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DECOMPOSITIONS OF CONTINUITY VIA GRILLS

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Abstract. In this paper, we introduce the notions of $\mathcal{G}$-α-open sets, $\mathcal{G}$-semi-open sets and $\mathcal{G}$-β-open sets in grill topological spaces and investigate their properties. Furthermore, by using these sets we obtain new decompositions of continuity.

1. Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [2], [3], [11] for details). In [10], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [5] have defined new classes of sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notions of $\mathcal{G}$-α-open sets, $\mathcal{G}$-semi-open sets and $\mathcal{G}$-β-open sets in grill topological spaces. We define grill α-continuous functions to obtain decompositions of continuity.

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2. **Preliminaries**

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$.

The definition of grill on a topological space, as given by Choquet [4], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological spaces X is said to be a grill on X if

(1) $\phi \notin \mathcal{G}$,
(2) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
(3) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For example let R be the set of all real numbers consider a subset $\mathcal{G}= \{A \subseteq R : m(A) \neq 0\}$, where $m(A)$ is the Lebesgue measure of A, then $\mathcal{G}$ is a grill. For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

Definition 2.1. [10] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is defined as follows:
$\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.

Proposition 2.1. [10] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$:*

(1) $A \subseteq B$ *implies that* $\Phi(A) \subseteq \Phi(B)$,
(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,
(3) $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.

Let $\mathcal{G}$ be a grill on a space X. Then we define a map $\Psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology

$\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X-U) = X-U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\text{-}Cl(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Example 2.1. [10] *Let τ denote the cofinite topology on an uncountable set X and let $\mathcal{G}$ be the grill of all uncountable subset of X. Then it is clearly $\tau \setminus \{\phi\} \subseteq \mathcal{G}$. We show that $\tau_{\mathcal{G}}$ is the cocountable topology which denoted by τ_{co} on X. If $V \in \tau_{\mathcal{G}}$, then $V = U - A$, where $U \in \tau$ and $A \notin \mathcal{G}$ implies that $(X-U)$ is finite and A is countable. Now $X - V = X \cap (X-V) = X \cap (X-(U \cap (X-A))) = X \cap ((X-U) \cup A) = (X-U) \cup A$ which is countable and hence $V \in \tau_{co}$. On the other hand if $V \in \tau_{co}$ implies that $X - V = A \notin \mathcal{G}$ and hence $V = X - A$, where $X \in \tau$ and $A \notin \mathcal{G}$ so $V \in \tau_{\mathcal{G}}$. Thus $\tau_{\mathcal{G}} = \tau_{co}$.*

Lemma 2.1. [10] *For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \mathcal{B}(\mathcal{G}, \tau) \subseteq \tau_{\mathcal{G}}$, where $\mathcal{B}(\mathcal{G}, \tau) = \{V - A : V \in \tau \text{ and } A \notin \mathcal{G}\}$ is an open base for $\tau_{\mathcal{G}}$.*

Example 2.2. *Let (X, τ) be a topological space. If $\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}$, then $\tau_{\mathcal{G}} = \tau$. Since for any $\tau_{\mathcal{G}}$-basic open set $V = X - A$ with $U \in \tau$ and $A \notin \mathcal{G}$, we have $A = \phi$, so that $V = U \in \tau$. Hence by Lemma 2.1 we have in this case $\tau = \mathcal{B}(\mathcal{G}, \tau) = \tau_{\mathcal{G}}$*

Definition 2.2. A subset A of a topological space X is said to be:

(1) α-open [9] if $A \subseteq Int(Cl(Int(A)))$,

(2) semi-open [6] if $A \subseteq Cl(Int(A))$,

(3) preopen [8] if $A \subseteq Int(Cl(A))$,

(4) β-open [1] if $A \subseteq Cl(Int(Cl(A)))$.

