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.

$\alpha, \beta, \gamma-$ORTHOGONALITY

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ABSTRACT. Orthogonality in inner product spaces can be expresed using the notion of norms. So many generalization of the concept of orthogonality was made in the context of Banach spaces. In this paper we introduce a new orthogonality relation in normed linear spaces, called $\alpha, \beta, \gamma-$ orthogonality wich generalised most of the known orthogonality. It is shown that $\alpha, \beta, \gamma-$ orthogonality is homogeneus if and only if the space is a real inner product space.

1. INTRODUCTION

In an inner product space $(X, \langle . \rangle)$, it is known that x and y are said to be orthogonal, if $\langle x, y\rangle = 0$. Orthogonality in inner product spaces is a binary relation that can be defined in many ways using the notion of norm. Over the years many mathematicians have tried to generalize this notion to arbitrary Banach spaces $(X, \|.\|)$. In 1934 Roberts [8] defined orthogonality relation for pairs of elements in Banach spaces. Two elements x , $y \in X$ are said to be orthogonal in the sense of Roberts denoted by $(x \perp y)\,(R)$ if and only if $\|x + ky\| = \|x - ky\|$, $\forall k \in \mathbb{R}$. Later in 1935

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, Birkhoff [3] gave another definition as, two elements x and y of X are said to be orthogonal if and only if $\|x\| \leq \|x+\lambda y\|$, $\forall\lambda \in \mathbb{R}$, and we write $(x \perp y)(B)$. In 1945 James [5] introduced Phythagorean and Isosceles orthogonality, two elements x and y of X are said to be orthogonal in Pythagorean sense, $(x \perp y)(P)$, if and only if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$, and they are orthogonal in Isosceles sense, $(x \perp y)(I)$ if and only if $\|x-y\| = \|x+y\|$.

In 1978 Kapoor and Parasad [7] considered $(ab)-$orthogonality. Two vectors x and y of X are said to be orthogonal in (ab) sense, $(x \perp y)(ab)$ if and only if
$\|ax+by\|^2 + \|x+y\|^2 = \|ax+y\|^2 + \|x+by\|^2$, $a, b \in (0,1)$.

In 1983, Andalafte, Diminnie and Freese [4] , generalize Phythagorean and Isosceles orthogonality by introducing $\alpha-$*orthogonality* $\alpha \neq 1$, as two elements x and y of X are said to be $\alpha-$orthogonal,
$(x \perp y)(\alpha)$ if and only if $(1+\alpha^2)\|x-y\|^2 = \|x-\alpha y\|^2 + \|\alpha x - y\|$. Clearly if $\alpha = 0, -1$ we have Phythagorean and Isosceles orthogonality. In (1985) they generalized this notation to $(\alpha, \beta)-$*orthogonality* as two elements x and y of X are said to be $(\alpha, \beta)-$ orthogonal, $(x \perp y)(\alpha, \beta)$ if and only if
$\|x-y\|^2 + \|\alpha x - \beta y\|^2 = \|x-\beta y\|^2 + \|y-\alpha x\|^2$, where $\alpha, \beta \neq 1$ [2] .

In 1988, $a-$Isosceles and $a-$Pythagorean orthogonalities were appeared by J. Alonso and C. Benitez [1], more precisely two elements x and y of X are said to be orthogonal in the $a-$Pythagorean sense, $(x \perp y)(aP)$, if and only if
$\|x-ay\|^2 = \|x\|^2 + a^2\|y\|^2$, and they are orthogonal in the $a-$Isosceles sense, $(x \perp y)(aI)$ if and only if $\|x-ay\| = \|x+ay\|$ for some fixed $a \neq 0$.

In 2010, Khalil and Alkhawalda introduced a new type of orthogonality i. e. distance orthogonality. We say x is distance orthogonal to y, $x \perp^d y$ if and only if, $d(x,[y]) = \|x\|$ and $d(y,[x]) = \|y\|$, where $[x]$ is the span of $\{x\}$ in X. Clearly, distance orthogonality is symmetric if $d(x,[y])$ is uniquely attained at 0 and $d(y,[x])$ is uniquely attained at 0, then x is d^*-orthogonal to y and write $x \perp^{d^*} y$. [6].

