

EXTREME TYPE POINTS IN CERTAIN BANACH SPACES

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ABSTRACT. In this paper, we introduce a new class of boundary points of the unit ball of Banach spaces. Such points are very close to being extreme points. We characterize such points in certain classical Banach spaces and some operator spaces.

1. INTRODUCTION

Let X be a Banach space, and $B_1[X]$ be the closed unit ball of X . An element $x \in B_1[X]$ is called extreme point if whenever $x = \frac{1}{2}(y + z)$,

with $y, z \in B_1[X]$, then $x = y = z$. Extreme points proved to be very important in the study of geometry of Banach spaces. In fact in the study of Isometries of operator spaces, the characterization of extreme points is very essential. We refer to [1], [2] and [3] on results on extreme points of certain Banach spaces.

However, there are Banach spaces, like $L^1[0, 1]$ and c_0 , whose unit balls have no extreme points. This raises the question "does there exists a class of points in the unit ball whose elements are close of being extreme?" It is the object of this paper,

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to introduce an extreme type points to be called "almost extreme points of class k " and give examples of such points in many classical spaces.

2. ALMOST EXTREME POINTS

In this section we define almost extreme points where k is a natural number.

Definition 2.1. . Let X be a Banach space and k be any natural number. A point $x \in S_1[X]$ is called an almost extreme point of class $k > 0$, if there is no $y \in X$ such that $\|y\| = \frac{1}{k}$, and $\|x \pm y\| = 1$.

In general a point $x \in S_m[X]$ is called an almost extreme point of class k if there is no $y \in X$ such that $\|y\| = \frac{1}{k}$, and $\|x \pm y\| = m$.

Almost extreme points are such points that are close to be extreme points. It is like a measure how far our point from being extreme.

Now let $\ell_2^\infty = (\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|(a, b)\|_\infty = \max\{|a|, |b|\}$.

Lemma 2.1. $(0, 1), (1, 0), (-1, 0), (0, -1)$ are not an almost extreme point of class k of $B_1(\ell_2^\infty)$ for any k .

Proof. We prove it for $(0, 1)$.

Take $y = (\frac{1}{k}, 0)$, so $\|y\| = \frac{1}{k}$.

Now $\|x \pm y\|_\infty = 1$. This implies that x is not an almost extreme point of class k for any k .

Similarly for $(1, 0), (-1, 0), (0, -1)$.

It should be remarked that the four points $(0, 1), (1, 0), (-1, 0), (0, -1)$ are the middle points of the extreme points $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$ of the unit ball of ℓ_2^∞ . □

Theorem 2.1. *If $x \in \ell_2^\infty$ such that $x = (\frac{n-1}{n}, 1)$, then x is an almost extreme point of class k for all k such that $1 \leq k \leq n-1$ only.*

Proof. First, let $k < n$.

Take any $y = (\alpha, \beta)$ such that $\|y\|_\infty = \frac{1}{k}$.

There exist 4 cases:

(1) $\alpha = \frac{1}{k}$, $|\beta| \leq \frac{1}{k}$. Then

$$\|x + y\|_\infty = \left\| \left(\frac{n-1}{n} + \alpha, 1 + \beta \right) \right\|_\infty > 1$$

since

$$\alpha = \frac{1}{k} > \frac{n-1}{n}$$

$$\frac{n-1}{n} + \alpha > \frac{n-1}{n} + \frac{1}{n} = 1.$$

(2) $\alpha = -\frac{1}{k}$, $|\beta| \leq \frac{1}{k}$. Then

$$\|x - y\|_\infty = \left\| \left(\frac{n-1}{n} - \alpha, 1 - \beta \right) \right\|_\infty > 1.$$

(3) $|\alpha| \leq \frac{1}{k}$, $\beta = \frac{1}{k}$. Then

$$\|x + y\|_\infty = \left\| \left(\frac{n-1}{n} + \alpha, 1 + \beta \right) \right\|_\infty > 1, \text{ since } 1 + \beta > 1.$$

(4) $|\alpha| \leq \frac{1}{k}$, $\beta = -\frac{1}{k}$.

$$\|x - y\|_\infty = \left\| \left(\frac{n-1}{n} - \alpha, 1 - \beta \right) \right\|_\infty > 1, \text{ since } 1 - \beta > 1.$$

So, $\|x \pm y\|_\infty \neq 1$ in all cases. Hence, x is an almost extreme point of class k for all k such that $1 \leq k \leq n-1$.