3. Properties of $\mathcal{G}$-α-Open Sets and $\mathcal{G}$-Semi-Open Sets

Definition 3.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is said to be

(1) Φ-open [5] if $A \subseteq Int(\Phi(A))$,

(2) $\mathcal{G}$-α-open if $A \subseteq Int(\Psi(Int(A)))$,

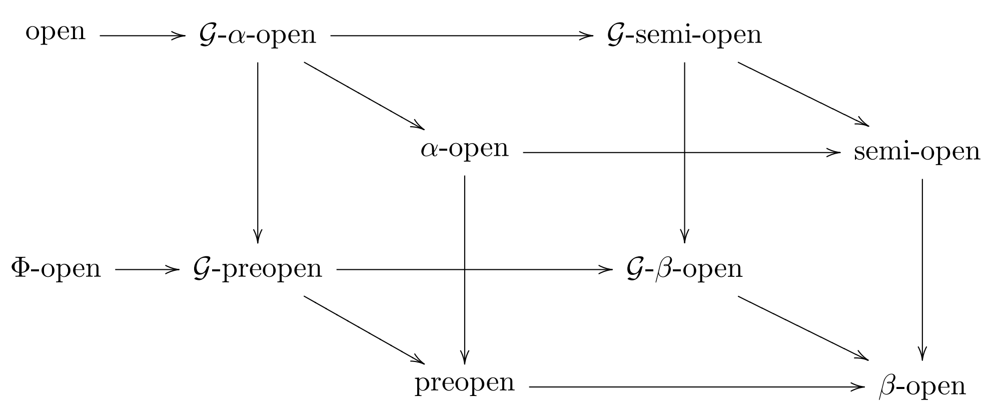
(3) $\mathcal{G}$-preopen [5] if $A \subseteq Int(\Psi(A))$,

(4) $\mathcal{G}$-semi-open if $A \subseteq \Psi(Int(A))$,

(5) $\mathcal{G}$-β-open if $A \subseteq Cl(Int(\Psi(A)))$.

The family of all $\mathcal{G}$-α-open (resp. $\mathcal{G}$-preopen, $\mathcal{G}$-semi-open, $\mathcal{G}$-β-open) sets in a grill topological space $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}\alpha O(X)$ (resp. $\mathcal{G}PO(X)$, $\mathcal{G}SO(X)$, $\mathcal{G}\beta O(X)$).

Remark 1. For several sets defined above, we have the following implications, where converses of implications need not be true as shown by below examples.



It is shown in [5] that openness and Φ-openness are independent of each other.

Example 3.1. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{c, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$. Then*

(1) *$A = \{b, c, d\}$ is a semi-open set which is not $\mathcal{G}$-semi-open.*

(2) *$A = \{b, c, d\}$ is a $\mathcal{G}$-β-open set which is not $\mathcal{G}$-semi-open.*

(3) $B = \{a, b\}$ *is a* $\mathcal{G}$*-semi-open set which is not preopen and hence it is not* $\mathcal{G}$*-preopen.*

(4) $C = \{a, b, c\}$ *is a* $\mathcal{G}$*-*α*-open set which is not open.*

Example 3.2. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ *and the grill* $G = \{\{b\}, \{a, b\}, \{a, b, c\}, \{c, b, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, X\}$. *Then* $A = \{a, c, d\}$ *is an* α*-open set and a* $\mathcal{G}$*-*β*-open set which is not* $\mathcal{G}$*-preopen. .*

Example 3.3. *Let* $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ *and the grill* $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. *Then*

(1) $A = \{a, c\}$ *is a* β*-open set which is not* $\mathcal{G}$*-*β*-open.*

(2) $B = \{a, b\}$ *is a* $\mathcal{G}$*-preopen set which is not* $\mathcal{G}$*-semi-open.*

Proposition 3.1. *For a subset of a grill topological space* $(X, \tau, \mathcal{G})$*, the following properties are hold:*

(1) *Every* $\mathcal{G}$*-*α*-open set is* α*-open.*

(2) *Every* $\mathcal{G}$*-semi-open set is semi-open.*

(3) *Every* $\mathcal{G}$*-*β*-open set is* β*-open.*

Theorem 3.1. *Let* A *be a subset of a grill topological space* $(X, \tau, \mathcal{G})$*. Then the following properties hold:*

(1) *A subset* A *of* X *is* $\mathcal{G}$*-*α*-open if and only if it is* $\mathcal{G}$*-semi-open and* $\mathcal{G}$*-pre-open,*

(2) *If* A *is* $\mathcal{G}$*-semi-open, then* A *is* $\mathcal{G}$*-*β*-open.*

(3) *If* A *is* $\mathcal{G}$*-preopen, then* A *is* $\mathcal{G}$*-*β*-open.*

Proof. (1) *Necessity.* This is obvious.

Sufficiency. Let A be $\mathcal{G}$-semi-open and $\mathcal{G}$-pre-open. Then we have $A \subseteq Int(\Psi(A)) \subseteq Int(\Psi(\Psi(Int(A)))) \subseteq Int(\Psi(Int(A)))$. This shows that A is $\mathcal{G}$-α-open.

