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TWO-DIMENSIONAL SINGULAR FREDHOLM INTEGRAL EQUATION WITH APPLICATIONS IN CONTACT PROBLEMS

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ABSTRACT. In this paper we discuss the solution of the Two-dimensional singular Fredholm integral equation (T-DFIE). The existence of a unique solution of the T-DFIE is discussed and proved using Banach Fixed point theorem. Then, using Toeplitz matrix method and Product Nystrom method, we obtain a linear algebraic system of equations (LAS). Some numerical cases, in contact problems when the kernel takes Cauchy kernel, logarithmic form and Carleman function, are solved.

1. INTRODUCTION

Many problems in mathematical physics, contact problems in the theory of elasticity, viscodynamics fluid and mixed problems of mechanics of continuous media reduce to the Fredholm integral equation (FIE) of the second kind and the kernel takes one of the following forms see [1, 2, 3]

$$k_{n,m}^{\upsilon,\lambda,\varepsilon}(x,y) = \frac{x^{\lambda}}{y^{\varepsilon+\lambda-1}} W_{n,m}^{\upsilon}(x,y), \quad (0 \le \varepsilon \le 1),$$

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$$W^{\upsilon}_{n,m}(x,y)=\int_0^\infty t^{\upsilon}J_n(tx)J_m(ty)dt, \tag{1.1}$$

where $J_n(z)$ is the Bessel function of the first kind.

Many different cases can be established from the formula (1.1) as:

1- Carleman kernel, $k(x,y)=|x-y|^{-\upsilon}, 0\le\upsilon\le 1, \varepsilon=0, \lambda=\frac{1}{2}, n=m=\pm\frac{1}{2}$,

$$k(x,y)=\sqrt{xy}\int_0^\infty t^{\upsilon}J_{\pm\frac{1}{2}}(tx)J_{\pm\frac{1}{2}}(ty)dt. \tag{1.2}$$

(for symmetric and skew symmetric respectively).

In [4] Artiunian shown the plane contact problem of the nonlinear theory of elasticity, in its first approximation, can be reduced to FIE of the first kind with Carleman kernel for symmetric and skew symmetric cases respectively. The important of Carleman kernel comes from the contact mixed problems in a half space in the theory of elasticity where in the modules of elasticity is changing according to the power law $\sigma_i=k_0\varepsilon_i^{\upsilon}(0\le\upsilon<1)$ where σ_i and $\varepsilon_i, i=1,2,3$ are, respectively, the stress and strain rate intensities, while k_0 and υ are physical constants.

2- Logarithmic kernel, $k(x,y)=-\ln|x-y|, \upsilon=\varepsilon=0, \lambda=\frac{1}{2}, n=m=\pm\frac{1}{2}$,

$$k(x,y)=\sqrt{xy}\int_0^\infty J_{\pm\frac{1}{2}}(tx)J_{\pm\frac{1}{2}}(ty)dt \tag{1.3}$$

(for symmetric and skew symmetric respectively).

Mkhitarian and Abdou [5, 6], using Krein's method, obtained the general formulae for the potential analytic functions of the FIE (even or odd cases) of the first kind with Carleman kernel and logarithmic kernel, respectively. Abdou and Ezz eldin, in [7], represented the plane contact problem in half-plane in the theory of elasticity, as an integral equation of the first kind with logarithmic kernel.

3- Elliptic integral form $k(x,y)=\frac{1}{x+y}E(\frac{2\sqrt{xy}}{x+y}), \lambda=\varepsilon=\upsilon=n=m=0,$

$$k(x,y)=\sqrt{xy}\int_0^\infty J_0(tx)J_0(ty)dt,\ (x,y)\in[a,b]. \tag{1.4}$$

Kovalenko [8] developed the FIE of the first kind in the mechanics mixed problems of continuous media and obtained an approximate solution, when the kernel of integral equation (1.1) satisfies the conditions $\varepsilon=\upsilon=m=0$ and $\lambda=1$.

4-Potential kernel, $k(x-\xi,y-\eta)=[(x-\xi)^2+(y-\xi)^2]^{\frac{-1}{2}}, \lambda=\frac{1}{2}, \varepsilon=\upsilon=0, n=m,$

$$k(x,y)=\sqrt{xy}\int_0^\infty J_m(tx)J_m(ty)dt,\ (m\geq 0) \tag{1.5}$$

5- The Generalized potential kernel,

$$k(x-\xi,y-\eta)=[(x-\xi)^2+(y-\xi)^2]^{-\upsilon}, \varepsilon=0, \lambda=\frac{1}{2}, n=m\geq 0,\ 0\leq\upsilon\leq 1,$$

$$k(x,y)=\sqrt{xy}\int_0^\infty t^\upsilon J_m(tx)J_m(ty)dt,\ (m\geq 0) \tag{1.6}$$

Abdou, in [9], solved the FIE of the second kind when the kernel in the potential function from equation (1.1) satisfies the previous conditions. The forms (1-5) are called Weber-Sonien forms, (see Abdou [7], Abdou and Salama [10, 11]. EL-Borai et al., in [12], studied the numerical solution for the two-dimensional Fredholm integral equation with weak singular kernel, but they have studied the problem on a rectangular path of the parties only. Bukhari in [15] used Numerical methods of singular integral equation for solving some problem in fluid dynamic. Abdou et al., in [16], studied the Toeplitz matrix method and nonlinear integral equation of Hammerstein typ.

2. Basic Equations of Two-Dimensional Singular Fredholm Integral Equation

Consider the three-dimensional semi-symmetric problem (Hertz contact problem) of two rigid surfaces having two different elastic materials occupying the domain

$$\Omega = \{\Omega : (x, y) \in \Omega : \sqrt{x^2 + y^2} \leq a,\ z = 0\}.$$

The problem is investigated from the semi-symmetric contact problem of two different elastic materials in three-dimensional when the modules of elasticity is changing, in the lower layer, according to the power law $\sigma_i = k_0 \varepsilon_i^{\upsilon} (0 \leq \upsilon < 1)$. Assume the upper surface is impressed into the lower surface by a constants force $P < \infty$, whose eccentricity of application e, and the frictional forces in the contact domain Ω between the two surfaces are so small in which it can be neglected. In the absence of the external forces, the components of the stresses and the strain satisfy both Hook's and Lame's laws.

Consider the singular integral equation of the second kind

$$\begin{aligned}(\lambda_1 + \lambda_2)\phi(x, y) - (\theta_1 + \theta_2) \int_\Omega \int_\Omega k(|x - u|; |y - v|)\phi(u, v) du dv \\ = [\delta - f_1(x, y) - f_2(x, y)]\end{aligned} \tag{2.1}$$

under the static condition:

$$\int_\Omega \int_\Omega \phi(x, y) dx dy = P < \infty,\ (P\ is\ a\ const), \tag{2.2}$$

where $f_i(x, y), i = 1, 2$ are the known functions describing the two surface, Ω is the contact domain between the two surfaces, $\phi(x, y)$ are the unknown normal stresses between $f_1(x, y)$ and $f_2(x, y), \lambda_i (i = 1, 2)$ are the coefficients bed of the compressible materials that depend on its geometry and its physical properties, δ is the rigid displacement under the action of a force P (P is a constant), $\theta_i = 1 - \upsilon_i^2/\pi E_i, i = 1, 2$ where υ_i are the Poisson's coefficients and E_i are the coefficients of Young.