If $k \geq n$, then we can take $y = (\alpha, 0)$ where $\alpha = \frac{1}{k}$.

$$\|x + y\|_\infty = \left\| \left(\frac{n-1}{n} + \alpha, 1 \right) \right\|_\infty = 1, \text{ because}$$

$$\alpha = \frac{1}{k} \leq \frac{1}{n}.$$

Consequently,

$$\frac{n-1}{n} + \alpha \leq \frac{n-1}{n} + \frac{1}{n} = 1.$$

Hence

$$\|x - y\|_{\infty} = \left\| \left(\frac{n-1}{n} - \alpha, 1 \right) \right\|_{\infty} = 1, \text{ since}$$

$$- \frac{1}{k} < \frac{1}{k} \leq \frac{1}{n}$$

and so

$$\frac{n-1}{n} - \frac{1}{k} < \frac{n-1}{n} + \frac{1}{k} \leq \frac{n-1}{n} + \frac{1}{n} = 1.$$

Similarly one can prove the same result for $x = (1, \frac{n-1}{n})$. \square

As a consequence of Theorem 1.3 we get $(\frac{9}{10}, 1)$ is an almost extreme point of class k for $k = 1, 2, \dots, 9$ —extreme but not an almost extreme point of class k for $k \geq 10$.

Let $\ell_2^1 = (\mathbb{R}^2, \|\cdot\|_1)$, where $\|(a, b)\|_1 = |a| + |b|$

Theorem 2.2. *If $x \in \ell_2^1$ and x is any of the points $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$ then x is not an almost extreme point of class k for all k .*

Proof. Let us prove the result for $(\frac{1}{2}, \frac{1}{2})$.

For any $k \in \mathbb{N}$, let $y = (\frac{1}{2k}, -\frac{1}{2k})$. Then $y \in \ell_2^1$ and $\|y\|_1 = \frac{1}{k}$.

$$\begin{aligned} \text{Now, } \|x + y\|_1 &= \left| \frac{1}{2} + \frac{1}{2k} \right| + \left| \frac{1}{2} - \frac{1}{2k} \right| \\ &= \frac{1}{2} + \frac{1}{2k} + \frac{1}{2} - \frac{1}{2k} = 1 \quad (\text{since } \frac{1}{2} \geq \frac{1}{2k}, \text{ so } \left| \frac{1}{2} - \frac{1}{2k} \right| = \frac{1}{2} - \frac{1}{2k}) \end{aligned}$$

Further,

$$\begin{aligned} \|x - y\|_1 &= \left| \frac{1}{2} - \frac{1}{2k} \right| + \left| \frac{1}{2} + \frac{1}{2k} \right| \\ &= \frac{1}{2} - \frac{1}{2k} + \frac{1}{2} + \frac{1}{2k} \quad (\text{since } \frac{1}{2} \geq \frac{1}{2k}, \text{ then } \left| \frac{1}{2} - \frac{1}{2k} \right| = \frac{1}{2} - \frac{1}{2k}). \end{aligned}$$

$$= 1.$$

Consequently $\|x \pm y\|_1 = 1$. That is x is not an almost extreme point of class k for all k .

This ends the proof. \square

Again, we should remark that the points under consideration are the middle points of the extreme points $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$, of the unit ball of ℓ_2^1 .

Lemma 1.2 and Theorem 1.4 suggests the following conjecture.

Conjecture. Let X be any Banach space, and $x \in B_1[X]$. If x is the middle point of two extreme points of $B_1[X]$, then x is not an almost extreme point of class k for any k .

3. AN ALMOST EXTREME POINT OF CLASS r - EXTREME POINT

In this section, r is any positive real number.