The purpose of this paper is to introduce $\alpha, \beta, \gamma-$ orthogonality, by which we generalize all the above mentioned orthogonality. Also we shall show that $\alpha, \beta, \gamma-$orthogonality is homogenous if and only if the space is a real inner product space. Hence if $\alpha, \beta, \gamma-$orthogonality is homogenous, then it is left and right additive.

2. $\alpha, \beta, \gamma-$ Orthogonality.

In this section we introduce $\alpha, \beta, \gamma-$ orthogonality, besides we shall study some properties of $\alpha, \beta, \gamma-$ orthogonality.

Definition 2.1. Let $(X, \|.\|)$ be a normed linear space over the reals. Let α, β, γ be a fixed real numbers such that $(\alpha, \beta, \gamma) \neq (0,0,0)$ and $\alpha \neq 1$, $\beta \neq \gamma$. For $x, y \in X$, we say that x is $(\alpha, \beta, \gamma)-$orthogonal to y, denoted by $(x \perp y)(\alpha, \beta, \gamma)$, if
$\|x - \gamma y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\gamma y - \alpha x\|^2$.

Remark 1. Let X be a Banach space and $x, y \in X$.
1) In the previous definition, the conditions $(\alpha, \beta, \gamma) \neq (0,0,0)$ and $\alpha \neq 1$, $\beta \neq \gamma$ are needed, since $(x \perp y)(0,0,0)$, $(x \perp y)\ (1, \beta, \gamma)$ and $(x \perp y)(\alpha, \beta, \beta)$, for all $x, y \in X$.
2) $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)(\alpha, \gamma, \beta)$.

Clearly, $(0 \perp y)(\alpha, \beta, \gamma)$ and $(x \perp 0)(\alpha, \beta, \gamma)$ for all $x, y \in X$, moreover we have the following lemma.

Lemma 2.1. *Let X be a Banach space and $x, y \in X$. Then*
a) $(x \perp x)(\alpha, \beta, \gamma)$ if and only if $x = 0$.
b) If X is a Hilbert space with inner product $\langle .,. \rangle$ then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $\langle x, y \rangle = 0$.
c) If $(x \perp y)(\alpha, \beta, \gamma)$, and $x \neq 0 \neq y$, then x, y are independent.

Proof. a) Suppose $(x \perp x)(\alpha, \beta, \gamma)$, then
$\left[(1-\gamma)^2 + (\alpha-\beta)^2\right] \|x\|^2 = \left[(1-\beta)^2 + (\gamma-\beta)^2\right] \|x\|^2$, which implies,
$[(1-\alpha)(\beta-\gamma)] \|x\|^2 = 0$, but $1 \neq \alpha, \beta \neq \gamma$, so $\|x\| = 0$.
b) Clearly $(x \perp y)(\alpha, \beta, \gamma)$ implies $\gamma \langle x, y\rangle + \alpha\beta \langle x, y\rangle = \beta \langle x, y\rangle + \gamma\alpha \langle x, y\rangle$ so
$[(1-\alpha)(\beta-\gamma)] \langle x, y\rangle = 0$, hence, $\langle x, y\rangle = 0$. The converse is obvious.
c) Suppose on the contrary that, $x = ry$, $r \in \mathbb{R}$. Since $(x \perp y)(\alpha, \beta, \gamma)$ and $x \neq 0 \neq y$, then $\left[(r-\gamma)^2 + (\alpha r-\beta)^2\right] \|y\|^2 = \left[(r-\beta)^2 + (\alpha r-\gamma)^2\right] \|y\|^2$. Hence
$\left[(r-\gamma)^2 + (\alpha r-\beta)^2\right] = \left[(r-\beta)^2 + (\alpha r-\gamma)^2\right]$, so $r(1-\alpha)(\beta-\gamma) = 0$, which implies $r = 0$, which is a contradiction. □

The following Lemmas give some properties of $\alpha, \beta, \gamma-$ orthogonality.

