NEW CONNECTIONS AMONG MULTIVALUED FUNCTIONS, HYPERSTRUCTURES AND FUZZY SETS

P. CORSINI AND B. DAVVAZ

ABSTRACT. One has considered three different hypergroups associated with a multivalued function $\Gamma : H \to D$ and one has calculated the corresponding membership functions and the associated join spaces.

1. INTRODUCTION

The interest for Multivalued Functions is increasing in the mathematical world. They have revealed themselves useful in several mathematical sectors as Analysis, Differential Equations, Physics, Game Theory, Algebra and Combinatorics. Especially in Algebra and Combinatorics, a particular case of Multivalued Functions, that is Hyperoperations, is studied in all continents since 1934, [1, 8, 12].

In this paper one establishes a connection between a Hyperstructure (i.e. a set endowed with hyperoperations) and a multivalued function , by considering the hypergroupoid defined as follows:

Given the sets U, D and given a multivalued function $\Gamma : U \to D$, that is a function Γ from U to the set $P(D)$ of non-empty subsets of D, the hyperoperation $\circ$ on U is

defined as follows:

$$\forall (x, y) \in U^2, \ x \circ_\Gamma y = \Gamma^{-1}\Gamma(x) \cup \Gamma^{-1}\Gamma(y).$$

For any multivalued function Γ, $\Gamma^{-1}\Gamma(x)$ can have different meanings: that one which have been considered in a precedent paper (see [4]):

(1) $\Gamma^{-1}\Gamma(x) = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$;

In this paper, one considers two others:

(2) $\Gamma^{-1}\Gamma(x) = \{y \mid \Gamma(y) \subseteq \Gamma(x)\}$;
(3) $\Gamma^{-1}\Gamma(x) = \{\mid \Gamma(z) \supseteq \Gamma(x)\}$;

another one is the following

(4) $\Gamma^{-1}\Gamma(x) = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$, supposing that

$$\Gamma(y) \cap \Gamma(x) \neq \emptyset \Rightarrow \{z \mid \Gamma(z) \cap \Gamma(x) \neq \emptyset\} = \{w \mid \Gamma(w) \cap \Gamma(y) \neq \emptyset\}.$$

2. Hyperoperations Defined by a Multivaued Function

Let Γ be a multivalued function, $\Gamma : H \to D$ is such that $\forall (x, y) \in H^2$ the following implication holds:

$$(3) \qquad \Gamma(x) \cap \Gamma(y) \neq \emptyset \Rightarrow \{z \mid \Gamma(z) \cap \Gamma(x) \neq \emptyset\} = \{w \mid \Gamma(w) \cap \Gamma(y) \neq \emptyset\}.$$

Let $(H; \circ_{3\Gamma})$ be the hypergroupoid defined, starting from a function Γ satisfying (3), as follows:

$$\forall x, \quad x \circ_{3\Gamma} x = \{v \mid \Gamma(v) \cap \Gamma(x) \neq \emptyset\}$$

$$\forall (x, y), \quad x \circ_{3\Gamma} y = x \circ_{3\Gamma} x \cup y \circ_{3\Gamma} y.$$

Let us consider the hyperoperations defined in a set H by a multivalued function $\Gamma : H \to D$, as follows

$$\forall x, \quad x \circ_\Gamma x = \Gamma^{-1}\Gamma(x)$$

$$\forall (x, y), \quad x \circ_\Gamma y = \Gamma^{-1}\Gamma(x) \cup \Gamma^{-1}\Gamma(y),$$

where by $\Gamma^{-1}\Gamma(x)$ we intend respectively:

(0) $A_0(x) = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$

(1) $A_1(x) = \{z \mid \Gamma(z) \subseteq \Gamma(x)\}$

(2) $A_2(x) = \{w \mid \Gamma(x) \subseteq \Gamma(w)\}$

(3) $A_3(x) = \{v \mid \Gamma(v) \cap \Gamma(x) \neq \emptyset\}$ and the condition (3) is satisfied.

Join spaces were introduced by W. Prenowitz and then applied by him and J. Jantosciak both in Euclidian and in non Euclidian geometry [13, 14]. Using this notion, several branches of non Euclidian geometry were rebuilt: descriptive geometry, projective geometry and spherical geometry. Then, several important examples of join spaces have been constructed in connection with binary relations, graphs, lattices, fuzzy sets, rough sets, see [2, 5, 7, 9, 10, 11, 15].