We can write equation (2.1) in the form

$$\phi(x, y) - \lambda \int_\Omega \int_\Omega k(|x-u|; |y-v|)\phi(u, v)dudv = f(x, y) \tag{2.3}$$

where,

$$\lambda = \frac{\theta_1 + \theta_2}{\lambda_1 + \lambda_2}, \; f(x, y) = \frac{\delta - f_1(x, y) - f_2(x, y)}{\lambda_1 + \lambda_2}.$$

Equation (2.3) represents a linear FIE of the second kind in two dimensional with the singular kernel. This formula is measured in the space $L_2(\Omega) \times L_2(\Omega)$, where Ω is the domain of integration. λ is a constant, may be complex, that has many physical meaning, the known function $k(|x-u|; |y-v|)$ is called the kernel of integral equation, which has a singular term and $f(x, y)$ is a known continuous function.

In general, we can write integral equation (2.3) in the form:

$$\mu\phi(x, y) - \lambda \int_\Omega \int_\Omega k(|x-u|; |y-v|)\phi(u, v)dudv = f(x, y) \tag{2.4}$$

The constant μ defines the kind of the integral equation.

3. The Existence and Uniqueness Solution

In order to guarantee the existence of a unique solution of equation (2.4), we assume the following conditions:

(i) The kernel $k(|x-u|;|y-v|) \in C([\Omega]\times[\Omega])$ and satisfies the discontinuity condition

$$[\int_\Omega\int_\Omega\int_\Omega\int_\Omega |k(|x-u|;|y-v)||^2\, dxdudydv]^{\frac{1}{2}} = A < \infty \qquad (A\ is\ a\ const)$$

(ii) The given function $f(x,y)$ is continuous with its derivatives and belongs to $J = C([\Omega]\times[\Omega])$, and its norm is defined as

$$\|f(x,y)\| = \max_{x,y\in J}[\int_\Omega\int_\Omega f^2(x,y)dxdy]^{\frac{1}{2}} = M, \quad \forall x,y\in\Omega$$

(iii) The unknown function $\phi(x,y)$ is satisfies the Lipschitz condition with respect to its argument and its normal is defined in $L_2(\Omega)\times L_2(\Omega)$ as:

$$\|\phi(x,y)\| = [\int_\Omega\int_\Omega |\phi(x,y)|^2\, dxdy]^{\frac{1}{2}} \le C\,\|\phi\|_2 \tag{3.1}$$

4. Banach Fixed Point Theorem

In this section, we prove the existence a unique solution of Eq. (2.4) using Banach fixed point theorem.

We write Eq. (2.4) in the integral operator form:

$$\overline{W}\phi(x,y) = \frac{1}{\mu}f(x,y) + W\phi(x,y) \tag{4.1}$$

where,

$$W\phi(x,y) = \frac{\lambda}{\mu}\int_\Omega\int_\Omega k(|x-u|;|y-v|)\phi(u,v)dudv \tag{4.2}$$

Theorem 4.1. *if the conditions (i) –(iii) are verified , then equation (2.4) has a unique solution in the Banach space* $C([\Omega] \times [\Omega])$ *.*

The proof of this theorem depends on the following two lemmas

Lemma 4.1. *Under the conditions (i) – (iii) , the operator* $\overline{W}$ *defined by (4.1), maps the space* $C([\Omega] \times [\Omega])$ *into itself .*

Proof. In view of the formula (4.1) and (4.2), then using condition (ii) , and applying Cauchy-Schwarz inequality, we have

$$\left\|\overline{W}\phi(x,y)\right\| \leq \frac{1}{|\mu|}||f(x,y)|| + \left|\frac{\lambda}{\mu}\right| || \int_{\Omega}\int_{\Omega} |k(|x-u|;|y-v|)|\,|\phi(u,v)|\,dudv||.$$

Using the conditions (i) and (iii), the above inequality takes the form

$$\left\|\overline{W}\phi(x,y)\right\| \leq \frac{M}{|\mu|} + \theta^* \left\|\phi(x,y)\right\|, \quad (\theta^* = \left|\frac{\lambda}{\mu}\right| AC). \tag{4.3}$$

Inequality (4.3) shows that, the operator $\overline{W}$ maps the space $C([\Omega]\times[\Omega])$ into itself.

Moreover, the inequality (4.3) involves that the operator W of Eq. (4.1) is bounded where,

$$\left\|W\phi(x,y)\right\| \leq \theta^* \left\|\phi(x,y)\right\|. \tag{4.4}$$

The inequalities (4.3) and (4.4) define that the operator $\overline{W}$ is bounded. □

Lemma 4.2. *If the conditions (i) and (iii) are satisfied, then the operator* $\overline{W}$ *is contractive in the Banach space* $C([\Omega] \times [\Omega])$*.*

Proof. For the two functions $\phi_1(x,y)$ and $\phi_2(x,y)$ in the space $C([\Omega] \times [\Omega])$ the formulas (4.1) and (4.2) lead to.

$$\left\|(\overline{W}\phi_1 - \overline{W}\phi_2)(x,y)\right\| \leq \left|\frac{\lambda}{\mu}\right| \left\|\int_\Omega \int_\Omega |k(|x-u|;|y-v|)|\,|\phi_1(u,v)-\phi_2(u,v)|\,dudv\right\|.$$

Using the condition (iii) and then applying Cauchy-Schwarz inequality, we have,

$$\left\|(\overline{W}\phi_1 - \overline{W}\phi_2)(x,y)\right\| \leq \theta^* \left\|\phi_1(x,y)-\phi_2(x,y)\right\|. \tag{4.5}$$

Inequality (4.5) shows that, the operator $\overline{W}$ is continuous in the space $C([\Omega]\times[\Omega])$. Also, $\overline{W}$ is a contraction operator, under the condition $\theta^* < 1$, in the Banach space $C([\Omega]\times[\Omega])$. Therefore, the operator $\overline{W}$ has a unique fixed point which is the unique solution of Eq.(2.4). □

4.1. **The Numerical Method for Solving T-DFIE.** In this section, we state the two numerical methods for solving integral equation (2.4) by using the Toeplitz matrix method and Product Nyström method.

4.2. **The Toeplitz Matrix Method.** Consider the linear integral equation (2.4) where $\Omega = [a,b]\times[c,d]$,

$$\mu\phi(x,y) - \lambda\int_a^b\int_c^d k(|x-u|;|y-v|)\phi(u,v)dudv = f(x,y), \tag{4.6}$$

The integral term of Eq. (4.6) can be written as:

(4.7)

$$\int_a^b\int_c^d k(|x-u|;|y-v|)\phi(u,v)dudv = \sum_{n=-N}^{N-1}\ \sum_{m=-M}^{M-1}\ \int_{nh}^{nh+h}\int_{mh'}^{mh'+h'} k(|x-u|;|y-v|)\phi(u,v)dudv$$

Here, we take $m=n$, where $h=(b-a)/N$, $h'=(d-c)/M$, we approximate the integral in the right hand side of Eq.(4.7) by

$$\int_{nh}^{nh+h}\int_{mh'}^{mh'+h'} k(|x-u|;|y-v|)\phi(u,v)dudv = A_{n,m}(x,y)\phi(nh,mh')+ B_{n,m}(x,y)\phi(nh+h,mh'+h')+R \tag{4.8}$$

where $A_{n,m}(x,y), B_{n,m}(x,y)$ are two arbitrary functions to be determined and R is the error.