(2) Since A is $\mathcal{G}$-semi-open and $\tau \subseteq \tau_{\mathcal{G}}$, we have $A \subseteq \Psi(Int(A)) \subseteq Cl(Int(A)) \subseteq Cl(Int(\Psi(A)))$. This shows that A is $\mathcal{G}$-β-open.

(3) The proof is obvious. □

Theorem 3.2. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-open if and only if $\Psi(A) = \Psi(Int(A))$.*

Theorem 3.3. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-open if and only if there exists $U \in \tau$ such that $U \subseteq A \subseteq \Psi(U)$.*

Proof. Let A be $\mathcal{G}$-semi-open, then $A \subseteq \Psi(Int(A))$. Take $U = Int(A)$. Then we have $U \subseteq A \subseteq \Psi(U)$. Conversely, let $U \subseteq A \subseteq \Psi(U)$ for some $U \in \tau$. Since $U \subseteq A$, we have $U \subseteq Int(A)$ and hence $\Psi(U) \subseteq \Psi(Int(A))$. Thus we obtain $A \subseteq \Psi(Int(A))$. □

Theorem 3.4. *If A is a $\mathcal{G}$-semi-open set in a grill topological space $(X, \tau, \mathcal{G})$ and $A \subseteq B \subseteq \Psi(A)$, then B is $\mathcal{G}$-semi-open in $(X, \tau, \mathcal{G})$.*

Proof. Since A be $\mathcal{G}$-semi-open, there exists an open set U of X such that $U \subseteq A \subseteq \Psi(U)$. Then we have $U \subseteq A \subseteq B \subseteq \Psi(A) \subseteq \Psi(\Psi(U)) = \Psi(U)$ and hence $U \subseteq B \subseteq \Psi(U)$. By Theorem 3.3, we obtain that B is $\mathcal{G}$-semi-open in $(X, \tau, \mathcal{G})$. □

Lemma 3.1. [10] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. If $U \in \tau$, then $U \cap \Phi(A) = U \cap \Phi(U \cap A)$ for any $A \subseteq X$.*

Lemma 3.2. *Let A be a subset of a grill topological space $(X, \tau, \mathcal{G})$. If $U \in \tau$, then $U \cap \Psi(A) \subseteq \Psi(U \cap A)$.*

Proof. Since $U \in \tau$, by Lemma 3.1 we obtain
$U \cap \Psi(A) = U \cap (A \cup \Phi(A)) = (U \cap A) \cup (U \cap \Phi(A) \subseteq (U \cap A) \cup \Phi(U \cap A) = \Psi(U \cap A)$. □

Proposition 3.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space.*

(1) *If $V \in \mathcal{G}SO(X)$ and $A \in \mathcal{G}\alpha O(X)$, then $V \cap A \in \mathcal{G}SO(X)$.*
(2) *If $V \in \mathcal{G}PO(X)$ and $A \in \mathcal{G}\alpha O(X)$, then $V \cap A \in \mathcal{G}PO(X)$.*

Proof. (1) Let $V \in \mathcal{G}SO(X)$ and $A \in G\alpha O(X)$. By using Lemma 3.2 we obtain

$$
\begin{aligned}
V \cap A &\subseteq \Psi(Int(V)) \cap Int(\Psi(Int(A))) \\
&\subseteq \Psi[Int(V) \cap Int(\Psi(Int(A)))] \\
&\subseteq \Psi[Int(V) \cap \Psi(Int(A))] \\
&\subseteq \Psi[\Psi[Int(V) \cap Int(A)]] \\
&\subseteq \Psi[Int(V \cap A)].
\end{aligned}
$$

This shows that $V \cap A \in \mathcal{G}SO(X)$.