Lemma 2.2. . *Let X be a Banach space and $x, y \in X$. Then,*
i) If $\alpha, \beta, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(\alpha x \perp \gamma\beta y)\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$.
ii) If $\beta, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp \gamma\beta y)\left(\alpha, \frac{1}{\beta}, \frac{1}{\gamma}\right)$.
iii) If $\alpha, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(\alpha x \perp \gamma^2 y)\left(\frac{1}{\alpha}, \frac{\beta}{\gamma^2}, \frac{1}{\gamma}\right)$.
iv) If $\alpha, \beta \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(\alpha x \perp \beta^2 y)\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{\gamma}{\beta^2}\right)$.
v) If $\alpha \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(\alpha x \perp y)\left(\frac{1}{\alpha}, \beta, \gamma\right)$.
vi) If $\beta \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp \beta y)\left(\alpha, 1, \frac{\gamma}{\beta}\right)$.
vii) If $\gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp \gamma^2 y)\left(\alpha, \frac{\beta}{\gamma^2}, \frac{1}{\gamma}\right)$.

Proof. The prove follows immediately from the definition of (α, β, γ) −orthogonality and since $\alpha \neq 1$, $\beta \neq \gamma$,then in (i) $\alpha \neq 1$ and$\frac{1}{\beta} \neq \frac{1}{\gamma}$, provided $\alpha, \gamma \neq 0$. By similar ideas one may show the other implications. □

Using the ideas in Lemma (2.2) we have,

Lemma 2.3. *Let X be a Banach space and $x, y \in X$. Then,*
i)If $\alpha, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $\left(\gamma y \perp \frac{1}{\gamma} x\right)\left(\frac{\beta}{\gamma}, \alpha\gamma, \gamma\right)$.

ii) If $\alpha, \beta \neq 0$, *then* $(x \perp y)(\alpha, \beta, \gamma)$ *if and only if* $(\beta y \perp \alpha^2 x)\left(\frac{\gamma}{\beta}, \frac{1}{\alpha^2}, \frac{1}{\alpha}\right)$.

In an inner product space it is known that, for all x, y, there exist a constant a such that $\langle x, ax+y\rangle = 0$. This property still valid in $(\alpha, \beta)-$*orthogonality*. Andalafte, Diminnie and Freese, Prove the following theorem.

Theorem 2.1. [2]. *Let X be a Banach space, for every $x, y \in X$, there exists a real constant a such that $(x \perp ax+y)(\alpha, \beta)$.*

Now we shall show that this property is still valid using $(\alpha, \beta, \gamma)-$orthogonality.

Theorem 2.2. *Let X be a Banach space , for all $x, y \in X$, there exist a real number a such that $(x \perp ax+y)(\alpha, \beta, \gamma)$.*

Proof. Let $(\alpha, \beta, \gamma) \neq \{(0,0,0)\}$ such that $\alpha \neq 1, \beta \neq \gamma$, and $x, y \in X$, we have two cases,

1) $\gamma \neq 0$. Then $\alpha, \frac{\beta}{\gamma} \neq 1$, by Theorem (2.1), there exist a real number r such that $(x \perp rx+\gamma y)\left(\alpha, \frac{\beta}{\gamma}\right)$, hence

$$\left\|x-\gamma\left(\frac{r}{\gamma}x+y\right)\right\|^2+\left\|\alpha x-\beta\left(\frac{r}{\gamma}x+y\right)\right\|^2=\left\|x-\beta\left(\frac{r}{\gamma}x+y\right)\right\|^2+\left\|\gamma\left(\frac{r}{\gamma}x+y\right)-\alpha x\right\|^2,$$

this complete the proof, were $a=\frac{r}{\gamma}$.

2) $\gamma = 0$, so $\beta \neq 0$, then there exist a real number r such that $(x \perp rx+\beta y)(\alpha, 0)$. So $\left\|x-\beta\left(\frac{r}{\beta}x+y\right)\right\|^2+\|\alpha x\|^2=\|x\|^2+\left\|\beta\left(\frac{r}{\beta}x+y\right)-\alpha x\right\|^2$. □

3. Homogeneity of $\alpha, \beta, \gamma-$ Orthogonality.

An orthogonality $\perp$ is homogeneous if $x \perp y$ implies $ax \perp by$ for all real constants a, b. It is left additive if $x \perp y$ and $z \perp y$ implies, $x+z \perp y$, and it is right additive if $x \perp y$ and $x \perp z$ implies, $x \perp y+z$. Now let us assume that $(\alpha, \beta, \gamma)-$orthogonality is homogeneous. We have the following,