Then, we put $\phi(u,v)=1.1, uv$ in Eq.(4.8) yields a set of two equations in terms of the two unknown functions $A_{n,m}(x,y)$ and $B_{n,m}(x,y)$. In this case $R=0$. By solving the results, the functions $A_{n,m}(x,y)$ and $B_{n,m}(x,y)$ take the forms

$$A_{n,m}(x,y) = \frac{1}{(nhh'+hmh'+hh')}[(nh+h)(mh'+h')I-J], \tag{4.9}$$

and

$$B_{n,m}(x,y) = \frac{1}{(nhh'+mhh'+hh')}[J-(nh)(mh')I], \tag{4.10}$$

where,

$$\begin{aligned} I(x,y) &= \int_{nh}^{nh+h}\int_{mh'}^{mh'+h'} k(|x-u|;|y-v|)dudv, \\ J(x,y) &= \int_{nh}^{nh+h}\int_{mh'}^{mh'+h'} uvk(|x-u|;|y-v|)dudv, \end{aligned}$$

hence, Eq. (4.7) becomes,

(4.11)

$$\begin{aligned}
&\int_a^b \int_c^d k(|x-u|;|y-v|)\phi(u,v)dudv \\
&= \sum_{n=-N}^{N-1} \sum_{m=-M}^{M-1} [A_{n,m}(x,y)\phi(nh,mh') + B_{n,m}(x,y)\phi(nh+h,mh'+h')] \\
&= \sum_{n=-N}^{N-1} \sum_{m=-M}^{M-1} A_{n,m}(x,y)\phi(nh,mh') + \\
&\qquad \sum_{n=-N}^{N} \sum_{m=-M}^{M} B_{(n-1)(m-1)}(x,y)\phi(nh,mh') \\
&= \sum_{n=-N}^{N} \sum_{m=-M}^{M} D_{n,m}(x,y)\phi(nh,mh')
\end{aligned}$$

where,

$$D_{n,m}(x,y) = \{ \begin{array}{cc} A_{-N,-M}(x,y) & n=-N, m=-M \\ A_{n,m}(x,y) + B_{n-1,m-1}(x,y) & -N < n < N, -M < n < M \\ B_{N-1,M-1}(x,y) & n=N, m=M \end{array} .$$

Thus, the integral equation (4.6) becomes:

$$\mu\phi(x,y) - \lambda \sum_{n=-N}^{N} \sum_{m=-M}^{M} D_{n,m}(x,y)\phi(nh,mh') = f(x,y)$$

If we put $x = kh, y = lh'$, then we get the following system of linear algebraic equations:

$$\mu\phi_{k,l} - \lambda \sum_{n=-N}^{N} \sum_{m=-M}^{M} D_{k\ln,m}\phi_{n,m} = f_{kl}, \quad -N \leq k \leq N, -M \leq l \leq M \tag{4.12}$$

where,

(4.13)

$$D_{k\ln,m} = \{ \begin{array}{ll} A_{-N,-M}(kh,lh') & n=-N, m=-M \\ A_{n,m}(kh,lh') + B_{n-1,m-1}(kh,lh') & -N<n<N, -M<n<M \\ B_{N-1,M-1}(kh,lh') & n=N, m=M \end{array} .$$

The matrix $D_{k\ln,m}$ may be written as $D_{k\ln,m} = G_{k\ln,m} - E_{k\ln,m}$ where,

$$G_{k\ln,m} = A_{n,m}(kh,lh') + B_{n-1,m-1}(kh,lh'), \tag{4.14}$$

is the Toeplitz matrix, and the matrix,

$$E_{k\ln,m} = \{ \begin{array}{ll} B_{-N-1,-M-1}(kh,lh') & n=-N, m=-M \\ 0 & -N<n<N, -M<n<M \\ A_{N,M}(kh,lh') & n=N, m=M \end{array} . \tag{4.15}$$

represents a matrix whose elements are zero except the first and last columns.

However, the solution of the system of equations (4.12) can be obtained in the form

$$\phi_{k,l} = [\mu I - \lambda(G_{k\ln,m} - E_{k\ln,m})]^{-1} f_{kl} \tag{4.16}$$

Where I is the identity matrix and $|\mu I - \lambda(G_{k\ln,m} - E_{k\ln,m})| \neq 0$.

The error term R is determined from Eq.(4.8) by letting $\phi(u,v) = u^2v^2$ to get

$$R = | \int_{nh}^{nh+h} \int_{mh'}^{mh'+h'} u^2v^2 k(|x-u|;|y-v|)dudv - A_{n,m}(x,y)(nh)^2(mh')^2$$

$$-B_{n,m}(x,y)(nh+h)^2(mh'+h')^2| = \bigcirc(h^3).$$

4.3. **The Existence and Uniqueness Solution of The Linear Algebraic System OF Toeplitz Matrix.** This section will be devoted, to prove the existence of a unique solution of the LAS (4.12) in the space l^∞. For this, we write Eq. (4.12) in the following integral operator form:

$$\overline{T}\phi_{k,l} = T\phi_{k,l} + \frac{1}{\mu} f_{k,l}, \tag{4.17}$$

$$T\phi_{k,l} = \frac{\lambda}{\mu}\sum_{n=-N}^{N}\sum_{m=-M}^{M} D_{kln,m}\phi_{nm}, \ (\mu \neq 0, -N \leq k \leq N, \ -M \leq l \leq M), \tag{4.18}$$

We consider the following theorem.

Theorem 4.2. *The algebraic system (4.12) in the Banach space l^∞ has a unique solution under the following conditions:*

$$\sup_{k,l} |f_{kl}| \leq H < \infty, \qquad (H \ is \ a \ const) \tag{4.19}$$

$$\sup_{N,M} \sum_{n=-N}^{N}\sum_{m=-M}^{M} |D_{kln,m}| \leq H^* \qquad (H^* \ is \ a \ const) \tag{4.20}$$

The known function $\phi(nh, mh')$, for the constants $Q > Q_1, Q > P_1$ satisfy

$$\sup_{n,m} |\phi(nh, mh')| \leq Q_1 \left\|\Phi\right\|_{l^\infty}, \tag{4.21}$$

$$\sup_{n,m} |\phi(nh, mh') - \psi(nh, mh')| \leq P_1 \left\|\Phi - \Psi\right\|_{l^\infty}, \tag{4.22}$$

where $\left\|\Phi\right\|_{l^\infty} = \sup_{n,m} |\phi_{n,m}|$ for each integer n, m. To prove this theorem, we introduce the following lemmas.