(2) Let $V \in \mathcal{G}PO(X)$ and $A \in \mathcal{G}\alpha O(X)$. By using Lemma 3.2 we obtain

$$
\begin{aligned}
V \cap A &\subseteq Int(\Psi(V)) \cap Int(\Psi(Int(A))) \\
&= Int[Int(\Psi(V)) \cap \Psi(Int(A))] \\
&\subseteq Int[\Psi[Int(\Psi(V)) \cap Int(A)]] \\
&\subseteq Int[\Psi[\Psi(V) \cap Int(A)]] \\
&\subseteq Int[\Psi[\Psi[V \cap Int(A)]]] \\
&\subseteq Int[\Psi[V \cap A]].
\end{aligned}
$$

This shows that $V \cap A \in \mathcal{G}PO(X)$. □

Corollary 3.1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space.*

(1) *If $V \in \mathcal{G}SO(X)$ and $A \in \tau$, then $V \cap A \in \mathcal{G}SO(X)$.*
(2) *If $V \in \mathcal{G}PO(X)$ and $A \in \tau$, then $V \cap A \in \mathcal{G}PO(X)$.*

Proposition 3.3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space.*

(1) *If $A, B \in \mathcal{G}\alpha O(X)$, then $A \cap B \in \mathcal{G}\alpha O(X)$.*
(2) *If $A_i \in \mathcal{G}\alpha O(X)$ for each $i \in I$, then $\cup_{i\in I} A_i \in \mathcal{G}\alpha O(X)$.*

Proof. (1) Let $A, B \in \mathcal{G}\alpha O(X)$. By Theorem 3.1 A is $\mathcal{G}$-semi-open and $\mathcal{G}$-pre-open and by Proposition 3.2 $A \cap B$ is $\mathcal{G}$-semi-open and $\mathcal{G}$-preopen. Therefore, $A \cap B \in G\alpha O(X)$.
(2) Let $A_i \in G\alpha O(X)$ for each $i \in I$. Then, we have

$$A_i \subseteq Int(\Psi(Int(A_i))) \subseteq Int(\Psi(Int(\cup_{i\in I} A_i))) \text{ and hence}$$
$$\cup_{i\in I} A_i \subseteq Int(\Psi(Int(\cup_{i\in I} A_i))).$$

This shows that $\cup_{i\in I} A_i \in \mathcal{G}\alpha O(X)$. □

Corollary 3.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then the family $\mathcal{G}\alpha O(X)$ is a topology for X such that $\tau \subseteq \mathcal{G}\alpha O(X) \subseteq \tau^\alpha$, where τ^α denotes the family of α-open sets of X.*

Proof. Since $\phi, X \in \mathcal{G}\alpha O(X)$, this is an immediate consequence of Propositions 3.1 and 3.3. □

Example 3.4. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and the grill*
$\mathcal{G} = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then
$\tau^\alpha = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}\}$ and
$\mathcal{G}\alpha O(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$ and hence $\tau \subsetneqq \mathcal{G}\alpha O(X) \subsetneqq \tau^\alpha$.

Remark 2. (1) The minimal grill is $\mathcal{G}= \{X\}$ in any a topological space (X, τ).
(2) The maximal grill is $\mathcal{G}= \mathcal{P}(X) \setminus \{\phi\}$ in any a topological space (X, τ).

The proofs of the following three corollary is straightforward, hence it is omitted.

Corollary 3.3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. If $\mathcal{G}= \mathcal{P}(X) \setminus \{\phi\}$. Then the following hold:*

(1) *A is $\mathcal{G}$-α-open if and only if A is α-open.*
(2) *A is $\mathcal{G}$-preopen if and only if A is preopen.*
(3) *A is $\mathcal{G}$-semi-open if and only if A is semi-open.*
(4) *A is $\mathcal{G}$-β-open if and only if A is β-open.*

Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $\mathcal{G}= \{X\}$, then $\Phi(A) = \phi$ for any subset A of X and $\Psi(A) = \tau_{\mathcal{G}}\text{-}Cl(A) = A$ and hence $\tau_{\mathcal{G}} = \tau_{dis}$, where τ_{dis} is the discrete topology on X.

Corollary 3.4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. If $\mathcal{G}= \{X\}$. Then the following hold:*

(1) *A is $\mathcal{G}$-α-open if and only if A is open.*
(2) *A is $\mathcal{G}$-preopen if and only if A is open.*
(3) *A is $\mathcal{G}$-semi-open if and only if A is open.*
(4) *A is $\mathcal{G}$-β-open if and only if A is semi-open.*

Corollary 3.5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. If $\Phi(A) = Cl(Int(Cl(A)))$ for any subset A of X. Then the following hold:*

(1) *A is $\mathcal{G}$-α-open if and only if A is α-open.*
(2) *A is $\mathcal{G}$-β-open if and only if A is β-open.*

Recall that (X, τ) is called submaximal if every dense subset of X is open.