Lemma 3.1. *Let X be a Banach space and $x, y \in X$. Then,*

i) If $\alpha, \beta, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$.

ii) If $\beta, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\alpha, \frac{1}{\beta}, \frac{1}{\gamma}\right)$.

iii) If $\alpha, \gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\frac{1}{\alpha}, \frac{\beta}{\gamma^2}, \frac{1}{\gamma}\right)$.

iv) If $\alpha, \beta \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{\gamma}{\beta^2}\right)$.

v) If $\alpha \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\frac{1}{\alpha}, \beta, \gamma\right)$.

vi) If $\beta \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\alpha, 1, \frac{\gamma}{\beta}\right)$.

vii) If $\gamma \neq 0$, then $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(x \perp y)\left(\alpha, \frac{\beta}{\gamma^2}, \frac{1}{\gamma}\right)$.

Proof. Let us prove (i), the others are similar. By homogeneity and since $\alpha, \beta, \gamma \neq 0$, we have $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $\left(\frac{x}{\alpha} \perp \frac{y}{\beta\gamma}\right)(\alpha, \beta, \gamma)$ which is by lemma (2.2) equivalent to $(x \perp y)\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$. □

The following proposition, is needed to prove our main result.

Proposition 3.1. *Suppose $(\alpha, \beta, \gamma)-$orthogonality is homogeneous, for x, y in the Banach space$X$, we have .*

i) If $\alpha \neq -1$, and $(x \perp y)(\alpha, \beta, \gamma)$, then $\|x - \gamma y\|^2 = (\gamma^2 - \beta^2)\|y\|^2 + \|x - \beta y\|^2$.

ii) If $|\beta| \neq 1$, and $(x \perp y)(\alpha, \beta, \gamma)$, then$\|x - \gamma y\|^2 = (1 - \alpha^2)\|x\|^2 + \|\gamma y - \alpha x\|^2$.

iii) If $|\gamma| \neq 1$, and $(x \perp y)(\alpha, \beta, \gamma)$, then$\|\alpha x - \beta y\|^2 = (\alpha^2 - 1)\|x\|^2 + \|x - \beta y\|^2$.

Proof. I) Suppose $\alpha \neq -1$, and $(x \perp y)(\alpha, \beta, \gamma)$, by lemma (3.1) (i), we may assume $|\alpha| < 1$. If $\alpha = 0$, the result follows. For $\alpha \neq 0$, let $p(n)$be the statement,

$p(n) : \|x - \gamma y\|^2 + \|\alpha^n x - \beta y\|^2 = \|x - \beta y\|^2 + \|\gamma y - \alpha^n x\|^2$, clearly $p(1)$ is true. Suppose $p(n)$ is true, by homogeneity $(\alpha^n x \perp y)(\alpha, \beta, \gamma)$, thus

$\|\alpha^n x - \gamma y\|^2 + \|\alpha^{n+1} x - \beta y\|^2 = \|\alpha^n x - \beta y\|^2 + \|\gamma y - \alpha^{n+1} x\|^2$, adding to $p(n)$, we obtain $\|x - \gamma y\|^2 + \|\alpha^{n+1} x - \beta y\|^2 = \|x - \beta y\|^2 + \|\gamma y - \alpha^{n+1} x\|^2$, which implies

$p(n+1)$, therefor $p(n)$ is true for all $n \in \mathbb{N}$. let $n \to \infty$, by continuity of the norm, the result follows.

(ii) and (iii) can be proved in similar arguments. □

Corollary 3.1. *Let X be a Banach space $x, y \in X$, and $(\alpha, \beta, \gamma)-$orthogonality is homogeneous, if $(x \perp y)(\alpha, \beta, \gamma)$, $|\gamma| \neq 1, \alpha \neq -1$, then $\|x - \beta y\| = \|x + \beta y\|$.*