Lemma 4.3. *If the conditions (4.19)-(4.21) are verified, then the operator* $\overline{T}$ *defined by equation (4.17) maps the space* l^∞ *into itself.*

Proof. Let U be the set of all functions $\Phi = \{\phi_{k,l}\}$ in l^∞ such that $|\Phi|_{l^\infty} \leq \beta$, β is a constant. Define the norm of the operator $\overline{T}\Phi$ in the space l^∞ by:

$$\left\|\overline{T}\Phi\right\|_{l^\infty} = \sup_{k,l}\left|\overline{T}\phi_{k,l}\right|, \text{ for each integer } k,l \tag{4.23}$$

From the formula (4.17) and (4.18) we get,

$$\left|\overline{T}\phi_{k,l}\right| \leq \left|\frac{\lambda}{\mu}\right| \sum_{n=-N}^{N}\sum_{m=-M}^{M} |D_{kln,m}| \sup_{n,m}|\phi(nh,mh')| + \frac{1}{|\mu|}\sup_{k,l}|f_{k,l}|.$$

Using the conditions (4.19) and (4.21) we have,

$$\left|\overline{T}\phi_{k,l}\right| \leq \left|\frac{\lambda}{\mu}\right| Q\left\|\Phi\right\|_{l^\infty} \sup_{N,M}\sum_{n=-N}^{N}\sum_{m=-M}^{M}|D_{kln,m}| + \frac{H}{|\mu|}.$$

Using the condition (4.20), the above inequality can be adapted in the form:

$$\sup_{k,l}\left|\overline{T}\phi_{k,l}\right| \leq \eta_1\left\|\Phi\right\|_{l^\infty} + \frac{H}{|\mu|}, \quad (\eta_1 = \left|\frac{\lambda}{\mu}\right| QH^*).$$

Since, the above inequality is true for each integer k,l, then with the aid of (4.23) we deduce

$$\left\|\overline{T}\Phi\right\|_{l^\infty} \leq \eta_1\left\|\Phi\right\|_{l^\infty} + \frac{H}{|\mu|}. \tag{4.24}$$

The inequality (4.24) shows that, the operator $\overline{T}$ maps the set U into itself, where

$$\beta = \frac{H}{(|\mu| - |\lambda|\, QH^*)} \tag{4.25}$$

Since $\beta > 0, H > 0$, therefore we have $\eta_1 < 1$. Also, the inequality (4.24) involves the bounded of the operator T where,

$$\|T\Phi\|_{l^\infty} \leq \eta_1 \|\Phi\|_{l^\infty} \tag{4.26}$$

Furthermore, the inequalities (4.24) and (4.25) define the bounded of the operator $\overline{T}$.

Lemma 4.4. *Under the two conditions (4.20) and (4.22), $\overline{T}$ is a contraction operator in the Banach space l^∞.*

□

Proof. For the two functions Φ and Ψ in l^∞, the formula (4.17) and (4.18) lead to

$$\left|\overline{T}\phi_{k,l} - \overline{T}\psi_{k,l}\right| \leq \left|\frac{\lambda}{\mu}\right| \sum_{n=-N}^{N} \sum_{m=-M}^{M} |D_{kln,m}| \sup_{n,m} |\phi(nh, mh') - \psi(nh, mh')| .$$

Using condition (4.22), we obtain :

$$\left|\overline{T}\phi_{k,l} - \overline{T}\psi_{k,l}\right| \leq \left|\frac{\lambda}{\mu}\right| Q \|\Phi - \Psi\|_{l^\infty} \sup_{N,M} \sum_{n=-N}^{N} \sum_{m=-M}^{M} |D_{kln,m}| .$$

Using the condition (4.20), we get

$$\left|\overline{T}\phi_{k,l} - \overline{T}\psi_{k,l}\right| \leq \eta_1 \|\Phi - \Psi\|_{l^\infty} .$$

The above inequality is true for each integer k, l, hence in view of (4.23) we have

$$\left\|\overline{T}\Phi - \overline{T}\Psi\right\|_{l^\infty} = \eta_1 \|\Phi - \Psi\|_{l^\infty} \tag{4.27}$$

The inequality (4.27) shows that, the operator $\overline{T}$ is continuous in the space l^∞, then $\overline{T}$ is a contraction operator under the condition $\eta_1 < 1$. □

4.4. **The Product Nystrom Method.** In this section, we discuss the solution of the T-DFIE of the second kind numerically using the Product Nyström method, (see Delves and Mohamed [13], and Atkinson [14]).

Consider the T-DFIE of the second kind

$$\mu\phi(x,y) - \lambda \int_a^b \int_b^c k(|x-u|;|y-v|)\phi(u,v)dudv = f(x,y) \tag{4.28}$$

when the kernel $k(|x-u|;|y-v|)$ is singular within the range of integration. We can often factor out the singularity in k by writing,

$$k(|x-u|;|y-v|) = \widetilde{k}(|x-u|;|y-v|)p(x,u;y,v) \tag{4.29}$$

where $p(x,u;y,v)$ and $\widetilde{k}(|x-u|;|y-v|)$ are badly behaved and well behaved functions of their arguments, respectively. We therefore rewrite (4.29) in the form:

$$\mu\phi(x,y) - \lambda \int_a^b \int_c^d p(x,u;y,v)\widetilde{k}(|x-u|;|y-v|)\phi(u,v)dudv = f(x,y) \tag{4.30}$$

We approximate the integral term in (4.30) when $x = x_i, y = y_s$ by

$$\begin{aligned} &\textstyle\int_a^b \int_c^d p(x_i,u;y_s,v)\widetilde{k}(|x_i-u|;|y_s-v|)\phi(u,v)dudv \approx \\ &\qquad\qquad \textstyle\sum_{j=0}^N \sum_{l=0}^M w_{ijsl}\widetilde{k}(|x_i-u|;|y_s-v|)\phi(u_j,v_l) \end{aligned} \tag{4.31}$$

where w_{ij} are the weights. Also, we approximate the integral term in (4.30) by a product integration from of Simpson's rule, we may write

$$\int_a^b \int_c^d p(x_i,u;y_s,v)\widetilde{k}(|x_i-u|;|y_s-v|)\phi(u,v)dudv =$$

$$\sum_{j=0}^{\frac{N-2}{2}} \sum_{l=0}^{\frac{M-2}{2}} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(x_i, u; y_s, v)\widetilde{k}(|x_i - u|; |y_s - v|)\phi(u, v)dudv,$$

where $x_i = u_i = a + ih$, $i = 0, 1, ..., N$ with $h = \frac{b-a}{N}$ and N even, also $y_s = v_s = c + sh'$,$s = 0, 1, ..., M$ with $h = \frac{d-c}{M}$ and M even. Now if we approximate the nonsingular part of the integrand over each interval $[u_{2j}, u_{2j+2}], [v_{2l}, v_{2l+2}]$ by the second degree Lagrange interpolation polynomial which interpolates it at the points $u_{2j}, u_{2j+1}, u_{2j+2}, v_{2l}, v_{2l+1}, v_{2l+2}$, to get