Definition 3.1. Let X be a normed space and $r > 0$. A point $x \in S_1[X]$ is called an almost extreme point of class k if there is no $y \in X$ such that $\|y\| = r$, and $\|x \pm y\| = 1$.

In general a point $x \in S_m[X]$ is called an almost extreme point of class k point if there is no $y \in X$ such that $\|y\| = r$, and $\|x \pm y\| = m$.

We let $r - ext(X)$ be the set of all almost extreme point of class k in $S_1[X]$.

Lemma 3.1. Let X be a Banach space and $x \in B_1[X]$. Then x is an extreme point of $B_1[X]$ if and only if x is an almost extreme point of class r for all $r > 0$.

Proof. Let x be an extreme point of $B_1[X]$ but not an almost extreme point of class r for some r_o . Then there exist $y \in X$ such that $\|y\| = r$ and $\|x \pm y\| = 1$. But this contradicts the assumption that x is an extreme point since in such a case $x = \frac{1}{2}[(x + y) + (x - y)]$. So x is an almost extreme point of class r for all $r > 0$.

For the converse, let x be an almost extreme point of class k for all $r > 0$. If x is not extreme then there is $y \in B_1[X]$ such that $\|x \pm y\| = 1$. This contradicts that x is an almost extreme point of class k for all $r \in \mathbb{R}$. So x is an extreme point.

This ends the proof. \square

Let $S_m[X] = \{x \in X : \|x\| = m\}$.

Theorem 3.1. *Let X be a normed space. If $x \in S_m[X]$ is not an almost extreme point of class m , then x is not an almost extreme point of class k for any $r \in (0, m)$.*

Proof. Let $x \in S_m[X]$ such that x is not an almost extreme point of class m . Then there exists $y \in S_m[X]$ such that $\|x \pm y\| = m$.

Let $z = \frac{r}{m}y$. So $\|z\| = \frac{r}{m}\|y\| = r$.

$$\begin{aligned} \|x - z\| &= \left\|x - \frac{r}{m}y\right\| = \left\|x - \frac{r}{m}x + \frac{r}{m}x - \frac{r}{m}y\right\| \\ &\leq \left\|x - \frac{r}{m}x\right\| + \left\|\frac{r}{m}x - \frac{r}{m}y\right\| \\ &\leq \left\|(1 - \frac{r}{m})x\right\| + \left\|\frac{r}{m}(x - y)\right\| \\ &= (1 - \frac{r}{m})\|x\| + \frac{r}{m}\|x - y\| \quad (\text{since } 0 < \frac{r}{m} < 1) \\ &= (1 - \frac{r}{m})m + \frac{r}{m}m \\ &= m - r + r = m. \end{aligned}$$

Now,

$$\begin{aligned} \|x - z\| &= \left\|x + \frac{r}{m}x - (\frac{r}{m}x + \frac{r}{m}y)\right\| \geq \left| \left\|(1 + \frac{r}{m})x\right\| - \left\|\frac{r}{m}(x + y)\right\| \right| \\ &= \left| (1 + \frac{r}{m})m - \frac{r}{m}m \right| \\ &= m + r - r = m. \end{aligned}$$

So $\|x - z\| = m$. Similarly $\|x + z\| = m$. Hence x is not an almost extreme point of class k for any $r \in (0, m)$.

This ends the proof. \square

Theorem 3.2. *If $x \in S_1[\ell_2^\infty]$ such that $x = (\beta, 1)$, with $0 < \beta \leq 1$, then x is an almost extreme point of class r for all r such that $1 - \beta < r < 1$.*

Proof. Let there is $y = (z, w)$ such that $\|y\|_\infty = r$. Then there are four cases:

(1) $z = r, |w| \leq r$.

$\|x + y\|_\infty = \|(\beta + r, 1 + w)\|_\infty > 1$, since $z = r > 1 - \beta$, and so $r + \beta > 1$.

(2) $z = -r, |w| \leq r$.

$\|x - y\|_\infty = \|(\beta - z, 1 - w)\|_\infty > 1$, since $z = -r < \beta - 1$, and so $r + \beta > 1$.