In order to define a join space, we need the following notation: If a, b are elements of a hypergroupoid $(H, \circ)$, then we denote $a/b = \{x \in H \mid a \in x \circ b\}$. Moreover, by A/B we intend the set $\bigcup\limits_{a \in A, b \in B} a/b$.

A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds for all elements a, b, c, d of H:

$$a/b \cap c/d \neq \emptyset \implies a \circ d \cap b \circ c \neq \emptyset.$$

Theorem 1. $(H; \circ_{1\Gamma})$ is a hypergroup, which is not generally a join space.

Proof. For all $(x, y, z) \in H^3$, we have

$$(x \circ_{1\Gamma} y) \circ_{1\Gamma} z = (x \circ_{1\Gamma} x \cup y \circ_{1\Gamma} y) \circ_{1\Gamma} z =$$

$$= (\{a \mid \Gamma(a) \subseteq \Gamma(x)\} \cup \{b \mid \Gamma(b) \subseteq \Gamma(y)\}) \circ_{1\Gamma} z.$$

Set

$$A = \{\lambda \mid \exists a : \Gamma(\lambda) \subseteq \Gamma(a), \Gamma(a) \subseteq \Gamma(x)\} \cup \{\mu \mid \exists b : \Gamma(\mu) \subseteq \Gamma(b), \Gamma(b) \subseteq \Gamma(y)\}.$$

We clearly have $(x \circ_{1\Gamma} y) \circ_{1\Gamma} z = A \cup \{c \mid \Gamma(c) \subseteq \Gamma(z)\}$.

We have

$$A \supseteq A' = \{a \mid \Gamma(a) \subseteq \Gamma(x)\{\cup\{b \mid \Gamma(b) \subseteq \Gamma(y)\}.$$

Indeed it is enough to set $\lambda = a, \mu = b$.

On the converse, $\Gamma(\lambda) \subseteq \Gamma(a)$ implies that $\Gamma(\lambda) \subseteq \Gamma(x)$, so $A \subseteq A'$.

By consequence, we have

$$(x \circ_{1\Gamma} y) \circ_{1\Gamma} z = \{a \mid \Gamma(a) \subseteq \Gamma(x)\} \cup \{b \mid \Gamma(b) \subseteq \Gamma(y)\} \cup \{c \mid \Gamma(c) \subseteq \Gamma(z)\} =$$

$$= x \circ_{1\Gamma} (y \circ_{1\Gamma} z).$$

In order to see that $(H; \circ_{1\Gamma})$ is not a join space, it is enough to consider the following example:

$H = \{1, 2, 3\}$ and $\Gamma : H \to \{u, v\}$ is defined as follows:

$$\Gamma(1) = u, \quad \Gamma(2) = v, \quad \Gamma(3) = \{u, v\}.$$

So, we obtain the hypergroupoid:

$\circ_{1\Gamma}$	1	2	3
1	1	1 2	1 2 3
2		2	1 2 3
3			1 2 3

We have $1/2 = \{x \mid 1 \in 2 \circ x\} = \{1,3\}, \quad 2/1 = \{y \mid 2 \in 1 \circ y\} = \{2,3\}$. So, $1/2 \cap 2/1 \neq \emptyset$, but we find also $1 \circ_\Gamma 1 \cap 2 \circ_\Gamma 2 = \emptyset$.

Theorem 2. $(H; \circ_{2\Gamma})$ is a hypergroup, which is not generally a join space.