$$\begin{aligned}
&\int_a^b \int_c^d p(u_i, u; v_s, v)\widetilde{k}(|u_i - u|; |v_s - v|)\phi(u, v)dudv = \\
&\quad \sum_{j=0}^{\frac{N-2}{2}} \sum_{l=0}^{\frac{M-2}{2}} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(u_i, u; v_s, v) \\
&\quad \times\{\tfrac{(u_{2j+1}-u)(v_{2l+1}-v)(u_{2j+2}-u)(v_{2l+2}-v)}{(2h^2)(2h'^2)}\widetilde{k}(|u_i - u_{2j}|; |v_s - v_{2l}|)\phi(u_{2j}, v_{2l}) \\
&\quad +\tfrac{(u-u_{2j})(v-v_{2l})(u_{2j+2}-u)(v_{2l+2}-v)}{(h^2)(h'^2)}\widetilde{k}(|u_i - u_{2j+1}|; |v_s - v_{2l+1}|)\phi(u_{2j+1}, v_{2l+1}) \\
&\quad \tfrac{(u-u_{2j})(v-v_{2l})(u-u_{2j+1})(v-v_{2l+1})}{(2h^2)(2h'^2)}\widetilde{k}(|u_i - u_{2j+2}|; |v_s - v_{2l+2}|)\phi(u_{2j+2}, v_{2l+2})\}dudv \\
&\quad = \sum_{j=0}^{N} \sum_{l=0}^{M} w_{ijsl}\widetilde{k}(|u_i - u_j|; |v_s - v_l|)\phi(u_j, v_l),
\end{aligned}$$

the weight functions w_{ijsl} are given by :

$$w_{i,s,0,0} = \tfrac{1}{4h^2h'^2} \int_{u_0}^{u_2} \int_{v_0}^{v_2} p(u_i, u; v_s, v)(u_1 - u)(v_1 - v)(u_2 - u)(v_2 - v)dudv,$$

$$w_{i,s,2j+1,2l+1} = \tfrac{1}{h^2h'^2} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(u_i, u; v_s, v)(u - u_{2j})(v - v_{2l})(u_{2j+2} - u)(v_{2l+2} - v)dudv,$$

$$\begin{aligned}
w_{i,s,2j,2l} = &\tfrac{1}{4h^2h'^2} \int_{u_{2j-2}}^{u_{2j}} \int_{v_{2l-2}}^{v_{2l}} p(u_i, u; v_s, v)(u - u_{2j-2})(v - v_{2l-2})(u - u_{2j-1})(v - v_{2l-2})dudv \\
&+\tfrac{1}{4h^2h'^2} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(u_i, u; v_s, v)(u_{2j+1} - u)(v_{2l+1} - v)(u_{2j+2} - u)(v_{2l+2} - v)dudv,
\end{aligned}$$

(4.32)

$$w_{i,s,N,M} = \frac{1}{4h^2h'^2}\int_{u_{N-2}}^{u_N}\int_{v_{M-2}}^{v_M} p(u_i,u;v_s,v)(u-u_{N-2})(v-v_{M-2})(u-u_{N-1})(v-v_{M-1})dudv.$$

If we define:

$$\begin{aligned}\alpha_{j,l}(u_i,v_s) &= \frac{1}{4h^2h'^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2l-2}}^{v_{2l}} p(u_i,u;v_s,v)(u-u_{2j-2})(v-v_{2l-2})\\ &\quad (u-u_{2j-1})(v-v_{2l-1})dudv,\\ \beta_{j,l}(u_i,v_s) &= \frac{1}{4h^2h'^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2l-2}}^{v_{2l}} p(u_i,u;v_s,v)(u_{2j-1}-u)(v_{2l-1}-v)\\ &\quad (u_{2j}-u)(v_{2l}-v)dudv,\end{aligned}$$

(4.33)

$$\gamma_{j,l}(u_i,v_s) = \tfrac{1}{4h^2h'^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2l-2}}^{v_{2l}} p(u_i-u,v_s-v)(u-u_{2j-2})(v-v_{2l-2})(u-u_{2j-1})(v-v_{2l-1})dudv.$$

It follows that

$$w_{i,0,s,0} = \beta_{1,1}(u_i,v_s), \quad w_{i,2j+1,s,2l+1} = 4\gamma_{j+1,l+1}(u_i,v_s),$$

$$w_{i,2j,s,2l} = \alpha_{j,l}(u_i,v_s) + \beta_{j+1,l+1}(u_i,v_s), \quad w_{i,N,s,M} = \alpha_{\frac{N}{2},\frac{M}{2}}(u_i,v_s). \tag{4.34}$$

If we define $\psi_K = \int_0^2\int_0^2 \xi^K\delta^K p(u_i,(u_{2j-2}+\xi h);v_s,(v_{2l-2}+\delta h'))d\xi d\delta$, $K=0,1,2$, and let $u_i-u_{2j-2}=(i-2j+2)h$, $v_s-v_{2l-2}=(s-2l+2)h'$ we have,

(4.35)

$$\psi_K = \int_0^2 \int_0^2 \xi^K \delta^K p((z-\xi)h, (g-\delta)h')d\xi d\delta, K = 0, 1, 2, \; z = i-2j+2, \; g = s-2l+2$$

Therefore, the integral equation (4.28) transformed into the following system of linear algebraic equations

$$\begin{aligned} \mu\phi(x_i, y_i) - \lambda \textstyle\sum_{j=0}^{N} \sum_{l=0}^{M} w_{ijsl}\widetilde{k}(|x_i - u_j|; |y_s - v_l|)\phi(u_j, v_l) \\ = f(x_i, y_s), \quad i = 0, 1, ..., N, \quad s = 0, 1, ..., M. \end{aligned} \tag{4.36}$$

Which can be written in matrix form.

5. Some Applications in Contact Problems

In this section, we apply the Toeplitz matrix method and Product Nystrom method to some contact problems when the kernel $k(|x-u|; |y-v|)$ takes the Cauchy kernel, logarithmic form, and Carleman function, with $\Omega = [-1, 1] \times [-1, 1]$).

Here, the Toeplitz matrix method and Product Nystrom method will be used to get the numerical solution for values of $\mu = 1$. We consider two rigid surfaces having two different materials. From equation (2.1), we get $\lambda = \frac{\theta_1+\theta_2}{\lambda_1+\lambda_2}$, and $\theta_i = \frac{1-\upsilon_i^2}{\pi E_i}$. In the theory of elasticity the relation between λ, υ, E and between μ, υ, E are given by $\lambda_i = \frac{2\mu\upsilon_i}{1-2\upsilon_i}, E_i = 2\mu(1+\upsilon_i), i = 1, 2$, where E is called Young modulus and λ, μ are called Lame constants. We use $N = 21, 41$ units, where $N = M$, and take $h = h'$. In the case 1, we choose $\upsilon_1 = 0.42, \upsilon_2 = 0.38, \; \lambda = 0.02269139783$, and in case 2, we choose $\upsilon_1 = 0.37, \upsilon_2 = 0.35, \lambda = 0.03933175622$. The error, in each case is computed. Maple 10 is used to carry out the computations.

Example1: (Cauchy kernel)

Consider the two-dimensional integral equation:

$$\phi(x,y) - \lambda \int_{-1}^{1}\int_{-1}^{1}(\tfrac{1}{x-u})(\tfrac{1}{y-v})\phi(u,v)dudv = f(x,y), \quad |x|,|y| \leq 1,$$

(exact solution $\phi(x,y) = \frac{xy}{50}$).