Lemma 3.3. [7] *If (X, τ) is submaximal, then $PO(X, \tau) = \tau$.*

Corollary 3.6. *If (X, τ) is submaximal, then for any gril $\mathcal{G}$ on X, $\tau = \alpha O(X) = PO(X, \tau) = \mathcal{G}PO(X) = \mathcal{G}\alpha O(X)$.*

Theorem 3.5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A, B subsets of X. If $U_\alpha \in \mathcal{G}SO(X, \tau)$ for each $\alpha \in \Delta$, then $\cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{G}SO(X, \tau)$.*

Proof. Since $U_\alpha \in \mathcal{G}SO(X, \tau)$, we have $U_\alpha \subseteq \Psi(Int(U_\alpha))$ for each $\alpha \in \Delta$. Thus we obtain $U_\alpha \subseteq \Psi(Int(U_\alpha)) \subseteq \Psi(Int(\cup_{\alpha\in\Delta} U_\alpha))$ and hence $\cup_{\alpha\in\Delta} U_\alpha \subseteq \Psi(Int(\cup_{\alpha\in\Delta}(U_\alpha)))$. This shows that $\cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{G}SO(X, \tau)$. □

Definition 3.2. A subset F of a grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-semi-closed (resp. $\mathcal{G}$-preclosed) if its complement is $\mathcal{G}$-semi-open (resp. $\mathcal{G}$-preopen).

Theorem 3.6. *If a subset A of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-closed, then $Int(\Psi(A)) \subseteq A$.*

Theorem 3.7. *If a subset A of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-preclosed, then $\Psi(Int(A)) \subseteq A$.*

Definition 3.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is called

(1) a g_1-set if $Int(\Psi(Int(A))) = Int(A)$,

(2) a g_2-set if $\Psi(Int(A)) = Int(A)$.

Definition 3.4. Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is called

(1) a G_1-set if $A = U \cap V$, where $U \in \tau$ and V is a g_1-set,

(2) a G_2-set if $A = U \cap V$, where $U \in \tau$ and V is a g_2-set.

Theorem 3.8. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset A of X, the following conditions are equivalent:*

(1) *A is open;*

(2) *A is $\mathcal{G}$-α-open and a G_1-set;*

(3) *A is $\mathcal{G}$-semi-open and a G_2-set.*

Proof. (1) $\Rightarrow$ (2) Let A be any open set. Then we have $A = Int(A) \subseteq Int(\Psi(Int(A)))$. Therefore A is $\mathcal{G}$-α-open and because X is a g_1-set, hence A is a G_1-set.

(2) $\Rightarrow$ (1) Let A be $\mathcal{G}$-α-open and a G_1-set. Let $A = U \cap C$, where U is open and $Int(\Psi(Int(C))) = Int(C)$. Since A is a $\mathcal{G}$-α-open set, we have

$$\begin{aligned} U \cap C &\subseteq Int(\Psi(Int(U \cap C))) \\ &= Int(\Psi(Int(U) \cap Int(C))) \\ &= Int(\Psi(U \cap Int(C))) \\ &\subseteq Int(\Psi(U) \cap \Psi(Int(C))) \\ &= Int(\Psi(U)) \cap Int(\Psi(Int(C))) \\ &= Int(\Psi(U)) \cap Int(C). \end{aligned}$$

Since $U \subseteq Int(\Psi(U))$, we have
$U \cap C = (U \cap C) \cap U \subseteq Int(\Psi(U)) \cap Int(C) \cap U = U \cap Int(C) = Int(U \cap C)$. Therefore, $A = U \cap C$ is an open set.

(1) ⇒ (3) This is obvious, because X is a g_2-set, then A is a G_2-set.

(3) ⇒ (1) Suppose that A is $\mathcal{G}$-semi-open and a G_2-set. Let $A = U \cap C$, where U is open and $\Psi(Int(C)) = Int(C)$. Since A is a $\mathcal{G}$-semi-open set, we have

$$\begin{aligned} U \cap C &\subseteq \Psi(Int(U \cap C)) \\ &= \Psi(Int(U) \cap Int(C)) \\ &= \Psi(U \cap Int(C)) \\ &\subseteq \Psi(U) \cap \Psi(Int(C)) \\ &= \Psi(U) \cap Int(C). \end{aligned}$$

Since $U \subseteq \Psi(U)$, we have
$U \cap C = (U \cap C) \cap U \subseteq \Psi(U) \cap Int(C) \cap U = U \cap Int(C) = Int(U \cap C)$.
Therefore, $A = U \cap C$ is an open set. □

The notion of $\mathcal{G}$-α-openness (resp. $\mathcal{G}$-semi-openness) is different from that of G_1-sets (resp. G_2-sets).

