$$* \left(1 + \sum_{k=2}^{v} (1-\beta)^{\alpha} (\frac{v-k+1}{v})^{\alpha} z^{k-1} \right)$$
$$= P(z) * Q(z).$$

Thus as $k \to v$, a small computation gives

$$\mathrm{Re}\{Q(z)\} = \mathrm{Re}\ \left\{ 1 + \sum_{k=2}^{\infty} \frac{(1-\beta)^{\alpha}}{v^{\alpha}} z^{k-1} \right\} > \frac{3v^{\alpha} - (1-\beta)^{\alpha}}{3v^{\alpha}}.$$

This ends the proof. □

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