Case1: $\lambda = 0.02269139783, \upsilon_1 = 0.42, \upsilon_2 = 0.38$:

Table (1) The results for Cauchy kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
	-1.00	-1.00	2.00000E-02	2.00302E-02	3.02201E-05	1.80157E-02	1.98420E-03
	-0.80	-0.80	1.28000E-02	2.56402E-02	1.28402E-02	9.85996E-03	2.94003E-03
	-0.60	-0.60	7.20000E-03	1.44306E-02	7.23066E-03	4.82688E-03	2.37311E-03
	-0.40	-0.40	3.20000E-03	6.41503E-03	3.21503E-03	1.56337E-03	1.63662E-03
	-0.20	-0.20	8.00000E-04	1.60353E-03	8.03532E-04	2.46074E-05	7.75392E-04
21	0.00	0.00	0.00000E+00	8.05386E-07	8.05386E-04	1.97371E-05	1.97371E-05
	0.20	0.20	8.00000E-04	1.60353E-03	8.03532E-04	1.66261E-03	8.62611E-04
	0.40	0.40	3.20000E-03	6.41503E-03	3.21503E-03	4.87843E-03	1.67843E-03
	0.60	0.60	7.20000E-03	1.44306E-02	7.23066E-03	9.61790E-03	2.41790E-03
	0.80	0.80	1.28000E-02	2.56402E-02	1.28402E-02	1.57902E-02	2.99025E-03
	1.00	1.00	2.00000E-02	2.00302E-02	3.02201E-05	2.20897E-02	2.08978E-03
	-1.00	-1.00	2.00000E-02	2.00529E-02	5.29974E-05	1.80384E-02	1.96160E-03
	-0.80	-0.80	1.28000E-02	2.56762E-02	1.28762E-02	9.89596E-03	2.90404E-03
	-0.60	-0.60	7.20000E-03	1.44511E-02	7.25112E-03	4.84738E-03	2.35262E-03
	-0.40	-0.40	3.20000E-03	6.42438E-03	3.22438E-03	1.57272E-03	1.62728E-03
	-0.20	-0.20	8.00000E-04	1.60619E-03	8.06196E-04	2.72674E-05	7.72732E-04

Continuation for Table (1) The results for Cauchy kernel when N=21, 41.

41	0.00	0.00	0.00000E+00	2.03144E-07	2.03144E-07	1.91348E-05	1.91348E-05
	0.20	0.20	8.00000E-04	1.60619E-03	8.06196E-04	1.66527E-03	8.65270E-04
	0.40	0.40	3.20000E-03	6.42438E-03	3.22438E-03	4.88778E-03	1.68778E-03
	0.60	0.60	7.20000E-03	1.44511E-02	7.25112E-03	9.63840E-03	2.43840E-03
	0.80	0.80	1.28000E-02	2.56762E-02	1.28762E-02	1.58262E-02	3.02620E-03
	1.00	1.00	2.00000E-02	2.00529E-02	5.29974E-05	2.21124E-02	2.11240E-03

Case2: $\lambda = 0.03933175622, \upsilon_1 = 0.37, \upsilon_2 = 0.35$:

Table (2) The results for Cauchy kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
	-1.00	-1.00	2.00000E-02	2.00302E-02	3.02197E-05	1.65562E-02	3.44375E-03
	-0.80	-0.80	1.28000E-02	2.56401E-02	1.28401E-02	7.69828E-03	5.10171E-03
	-0.60	-0.60	7.20000E-03	1.44305E-02	7.23059E-03	3.07720E-03	4.12279E-03
	-0.40	-0.40	3.20000E-03	6.41500E-03	3.21500E-03	3.55846E-04	2.84415E-03
	-0.20	-0.20	8.00000E-04	1.60352E-03	8.03524E-04	6.37857E-04	1.62142E-04
21	0.00	0.00	0.00000E+00	8.05374E-07	8.05374E-07	2.77100E-05	2.77100E-05
	0.20	0.20	8.00000E-04	1.60352E-03	8.03524E-04	2.28880E-03	1.48880E-03
	0.40	0.40	3.20000E-03	6.41500E-03	3.21500E-03	6.10467E-03	2.90467E-03
	0.60	0.60	7.20000E-03	1.44305E-02	7.23059E-03	1.13879E-02	4.18792E-03
	0.80	0.80	1.28000E-02	2.56401E-02	1.28401E-02	1.79815E-02	5.18156E-03
	1.00	1.00	2.00000E-02	2.00302E-02	3.02197E-05	2.36303E-02	3.63030E-03
	-1.00	-1.00	2.00000E-02	2.00529E-02	5.29966E-05	1.65789E-02	3.42110E-03
	-0.80	-0.80	1.28000E-02	2.56761E-02	1.28761E-02	7.73428E-03	5.06572E-03
	-0.60	-0.60	7.20000E-03	1.44510E-02	7.25105E-03	3.09770E-03	4.10230E-03

Continuation for Table (2) The results for Cauchy kernel when N=21, 41.

41	-0.40	-0.40	3.20000E-03	6.42435E-03	3.22435E-03	3.65196E-04	2.83480E-03
	-0.20	-0.20	8.00000E-04	1.60618E-03	8.06189E-04	6.40517E-04	1.59483E-04
	0.00	0.00	0.00000E+00	2.03141E-07	2.03141E-07	2.71077E-05	2.71077E-05
	0.20	0.20	8.00000E-04	1.60618E-03	8.06189E-04	2.29146E-03	1.49146E-03
	0.40	0.40	3.20000E-03	6.42435E-03	3.22435E-03	6.11402E-03	2.91402E-03
	0.60	0.60	7.20000E-03	1.44510E-02	7.25105E-03	1.14084E-02	4.20840E-03
	0.80	0.80	1.28000E-02	2.56761E-02	1.28761E-02	1.80175E-02	5.21750E-03
	1.00	1.00	2.00000E-02	2.00529E-02	5.29966E-05	2.36530E-02	3.65300E-03

Comment, from the above results obtained from the Toeplitz matrix method and Product Nyström method, we note that:

1- when the values of λ is fixed, the error values $Err.T.$, and $Err.N.$ are increase as well as N increases for the two different materials $(\upsilon_1 = 0.42, \upsilon_2 = 0.38)$, $(\upsilon_1 = 0.37, \upsilon_2 = 0.35)$, see tables (1-2).

2-when the values of N is fixed, the error values $Err.T.$, are decrease, while the error values $Err.N.$ are increase with the increasing of υ and λ, for each materials.

3-by comparing the results obtain from the Toeplitz matrix method and Product Nystrom method, we note that:

i-The minimum value of the error in Toeplitz matrix method is less than the minimum value of the error in Product Nystrom method.

ii- The Toeplitz matrix method is better to evaluate the approximate solution than the Product Nystrom method.