(3) $|z| \leq r, w = r$.

$\|x + y\|_\infty = \|(\beta + z, 1 + w)\|_\infty > 1$, since $1 + w = 1 + r > 1$.

(4) $|z| \leq r, w = -r$.

$\|x - y\|_\infty = \|(\beta - z, 1 - w)\|_\infty > 1$, since $1 - w = 1 + r > 1$.

So $x = (\beta, 1)$ is an almost extreme point of class r such that $1 - \beta < r < 1$.

This ends the proof. \square

Similarly one can prove the same result for $x = (\beta, -1)$, $x = (1, \beta)$, $x = (-1, \beta)$.

Lemma 3.2. *If $x \in S_1[\ell_2^\infty]$ and $x = (\beta, 1)$, then x is not an almost extreme point of class r for all r such that $r \leq 1 - \beta$.*

Proof. Take $y = (r, 0)$. Then $\|x + y\|_\infty = \|(\beta, 1) + (r, 0)\|_\infty = 1$ because $r \leq 1 - \beta$, and so $\beta + r \leq 1$.

Similarly $\|x - y\|_\infty = \|(\beta, 1) - (r, 0)\|_\infty = 1$ since $0 < r$, and so $\beta - r < \beta < 1$.

Hence $\|x \pm y\|_\infty = 1$. That is x is not an almost extreme point of class r .

This ends the proof. \square

Theorem 3.3. *Let $0 < r < 1$. Then almost extreme points of class r in $B_1[c_0] = \emptyset$.*

Proof. Let $(a_n) \in c_0$ such that $\|(a_n)\|_\infty = 1$. Then $\lim_{n \rightarrow \infty} a_n = 0$. This implies there exist n_0 such that $a_{n_0} < \frac{r}{10}$, and $r \pm \frac{r}{10} < 1$. Take

$$y = (0, 0, \dots, \underset{n_0\text{-coordinate}}{r}, 0, 0, \dots), \text{ so } y \in c_0 \text{ and } \|y\|_\infty = r.$$

Now $\|a \pm y\|_\infty = \left\| (a_1, a_2, \dots, r \pm \frac{r}{10}, a_{n_0+1}, \dots) \right\|_\infty = 1$. This implies almost extreme points of class r in $B_1[c_0] = \emptyset$

This ends the proof. \square

Lemma 3.3. *Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $f(t_o) = 0$ for some $t_o \in [0, 1]$ and $\|f\|_\infty = 1$. Then f is not an almost extreme point of class 1.*

Proof. Define $g : [0, 1] \rightarrow [0, 1]$ such that $g(t) = 1 - f(t)$. Then g is continuous and $\|g\|_\infty = 1$ because $g(t_o) = 1 - f(t_o) = 1$ and $g(t) \geq 0$.

Now,

$$\|f + g\|_\infty = \|f + 1 - f\|_\infty = 1.$$

$$\|f - g\|_\infty = \|2f - 1\|_\infty = \sup_{t \in [0, 1]} |(2f - 1)(t)|.$$

Now, $\sup_{t \in [0, 1]} |2f(t) - 1| = 1$ since $\|f\|_\infty = 1$ and $f(t) \geq 0$. Hence $\exists t_o \in [0, 1]$ such that $f(t_o) = 1$. So $|2f(t_o) - 1| = 1$.

This ends the proof. \square

We should remark that $f(x) = x^2$ is not an almost extreme point of class r for any $r \in (0, 1)$.

Problem A. Let $f \in C[0, 1]$, and $f(x) = \alpha$, such that $0 < \alpha < 1$, for all $x \in [0, 1]$. Must f be an almost extreme point of class r for some $r \in (1 - \alpha, 1)$

Let ℓ_n^p be R^n with the norm $\|(x_1, \dots, x_n)\| = (\sum |x_i|^p)^{\frac{1}{p}}$. For any Banach space X , we let $L(X)$ denote the space of all bounded linear operators from X to X .