Proof. For all $(x, y, z) \in H^3$, we have

$$(x \circ_{2\Gamma} y) \circ_{2\Gamma} z = (x \circ_{2\Gamma} x \cup y \circ_{2\Gamma} y) \circ_{2\Gamma} z =$$

$$= (\{u \mid \Gamma(x) \subseteq \Gamma(u)\} \cup \{v \mid \Gamma(y) \subseteq \Gamma(v)\}) \circ_{2\Gamma} z =$$

$$= (\bigcup_{\Gamma(x) \subseteq \Gamma(u)} u \circ_{2\Gamma} u \cup \bigcup_{\Gamma(y) \subseteq \Gamma(v)} v \circ_{2\Gamma} v) \circ_{2\Gamma} z =$$

$$= (\{\lambda \mid \exists u : \Gamma(x) \subseteq \Gamma(u), \ \Gamma(u) \subseteq \Gamma(\lambda)\} \cup$$

$$\cup \{\mu \mid \exists v : \Gamma(y) \subseteq \Gamma(v), \ \Gamma(v) \subseteq \Gamma(\mu)\} \cup \{c \mid \Gamma(z) \subseteq \Gamma(c)\}.$$

By a similar way as in Theorem 1, one sees that setting

$$B' = \{\lambda \mid \exists u : \Gamma(x) \subseteq \Gamma(u) \subseteq \Gamma(\lambda)\} \cup \{\mu \mid \exists v : \Gamma(y) \subseteq \Gamma(v) \subseteq \Gamma(\mu)\}$$

one has $B = B' = \{u \mid \Gamma(u) \supseteq \Gamma(x)\} \cup \{v \mid \Gamma(v) \supseteq \Gamma(y)\}$. By consequence,

$$(x \circ_{2\Gamma} y) \circ_{2\Gamma} z = x \circ_{2\Gamma} (y \circ_{2\Gamma} z).$$

Also in $(H; \circ_{2\Gamma})$ every element is an identity, so $(H; \circ_{2\Gamma})$ is a hypergroup.

Let us see now an example of a $\circ_{2\Gamma}$ hypergroup H^*, which is not a join space.

Let $\Gamma : H^* \to D$ be a multivalued function from $H^* = \{1, 2, 3, 4\}$ to $D = \{d_1, d_2, d_3, d_4\}$ defined by

$$\Gamma(1) = \{d_1\}, \ \ \Gamma(2) = \{d_1, d_2\}, \ \ \Gamma(3) = \{d_1, d_4\}, \ \ \Gamma(4) = \{d_4\}.$$

We have clearly

$$1 \circ_{2\Gamma} 1 = \{x \mid \Gamma(1) \subseteq \Gamma(x)\} = \{1, 2, 3\}, \ \ 2 \circ_{2\Gamma} 2 = \{y \mid \Gamma(2) \subseteq \Gamma(y)\} = \{2\},$$

$$3 \circ_{2\Gamma} 3 = \{z \mid \Gamma(3) \subseteq \Gamma(z)\} = \{3\}, \ \ 4 \circ_{2\Gamma} 4 = \{v \mid \Gamma(4) \subseteq \Gamma(v)\} = \{3, 4\}.$$

So, we obtain the hypergroupoid:

$\circ_{2\Gamma}$	1	2	3	4
1	1 2 3	1 2 3	1 2 3	H^*
2		2	2 3	2 3 4
3			3	3 4
4				3 4

which is a hypergroup, but it is not a join space. Indeed, we have

$2/3 = \{x \mid 2 \in 3 \circ x\} = \{1, 2\}, \ \ 3/2 = \{y \mid 3 \in 2 \circ y\} = \{1, 3, 4\}$. So, $2/3 \cap 3/2 \neq \emptyset$, but we have also $2 \circ_{2\Gamma} 2 \cap 3 \circ_{2\Gamma} 3 = \emptyset$.

By consequence, $(H^*; \circ_{2\Gamma})$ is not generally a join space.

Let (i) be the following condition:

$$(i) \ \ \ \ \forall x \in H, \ \forall y \in H, \ \Gamma(x) \subseteq \Gamma(y) \Rightarrow \Gamma(x) = \Gamma(y)$$

and let (δ) be the condition:

$$(\delta) \ \ \ \forall (i, j), \ x_i \circ_{1\Gamma} x_i \cap x_j \circ_{1\Gamma} x_j \neq \emptyset \Rightarrow x_i \circ_{1\Gamma} x_i = x_j \circ_{1\Gamma} x_j.$$

Then $\forall(x, y)$ if $\Gamma(y) \subseteq \Gamma(x)$ then we have $y \in x \circ_{1\Gamma} x$ whence $y \circ_{1\Gamma} y \cap x \circ_{1\Gamma} x \neq \emptyset$, so $y \circ_{1\Gamma} y = x \circ_{1\Gamma} x$. By consequence (δ) implies (i).