Example2: (logarithmic kernel)

Consider the integral equation

$$\phi(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}\ln|x-y|\ln|y-v|\,\phi(u,v)dudv=f(x,y),$$
$$(\text{ exact solution } \phi(x,y)=\tfrac{xy}{50}).$$

Case1: $\lambda=0.02269139783, \upsilon_1=0.42, \upsilon_2=0.38$:

Table (3) The results for logarithmic kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
	-1.00	-1.00	2.00000E-02	1.95998E-02	4.00149E-04	1.95476E-02	4.52360E-04
	-0.80	-0.80	1.28000E-02	1.22461E-02	5.53835E-04	1.22525E-02	5.47424E-04
	-0.60	-0.60	7.20000E-03	6.76747E-03	4.32528E-04	6.71265E-03	4.87347E-04
	-0.40	-0.40	3.20000E-03	2.97307E-03	2.26921E-04	2.84974E-03	3.50250E-04
	-0.20	-0.20	8.00000E-04	7.42496E-04	5.75033E-05	6.15385E-04	1.84614E-04
21	0.00	0.00	0.00000E+00	1.00592E-05	1.00592E-05	1.89573E-06	1.89573E-06
	0.20	0.20	8.00000E-04	7.42496E-03	5.75033E-05	9.80035E-04	1.80035E-04
	0.40	0.40	3.20000E-03	2.97307E-03	2.26921E-04	3.54535E-03	3.45358E-04
	0.60	0.60	7.20000E-03	6.76747E-03	4.32528E-04	7.67624E-03	4.76245E-04
	0.80	0.80	1.28000E-02	1.22461E-02	5.53835E-04	1.33436E-02	5.43679E-04
	1.00	1.00	2.00000E-02	1.95998E-02	4.00149E-04	2.04499E-02	4.49927E-04
	-1.00	-1.00	2.00000E-02	1.95713E-02	4.28604E-04	1.95191E-02	4.80900E-04
	-0.80	-0.80	1.28000E-02	1.21986E-02	6.01320E-04	1.22050E-02	5.95000E-04
	-0.60	-0.60	7.20000E-03	6.73661E-03	4.63386E-04	6.68179E-03	5.18210E-04
	-0.40	-0.40	3.20000E-03	2.95681E-03	2.43183E-04	2.83348E-03	3.66520E-04
	-0.20	-0.20	8.00000E-04	7.35797E-04	6.42025E-05	6.08686E-04	1.91314E-04
41	0.00	0.00	0.00000E+00	4.76594E-06	4.76594E-06	3.39753E-03	3.39753E-06

Continuation for Table (3) The results for logarithmic kernel when N=21, 41.

	0.20	0.20	8.00000E-04	7.35795E-04	6.42024E-05	5.70913E-03	4.90913E-03
	0.40	0.40	3.20000E-03	2.95681E-03	2.43183E-04	3.52909E-03	3.29090E-04
	0.60	0.60	7.20000E-03	6.73661E-03	4.63387E-04	7.64538E-03	4.45380E-04
	0.80	0.80	1.28000E-02	1.21986E-02	6.01320E-04	1.32961E-02	4.96100E-04
	1.00	1.00	2.00000E-02	1.95713E-02	4.28604E-04	2.04214E-02	4.21400E-04

Case2: $\lambda = 0.03933175622, \upsilon_1 = 0.37, \upsilon_2 = 0.35$:

Table (4) The results for logarithmic kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
	-1.00	-1.00	2.00000E-02	1.93043E-02	6.95631E-04	1.92183E-02	7.81697E-04
	-0.80	-0.80	1.28000E-02	1.18349E-02	9.65057E-04	1.18540E-02	9.45936E-04
	-0.60	-0.60	7.20000E-03	6.44595E-03	7.54045E-03	6.35799E-03	8.42004E-04
	-0.40	-0.40	3.20000E-03	2.80401E-03	3.95982E-04	2.59554E-03	6.04456E-04
	-0.20	-0.20	8.00000E-04	6.99178E-04	1.00821E-04	4.82405E-04	3.17594E-04
21	0.00	0.00	0.00000E+00	1.65848E-05	1.65848E-05	1.32055E-06	1.32055E-06
	0.20	0.20	8.00000E-04	6.99178E-04	1.00821E-04	1.11363E-03	3.13635E-04
	0.40	0.40	3.20000E-03	2.80401E-03	3.95982E-04	3.79969E-03	5.99696E-04
	0.60	0.60	7.20000E-03	6.44595E-03	7.54045E-04	8.02610E-03	8.26106E-04
	0.80	0.80	1.28000E-02	1.18349E-02	9.65057E-04	1.37425E-02	9.42552E-04
	1.00	1.00	2.00000E-02	1.93043E-02	6.95631E-04	2.07795E-02	7.79577E-04
	-1.00	-1.00	2.00000E-02	1.92560E-02	7.43950E-04	1.91700E-02	8.30000E-04
	-0.80	-0.80	1.28000E-02	1.17548E-02	1.04511E-03	1.17739E-03	1.02610E-03
	-0.60	-0.60	7.20000E-03	6.39441E-03	8.05587E-04	6.30645E-03	8.93550E-04
	-0.40	-0.40	3.20000E-03	2.77703E-03	4.22962E-04	2.56856E-03	6.31440E-04
	-0.20	-0.20	8.00000E-04	6.88093E-04	1.11906E-04	4.71320E-04	3.28680E-04

Continuation for Table (4) The results for logarithmic kernel when N=21, 41.

41	0.00	0.00	0.00000E+00	7.80419E-06	7.80419E-06	7.46006E-06	7.46006E-06
	0.20	0.20	8.00000E-04	6.88093E-04	1.11906E-04	1.10254E-03	3.02545E-04
	0.40	0.40	3.20000E-03	2.77703E-03	4.22962E-04	3.88271E-03	5.72710E-04
	0.60	0.60	7.20000E-03	6.39441E-03	8.05587E-04	7.97456E-03	7.74560E-04
	0.80	0.80	1.28000E-02	1.17548E-02	1.04511E-03	1.36624E-02	8.62400E-04
	1.00	1.00	2.00000E-02	1.92560E-02	7.43950E-04	2.07312E-02	7.31200E-04

Comment, from the above results obtained from the Toeplitz matrix method and Product Nystrom method, we note that:

1- When the values of λ and υ are fixed, the error values $Err.T.$, and $Err.N.$ are increases as well as N increases for the two different materials
($\upsilon_1 = 0.42, \upsilon_2 = 0.38$),($\upsilon_1 = 0.37, \upsilon_2 = 0.35$), see tables (3-4).

2-When the value of N is fixed, the error values $Err.T.$, and $Err.N.$ are increase with the increasing of υ and λ, for each materials.

3-By comparing the results obtain from the Toeplitz matrix method and Product Nystrom method, we note that:

The error in the evaluation of the approximate solution, by using the Toeplitz matrix method, is less than the error in the evaluation of the approximate solution, using the Product Nyström method in all cases.

Example 3: (Carleman kernel)

Consider the integral equation

$$\phi(x,y) - \lambda \int_{-1}^{1}\int_{-1}^{1} |x-y|^{-\upsilon_1} |y-v|^{-\upsilon_2} \phi(u,v)dudv = f(x,y),$$

(exact solution $\phi(x,y) = \frac{xy}{50}$).