Theorem 3.4. *Let $T \in S_1[L(\ell_n^p)]$, $1 < p < \infty$, such that $T = \sum_{i=1}^n \delta_i \otimes w_i$, with $\{w_1, w_2, \dots, w_n\}$ are linearly independent. Then T is an almost extreme point of class 1.*

Proof. Let T be not an almost extreme point of class 1. Then there is some $S \in S_1[L(\ell_n^p)]$ such that $\|T \pm S\| = 1$.

Since T and S are operators on a finite dimension Banach space, then T and S attain their norms.

That is there exist $x, y \in S_1[\ell_n^p]$ such that $\|T(x)\| = \|S(y)\| = 1$.

But $\|Ty \pm Sy\| = \|(T \pm S)y\| \leq \|(T \pm S)\| \|y\| = \|(T \pm S)\| \leq 1$. Since

$\|S(y)\| = 1$, then the uniform convexity of ℓ_n^p implies that $\|Ty\| = 0$. So

$Ty = \sum_{i=1}^n \langle \delta_i, y \rangle w_i = 0$. Since the w_i 's are linearly independent, it follows that $\langle \delta_i, y \rangle = 0$ for all $i = 1, \dots, n$. This implies that $y = 0$, which contradicts the assumption on y . So T is an almost extreme point of class 1.

This ends the proof. □

Theorem 3.5. *Let $T = \sum_{i=1}^k \delta_i \otimes u_i$, with $\|T\| = 1$, $\{u_1, u_2, \dots, u_n\}$ independent, $k \leq n$ and $\cup_{i=1}^k \text{supp}(u_i) = \{1, 2, \dots, n\}$. Then T is 1-extreme operator in $S_1[L(\ell_n^p)]$ where $2 < p < \infty$.*

Proof. Assume if possible that T is not an almost extreme point of class 1. Then there exist $S \in S_1[L(\ell_n^p)]$ such that $\|T \pm S\| = 1$. Being operators on finite dimensional normed space, there exist $x, y \in S_1[\ell_n^p]$ such that $\|T(x)\| = \|S(y)\| = 1$.

But then $T(y) = S(x) = 0$ because of the uniform convexity of ℓ_n^p and $\|Tx \pm Sx\| = 1 = \|Ty \pm Sy\|$.

Since the u_i^s are independent and $Ty = 0$, it follows that $\text{supp}(y) \subset \{k+1, \dots, n\}$, noting that $(\sum_{i=1}^k \delta_i \otimes u_i)(y) = \sum_{i=1}^k \langle \delta_i, y_i \rangle u_i$. So $\langle \delta_i, y_i \rangle = 0$ for all $i = 1, \dots, k$. Further x and y have disjoint support.

Now, since $\|T \pm S\| = 1$, and x, y have disjoint support, then

$$\begin{aligned} \|(T+S)(x+y)\|^p + \|(T-S)(x+y)\|^p &\leq \|T+S\|^p \|(x+y)\|^p + \|T-S\|^p \|(x+y)\|^p \\ &= 2 \|(x+y)\|^p = 2(\|x\|^p + \|y\|^p). \end{aligned}$$

Hence

$$\|Tx + Sy\|^p + \|Tx - Sy\|^p \leq 2 \|(x+y)\|^p = 2(\|x\|^p + \|y\|^p) = 4.$$

Since $p > 2$, we can use Clarkson's inequalities and use the fact that

$$\|x\| = \|Tx\| = \|Sy\| = \|y\|$$

to get:

$$2(\|Tx\|^p + \|Sy\|^p) \leq \|Tx + Sy\|^p + \|Tx - Sy\|^p \leq 4 = 2(\|Tx\|^p + \|Sy\|^p).$$

This implies that

$$\|Tx + Sy\|^p + \|Tx - Sy\|^p = 2(\|Tx\|^p + \|Sy\|^p).$$

This can happen only if Tx and Sy have disjoint support. Which is not true since $\cup_i \text{supp}(u_i) = \{1, 2, \dots, n\}$. Hence there is no such S , and hence T is an almost extreme point of class 1.

This ends the proof. □

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