3. On the Hypergroups $(H; \circ_{1\Gamma})$ and $(H; \circ_{2\Gamma})$

Proposition 1. If the condition (i) is satisfied, then $(H; \circ_{1\Gamma})$ is a join space.

Proof. Indeed, if $a/b \cap c/d \neq \emptyset$. Then there exists $x \in H$ such that

$$a \in b \circ_{1\Gamma} x = b \circ_{1\Gamma} b \cup x \circ_{1\Gamma} x$$

and

$$c \in d \circ_{1\Gamma} x = d \circ_{1\Gamma} d \cup x \circ_{1\Gamma} x.$$

If $a \in b \circ_{1\Gamma} b = \{y \mid \Gamma(y) \subseteq \Gamma(b)\}$, then it follows that $\Gamma(y) = \Gamma(b)$. Therefore $\Gamma(a) = \Gamma(b)$, whence $b \in a \circ_{1\Gamma} a$, so $b \in a \circ_{1\Gamma} d$, hence

$$b \in (a \circ_{1\Gamma} a \cup d \circ_{1\Gamma} d) \cap (b \circ_{1\Gamma} b \cup c \circ_{1\Gamma} c).$$

If $a \in x \circ_{1\Gamma} x$, then it follows that $\Gamma(a) = \Gamma(x)$.

If $c \in x \circ_{1\Gamma} x$, then we have $\Gamma(c) = \Gamma(x)$.

So $\Gamma(c) = \Gamma(a)$, whence $a \in a \circ_{1\Gamma} a \subseteq a \circ_{1\Gamma} d$, $a \in c \circ_{1\Gamma} c \subseteq c \circ_{1\Gamma} b$. It follows that $a \circ_{1\Gamma} d \cap b \circ_{1\Gamma} c \neq \emptyset$.

Proposition 2. If the condition (i) holds, then $(H; \circ_{2\Gamma})$ is a join space.

Proof. Indeed, if $a/b \cap c/d \neq \emptyset$, then there exists $x \in H$ such that

$$a \in b \circ_{2\Gamma} x = b \circ_{2\Gamma} b \cup x \circ_{2\Gamma} x$$

and

$$c \in d \circ_{2\Gamma} x = d \circ_{2\Gamma} d \cup x \circ_{2\Gamma} x.$$

We have $b \circ_{2\Gamma} b = \{z \mid \Gamma(b) \subseteq \Gamma(z)\}$, $d \circ_{2\Gamma} d = \{u \mid \Gamma(d) \subseteq \Gamma(u)\}$ and $x \circ_{2\Gamma} x = \{\lambda \mid \Gamma(x) \subseteq \Gamma(\lambda)\}$.

If $a \in b \circ_{2\Gamma} b$, then it follows that $\Gamma(b) \subseteq \Gamma(a)$.

If $c \in d \circ_{2\Gamma} d$, then $\Gamma(d) \subseteq \Gamma(c)$.

So $\Gamma(a) = \Gamma(b)$, or respectively $\Gamma(d) = \Gamma(c)$. Since

$$a \circ_{2\Gamma} d = \{v \mid \Gamma(a) \subseteq \Gamma(v)\} \cup \{w \mid \Gamma(d) \subseteq \Gamma(w)\}$$

and

$$b \circ_{2\Gamma} c = \{s \mid \Gamma(b) \subseteq \Gamma(s)\} \cup \{t \mid \Gamma(c) \subseteq \Gamma(t)\}$$

it follows that $\Gamma(a) = \Gamma(b)$ implies that $a \in b \circ_{2\Gamma} c$, so $a \in a \circ_{2\Gamma} d \cap b \circ_{2\Gamma} c$.

If $a \in x \circ_{2\Gamma} x$ and $c \in x \circ_{2\Gamma} x$, then $\Gamma(a) = \Gamma(x) = \Gamma(c)$. It follows that

$$a \in b \circ_{2\Gamma} c \cap a \circ_{2\Gamma} d = (b \circ_{2\Gamma} b \cup c \circ_{2\Gamma} c) \cap (a \circ_{2\Gamma} a \cup d \circ_{2\Gamma} d).$$

Hence, in every case, the join space condition is satisfied.