Proof. Since $|\gamma| \neq 1$, by (3.1), we have $\|\alpha x - \beta y\|^2 = (\alpha^2 - 1)\|x\|^2 + \|x - \beta y\|^2 \,(\star)$. If $\alpha = 0$, the result follows by $(\star)$. For $\alpha \neq 0$, as before we may assume $|\alpha| < 1$. Let $p(n)$ be the statement, $p(n) : \|\alpha^n x - \beta y\|^2 = (\alpha^{2n} - 1)\|x\|^2 + \|x - \beta y\|^2$, clearly $p(1)$ is $(\star)$. Suppose $p(n)$ is true, by homogeneity $(\alpha^n x \perp y)(\alpha, \beta, \gamma)$, thus by $(\star)$ again we have, $\|\alpha^{n+1} x - y\|^2 = (\alpha^2 - 1)\alpha^{2n}\|x\|^2 + \|\alpha^n x - \beta y\|^2$. By substitution in $p(n)$, we obtain $\|\alpha^{n+1} x - y\|^2 = (\alpha^2 - 1)\alpha^{2n}\|x\|^2 + (\alpha^{2n} - 1)\|x\|^2 + \|x - \beta y\|^2$, which is $p(n+1)$, therefor $p(n)$ is true for all $n \in \mathbb{N}$. Let $n \to \infty$. By continuity of the norm, we have $\|x - \beta y\|^2 = \beta^2\|y\|^2 + \|x\|^2$, since $(x \perp -y)(\alpha, \beta, \gamma)$, so
$\|x + \beta y\|^2 = \beta^2\|y\|^2 + \|x\|^2$. □

4. $\alpha, \beta, \gamma-$ Orthogonality and Others.

Now we shall study the relation between $(\alpha, \beta, \gamma)-$orthogonality and the other orthogonality. The proof of the following lemma is clear by the definition.

Lemma 4.1. *Let X be a Banach space and $x, y \in X$. Then,*

1) If $(x \perp y)(\alpha, \beta, 1)$ then $(x \perp y)(\alpha, \beta)$.

2) If $(x \perp y)(\alpha, \alpha, 1)$ then $(x \perp y)(\alpha)$.

3) If $(x \perp y)(0, 0, 1)$ then $(x \perp y)(P)$.

4) If $(x \perp y)(-1, 0, -1)$ then $(x \perp y)(I)$.

5) If $(x \perp y)(-1, 0, a)$ then $(x \perp y)(aI)$.

6) If $(x\perp y)(0,0,a)$ *then* $(x\perp y)(aP)$.

7) If $(x\perp y)(a,b,-1)$ *then* $(x\perp y)(ab)$.

Also we have the following lemma,

Lemma 4.2. *Let* X *be a Banach space and* $x,y\in X$. *Then,*

i) If $(x\perp y)(-1,0,\gamma)$, *for every* $\gamma\neq 0$, *then* $(x\perp y)(R)$.

ii) If $(x\perp y)(0,0,\gamma)$, *for every* $\gamma\neq 0$, *then* $(x\perp y)(B)$.

iii) If $(x\perp y)(0,0,\gamma)$, *for every* $\gamma\neq 0$, *then* $(x\perp y)(d)$.

Proof. The prove of (i) is clear and the prove of (ii) is a part of the prove of (iii). Now suppose $(x\perp y)(0,0,\gamma)$, for every $\gamma\in\mathbb{R}\diagdown\{0\}$, then $\|x-\gamma y\|^2=\|x\|^2+\|\gamma y\|^2\geq\|x\|^2$ $(\star)$, so $\inf_{\lambda\in\mathbb{R}}\|x-\lambda y\|^2=\|x\|^2$. Moreover $(x\perp y)(0,0,\gamma)$, for every $\gamma\neq 0$, if and only if $(x\perp y)\left(0,0,\frac{1}{\lambda}\right)$, for every $\lambda\neq 0$, so $\left\|x-\frac{1}{\lambda}y\right\|^2=\|x\|^2+\left\|\frac{1}{\lambda}y\right\|^2$, which is equivalent to $\|\lambda x-y\|^2=\|\lambda x\|^2+\|y\|^2$, therefore $\inf_{\lambda\in\mathbb{R}}\|\lambda x-y\|^2=\|y\|^2$. □

The above discussion shows, for a certain conditions on α,β,γ, we obtained the mentioned orthogonality. A question is that, if $(x\perp y)(\alpha_0,\beta_0,\gamma_0)$, for some $(\alpha_0,\beta_0,\gamma_0)\neq(0,0,0)$ and $\alpha_0\neq 1$, $\beta_0\neq\gamma_0$, which one of the mentioned orthogonalities is satisfied. To answer this question the homogeneity of $\alpha,\beta,\gamma-$ orthogonality is needed, so from now on we assume that $(\alpha,\beta,\gamma)-$orthogonality is homogeneous. The following lemma shows that $(\alpha,\beta,\gamma)-$orthogonality implies $(\alpha,\delta)-$orthogonality for some δ.