Remark 3. (1) In Example 3.1, $A = \{a, b\}$ is a g_1-set and hence a G_1-set but it is not $\mathcal{G}$-α-open. And $B = \{a, b, c\}$ is $\mathcal{G}$-α-open but it is not a G_1-set.
(2) In Example 3.2, $A = \{a, c, d\}$ is a g_2-set and hence a G_2-set but it is not $\mathcal{G}$-semi-open.
(3) In Example 3.1, $B = \{a, b, c\}$ is $\mathcal{G}$-semi-open but it is not a G_2-set.

4. **Decompositions of Continuity**

Definition 4.1. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be grill α-continuous (resp. grill semi-continuous, grill pre-continuous [5]) if the inverse image of each open set of Y is $\mathcal{G}$-α-open (resp. $\mathcal{G}$-semi-open, $\mathcal{G}$-preopen).

Theorem 4.1. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is grill α-continuous;*
(2) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $W \in \mathcal{G}\alpha O(X)$ containing x such that $f(W) \subseteq V$;*
(3) *The inverse image of each closed set in Y is $\mathcal{G}$-α-closed;*
(4) *$Cl(Int_{\mathcal{G}}(Cl(f^{-1}(B)))) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$;*
(5) *$f(Cl(Int_{\mathcal{G}}(Cl(A)))) \subseteq Cl(f(A))$ for each $A \subseteq X$.*

Proof. The implications follow easily from the definitions. □

Corollary 4.1. *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be grill α-continuous, then*

(1) *$f(\Psi(U)) \subseteq Cl(f(U))$ for each $U \in \mathcal{G}PO(X)$.*
(2) *$\Psi(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for each $V \in \mathcal{G}PO(Y)$.*

Theorem 4.2. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill α-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is grill α-continuous.*

Definition 4.2. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{H})$ is said to be grill irresolute if $f^{-1}(V)$ is $\mathcal{G}$-semi-open in $(X, \tau, \mathcal{G})$ for each $\mathcal{G}$-semi-open V of $(Y, \sigma, \mathcal{H})$.

Remark 4. It is obvious that continuity implies grill semi-continuity and grill semi-continuity implies semi-continuity.

Theorem 4.3. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following are equivalent:*

(1) *f is grill semi-continuous.*
(2) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \mathcal{G}SO(X)$ containing x such that $f(U) \subseteq V$.*
(3) *The inverse image of each closed set in Y is $\mathcal{G}$-semi-closed.*

Theorem 4.4. *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{H})$ be grill semi-continuous and $f^{-1}(\Psi(V)) \subseteq \Psi(f^{-1}(V))$ for each $V \in \sigma$. Then f is grill irresolute.*

Proof. Let B be any $\mathcal{G}$-semi-open set of $(Y, \sigma, \mathcal{H})$. By Theorem 3.3, there exists $V \in \sigma$ such that $V \subseteq B \subseteq \Psi(V)$. Therefore, we have $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(\Psi(V)) \subseteq \Psi(f^{-1}(V))$. Since f is grill semi-continuous and $V \in \sigma$, $f^{-1}(V) \in \mathcal{G}SO(X)$ and hence by Theorem 3.4, $f^{-1}(B)$ is a $\mathcal{G}$-semi-open set of $(X, \tau, \mathcal{G})$. This shows that f is grill irresolute. □

Theorem 4.5. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill semi-continuous if and only if the graph function $g : X \to X \times Y$ is grill semi-continuous.*

Theorem 4.6. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill α-continuous if and only if it is grill semi-continuous and grill pre-continuous.*

Proof. This is an immediate consequence of Theorem 3.1. □

Theorem 4.7. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill α-continuous if and only if $f : (X, \mathcal{G}\alpha O(X)) \to (Y, \sigma)$ is continuous.*

Proof. This is an immediate consequence of Corollary 3.2. □

Definition 4.3. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be G_1-continuous (resp. G_2-continuous) if the inverse image of each open set of Y is G_1-open (resp. G_2-open) in $(X, \tau, \mathcal{G})$.

Theorem 4.8. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) *f is continuous;*

(2) *f is grill α-continuous and G_1-continuous;*

(3) *f is grill semi-continuous and G_2-continuous.*

Proof. This is an immediate consequence of Theorem 3.8. □

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