Case1:$\lambda = 0.02269139783, \upsilon_1 = 0.42, \upsilon_2 = 0.38$:

Table (5) The results for Carleman kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
21	-1.00	-1.00	2.00000E-02	1.99095E-02	9.04508E-05	2.00663E-02	6.63310E-05
	-0.80	-0.80	1.28000E-02	1.26275E-02	1.72412E-04	1.28304E-02	3.04480E-05
	-0.60	-0.60	7.20000E-03	7.07720E-03	1.22798E-04	7.27872E-03	7.87297E-05
	-0.40	-0.40	3.20000E-03	3.15818E-03	4.18115E-05	3.32291E-03	1.22915E-04
	-0.20	-0.20	8.00000E-04	8.21193E-04	2.11938E-05	9.86085E-04	1.86085E-04
	0.00	0.00	0.00000E+00	6.96363E-05	6.96363E-05	2.4815E-04	2.48415E-04
	0.20	0.20	8.00000E-04	8.21193E-04	2.11938E-05	1.11452E-03	3.14528E-04
	0.40	0.40	3.20000E-03	3.15818E-03	4.18115E-05	3.58188E-03	3.81885E-04
	0.60	0.60	7.20000E-03	7.07720E-03	1.22798E-04	7.64174E-03	4.41745E-04
	0.80	0.80	1.28000E-02	1.26275E-02	1.72412E-04	1.32955E-02	4.95576E-04
	1.00	1.00	2.00000E-02	1.99095E-02	9.04508E-05	2.05448E-02	5.44860E-04
41	-1.00	-1.00	2.00000E-02	1.98652E-02	1.34713E-04	2.00220E-02	2.20000E-05
	-0.80	-0.80	1.28000E-02	1.25518E-02	2.48119E-04	1.27547E-02	4.53000E-05
	-0.60	-0.60	7.20000E-03	7.02149E-03	1.78509E-04	7.2301E-03	2.30100E-05
	-0.40	-0.40	3.20000E-03	3.11929E-03	8.07040E-05	3.28402E-03	8.40200E-05
	-0.20	-0.20	8.00000E-04	7.93977E-04	6.02225E-06	9.58869E-04	1.58869E-04
	0.00	0.00	0.00000E+00	3.23367E-05	3.23367E-05	2.11115E-04	2.11115E-04
	0.20	0.20	8.00000E-04	7.93977E-04	6.02225E-06	1.08730E-03	2.87304E-04
	0.40	0.40	3.20000E-03	3.11929E-03	8.07040E-05	3.54299E-03	3.42990E-04
	0.60	0.60	7.20000E-03	7.02149E-03	1.78509E-04	7.58603E-03	3.86030E-04

Continuation for Table (5) The results for Carleman kernel when N=21, 41.

	0.80	0.80	1.28000E-02	1.25518E-02	2.48119E-04	1.32198E-02	4.19800E-04
	1.00	1.00	2.00000E-02	1.98652E-02	1.34713E-04	2.05005E-02	5.00500E-04

Case2: $\lambda = 0.03933175622, \upsilon_1 = 0.37, \upsilon_2 = 0.35$:

Table (6) The results for Carleman kernel when N=21, 41.

N	x	y	Exact sol.	Appr. sol.T.	Err. T.	Appr. sol. N.	Err. N.
	-1.00	-1.00	2.00000E-02	1.98960E-02	1.03970E-04	2.02001E-02	2.00163E-04
	-0.80	-0.80	1.28000E-02	1.26165E-02	1.83457E-04	1.29704E-02	1.70453E-04
	-0.60	-0.60	7.20000E-03	7.07334E-03	1.26657E-04	7.43285E-03	2.32851E-04
	-0.40	-0.40	3.20000E-03	3.17040E-03	2.95929E-05	3.48630E-03	2.86302E-04
	-0.20	-0.20	8.00000E-04	8.46458E-04	4.64589E-05	1.16478E-03	3.64783E-04
21	0.00	0.00	0.00000E+00	1.17060E-04	1.17060E-04	4.40904E-04	4.40904E-04
	0.20	0.20	8.00000E-04	8.46458E-04	4.64589E-05	1.32347E-03	5.23476E-04
	0.40	0.40	3.20000E-03	3.17040E-03	2.95929E-05	3.80712E-03	5.23476E-04
	0.60	0.60	7.20000E-03	7.07334E-03	1.26657E-04	7.88339E-03	6.07128E-04
	0.80	0.80	1.28000E-02	1.26165E-02	1.83457E-04	1.35531E-02	6.83396E-04
	1.00	1.00	2.00000E-02	1.98960E-02	1.03970E-04	2.08294E-02	7.53171E-04
	-1.00	-1.00	2.00000E-02	1.98352E-02	1.64795E-04	2.01393E-02	8.29492E-04
	-0.80	-0.80	1.28000E-02	1.25097E-02	2.90207E-04	1.28636E-02	1.39300E-04
	-0.60	-0.60	7.20000E-03	6.99173E-03	2.08260E-04	7.35124E-03	6.36000E-05
	-0.40	-0.40	3.20000E-03	3.11043E-03	8.95604E-05	3.42633E-03	1.51240E-04
	-0.20	-0.20	8.00000E-04	8.01997E-04	1.99783E-06	1.12031E-03	2.26330E-04
41	0.00	0.00	0.00000E+00	5.40886E-05	5.40886E-05	3.27900E-03	3.77932E-04
	0.20	0.20	8.00000E-04	8.01997E-04	1.99783E-06	1.27900E-03	4.79009E-04

Continuation for Table (6) The results for Carleman kernel when N=21, 41.

	0.40	0.40	3.20000E-03	3.11043E-03	8.95604E-05	3.74715E-03	5.47150E-04
	0.60	0.60	7.20000E-03	6.99173E-03	2.08260E-04	7.80178E-03	6.01780E-04
	0.80	0.80	1.28000E-02	1.25097E-02	2.90207E-04	1.34463E-02	6.463006E-04
	1.00	1.00	2.00000E-02	1.98352E-02	1.64795E-94	2.07685E-02	7.68580E-04

Comment, from the above results obtained from the Toeplitz matrix method and Product Nyström method, we note that:

1- When the values of λ and υ are fixed, the error values $Err.T.$, and $Err.N.$ are increases as well as N increases for the two different materials
$(\upsilon_1 = 0.42, \upsilon_2 = 0.38)$,$(\upsilon_1 = 0.37, \upsilon_2 = 0.35)$, see tables (5-6).

2-when the values of N is fixed , the error values $Err.T.$, and $Err.N.$ are increase with the increasing of υ and λ, for each materials.

3-by comparing the results obtain from the Toeplitz matrix method and Product Nyström method, we note that:

In the Carleman function, the error values in the evaluation of the approximate solution by using the Toeplitz matrix method is less than the error by Product Nyström method.

6. Conclusion

In general, we can apply the Toeplitz matrix method and Product Nyström method for different values of Poisson ratio corresponding to the different materials. From discussion we concludes that:

1-The minimum error $Err.T.$ of the Toeplitz matrix method, for different values of υ, always less than the corresponding minimum error $Err.N.$ of the Product Nyström method. So, the Toeplitz matrix method is the best for solving theT-DFIE.

2-The maximum errors $Err.T.$ by the Toeplitz matrix method is less than the maximum errors by the Product Nyström method.

3-The error $Err.T.$, and $Err.N.$ of the T-DFIE with Carleman kernel is less than both error $Err.T.$, and $Err.N.$ of T-DFIE with a logarithmic kernel.

4- The results errors $Err.T.$, and $Err.N.$ with Carleman kernel are better than the errors $Err.T.$, , $Err.N.$ with logarithmic kernel at values of $0.3 \leq \nu \leq 0.5$.

5- From the previous results, we note that the results errors $Err.T.$, and $Err.N.$ with Carleman kernel are better than the errors $Err.T.$, and $Err.N.$ with logarithmic kernel and Cauchy kernel.

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