Lemma 4.3. *Let* X *be a Banach space and* $x,y\in X$. *Then, For* $\beta_0\neq 0$, *if* $(x\perp y)(\alpha_0,\beta_0,\gamma_0)$, *then there exist* δ_0, *such that* $(x\perp y)(\alpha_0,\delta_0)$. *Moreover* $(\alpha_0,\delta_0)-$ *orthogonality is homogeneous.*

Proof. Let $\beta_0 \neq 0$, and $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, by (3.1), (vi) $(x \perp y)\left(\alpha_0, 1, \frac{\gamma_0}{\beta_0}\right)$, hence, by (1), (2) $(x \perp y)\left(\alpha_0, \frac{\gamma_0}{\beta_0}, 1\right)$. Thus by (4.1), (1), and homogeneity of $(\alpha_0, \beta_0, \gamma_0)$ $-$orthogonality the result follows, where $\delta_0 = \frac{\gamma_0}{\beta_0}$. □

Andalafte, Diminnie and Freese in [2] Prove the following.

Theorem 4.1. [2]. *If $\alpha, \beta \neq -1$ and $(\alpha, \beta)-$ orthogonality is homogeneous, then $(x \perp y)(\alpha, \beta)$ implies $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.*

Theorem 4.2. [2]. *If $(\alpha, \beta)-$ orthogonality is homogeneous, then $(x \perp y)(\alpha, \beta)$ implies $\|x - y\| = \|x + y\|$.*

Theorem 4.3. [2] .*If $(\alpha, \beta)-$ orthogonality is homogeneous in a normed linear space $(X, \|.\|)$ then $(X, \|.\|)$ is a real inner product space.*

Now we shall proof one of our main theorems in this paper.

Theorem 4.4. *Let X be a Banach space and $x, y \in X$. Then,*
i) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(P)$.
ii) For $\beta_0 \neq 0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(I)$.
iii) For $\beta_0 \neq 0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(R)$.
iv) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(B)$.
v) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(d)$.
vi) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(aP)$
vii) For $\beta_0 \neq 0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(aI)$.
viii) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(\alpha)$.
ix) For $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(\alpha, \beta)$.
x) For $\beta_0 \neq 0$, if $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, then $(x \perp y)(ab)$.

Proof. i) Let $\beta_0 \neq 0$, $\alpha_0 \neq -1$, $\beta_0 \neq -\gamma_0$, and $(x \perp y)(\alpha_0, \beta_0, \gamma_0)$, by Lemma (4.3), $(x \perp y)(\alpha_0, \delta_0)$, where $\alpha_0 \neq -1$, $\delta_0 = \frac{\gamma_0}{\beta_0} \neq -1$, by Theorem(4.1),
$\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

Now to prove the other parts, let $\beta_0 \neq 0$, by Lemma (4.3) we have, $(x \perp y)(\alpha_0, \delta_0)$.

So,

ii) $\|x - y\| = \|x + y\|$, by Theorem (4.2).

iii) Let $\lambda \in \mathbb{R}$, sine $(\alpha_0, \delta_0)-$ orthogonality is homogeneous, then $(x \perp \lambda y)(\alpha_0, \delta_0)$. By (ii) $\|x - \lambda y\| = \|x + \lambda y\|$.

iv) By (iii) $\|x - \lambda y\| = \|x + \lambda y\|$, and by (i) $\|x - \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2$, for all $\lambda \in \mathbb{R}$, so $\|x - \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$.

v) Let $\lambda \in \mathbb{R}$, since $(\alpha_0, \delta_0)-$ orthogonality is homogeneous, then $(x \perp \lambda y)(\alpha_0, \delta_0)$ and $(\lambda x \perp y)(\alpha_0, \delta_0)$, as in (iv) $\|x - \lambda y\| \geq \|x\|$ and $\|\lambda x - y\| \geq \|y\|$ for all $\lambda \in \mathbb{R}$, hence $\inf\limits_{\lambda \in \mathbb{R}} \|x - \lambda y\|^2 = \|x\|^2$ and $\inf\limits_{\lambda \in \mathbb{R}} \|\lambda x - y\|^2 = \|y\|^2$.

vi) By (i) since $(x \perp ay)(\alpha_0, \delta_0)$, $a \in \mathbb{R}$.

vii) By (ii) since $(x \perp ay)(\alpha_0, \delta_0)$, $a \in \mathbb{R}$.

viii) Since $(\alpha_0, \delta_0)-$ orthogonality is homogeneous, then $(x \perp \alpha y)(\alpha_0, \delta_0)$ and $(\alpha x \perp y)(\alpha_0, \delta_0)$, so by (i) $\|x - \alpha y\|^2 = \|x\|^2 + \alpha^2 \|y\|^2$ and $\|\alpha x - y\|^2 = \alpha^2 \|x\|^2 + \|y\|^2$, by summing we have $\|x - \alpha y\|^2 + \|\alpha x - y\|^2 = (\alpha^2 + 1)\left(\|x\|^2 + \|y\|^2\right) = (\alpha^2 + 1)\|x + y\|^2$.

ix) Let $\alpha, \beta \in \mathbb{R}$, by homogeneity of $(\alpha_0, \delta_0)-$ orthogonality, we have $(\alpha x \perp \beta y)(\alpha_0, \delta_0)$, this condition, by (i) implies, $\|\alpha x - \beta y\|^2 = \alpha^2 \|x\|^2 + \beta^2 \|y\|^2$, whence
$\|\alpha x - \beta y\|^2 + \|x - y\|^2 = \left(\alpha^2 \|x\|^2 + \|y\|^2\right) + \left(\|x\|^2 + \beta^2 \|y\|^2\right) = \|\alpha x - y\|^2 + \|x - \alpha y\|^2$.

x) Follows by replacing α, β by a, b and the fact that, $(x \perp y)(\alpha_0, \delta_0)$ if and only if $(x \perp -y)(\alpha_0, \delta_0)$. □

Theorem 4.5. *Let X be a Banach space and $x, y \in X$. Then, $(x \perp y)(\alpha, \beta, \gamma)$ if and only if $(y \perp x)(\alpha, \beta, \gamma)$, that is $\alpha, \beta, \gamma-$ orthogonality is symmetric.*

Proof. Suppose $(x \perp y)(\alpha, \beta, \gamma)$, then by homogeneity

$(\gamma x \perp y)(\alpha, \beta, \gamma)$, $(\beta x \perp \alpha y)(\alpha, \beta, \gamma)$, $(\beta x \perp y)(\alpha, \beta, \gamma)$, and $(\gamma x \perp \alpha y)(\alpha, \beta, \gamma)$, hence by (4.3) (i), we have

1)- $\|\gamma x - y\|^2 = \gamma^2 \|x\|^2 + \|y\|^2$,

2)- $\|\beta x - \alpha y\|^2 = \beta^2 \|x\|^2 + \alpha^2 \|y\|^2$,

3) $\|\beta x - y\|^2 = \beta^2 \|x\|^2 + \|y\|^2$,

4) $\|\gamma x - \alpha y\|^2 = \gamma^2 \|x\|^2 + \alpha^2 \|y\|^2$.

Clearly $(1) + (2) = (3) + (4)$. So $(y \perp x)(\alpha, \beta, \gamma)$. □

By Theorem (4.3), and lemma (4.3) we have the following main theorem.

Theorem 4.6. *If $(\alpha_0, \beta_0, \gamma_0)-$ orthogonality is homogeneous in a normed linear space $(X, \|.\|)$, then $(X, \|.\|)$ is a real inner product space.*

An obvious consequences of (4.6) is the following corollary.

Corollary 4.1. . *If $(\alpha_0, \beta_0, \gamma_0)-$ orthogonality is homogeneous in a normed linear space $(X, \|.\|)$, then it is booth left and right additive..*

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