Here we give an example of a hypergroup $(H; \circ_{2\Gamma})$ which is not (i), but it is a join space. Let now $\Gamma : I(4) \to D$ be a multivalued function from $I(4) = \{1, 2, 3, 4\}$ to $D = \{u, v, w, z\}$ defined by

$$\Gamma(1) = \{u, v, w\}, \ \Gamma(2) = \{u, v\}, \ \ \Gamma(3) = \{v, z\}, \ \ \Gamma(4) = \{z\}.$$

We find

$$1 \circ_{2\Gamma} 1 = \{1\}, \ 2 \circ_{2\Gamma} 2 = \{1, 2\},$$

$$3 \circ_{2\Gamma} 3 = \{3\}, \ 4 \circ_{2\Gamma} 4 = \{z \mid \Gamma(4) \subseteq \Gamma(z)\} = \{3, 4\}.$$

So, we obtain the hypergroupoid:

$I(4), \circ_{2\Gamma}$	1	2	3	4
1	1	1 2	1 3	1 3 4
2	1 2	1 2	1 2 3	H
3	1 3	1 2 3	3	3 4
4	1 3 4	H	3 4	3 4

To check that the join space property is satisfied, it is sufficient to check it in the following cases (I), (II), (III).

(I)

   1) $1/2 \cap a/b \neq \emptyset \Rightarrow 1 \in 1 \circ_{2\Gamma} b \cap 2 \circ_{2\Gamma} a.$

   2) $3/4 \cap c/d \neq \emptyset \Rightarrow 3 \in 3 \circ_{2\Gamma} d \cap 4 \circ_{2\Gamma} c.$

   3) $2/1 \cap a/b \neq \emptyset \Rightarrow 1 \in 2 \circ_{2\Gamma} b \cap 1 \circ_{2\Gamma} a.$

   4) $4/3 \cap c/d \neq \emptyset \Rightarrow 3 \in 4 \circ_{2\Gamma} d \cap 3 \circ_{2\Gamma} c.$

(II)

   1) $a/a \cap b/c \neq \emptyset \Rightarrow a \in a \circ_{2\Gamma} c \cap a \circ_{2\Gamma} b.$

   2) $a/b \cap a/c \neq \emptyset \Rightarrow a \in a \circ_{2\Gamma} c \cap a \circ_{2\Gamma} b.$

   3) $b/a \cap c/a \neq \emptyset \Rightarrow b \circ_{2\Gamma} a \cap c \circ_{2\Gamma} a \neq \emptyset.$

(III)

   1) $a/2 \cap 4/b \neq \emptyset \Rightarrow a \circ_{2\Gamma} b \cap 2 \circ_{2\Gamma} 4 = a \circ_{2\Gamma} b \neq \emptyset.$

   2) $a/4 \cap 2/b \neq \emptyset \Rightarrow a \circ_{2\Gamma} b \cap 4 \circ_{2\Gamma} 2 = a \circ_{2\Gamma} b \neq \emptyset$

Theorem 3. Let $\Gamma : H \to D$ be a multivalued fuction, such that the condition (i) is satisfied. Then we have $(H; \circ_{2\Gamma}) = (H; \circ_{1\Gamma})$ is a join space and $\forall (x, y)$

    1) if $\Gamma(x) = \Gamma(y)$, then $x/y = H$,

    2) if $\Gamma(x) \neq \Gamma(y)$, then $x/y = x \circ_{2\Gamma} x = \{z \mid \Gamma(z) = \Gamma(x)\}$.

Proof. By Proposition 1, it follows that $(H; \circ_{2\Gamma}) = (H; \circ_{1\Gamma})$ is a join space.

1) It is enough to remark that

$$x/y = \{a \mid x \in a \circ_{2\Gamma} y = a \circ_{2\Gamma} a \cup y \circ_{2\Gamma} y\}$$

and since $\Gamma(x) = \Gamma(y)$, $x \in y \circ_{2\Gamma} y$.

2) We have $x/y = \{z \mid x \in y \circ_{2\Gamma} z = y \circ_{2\Gamma} y \cup z \circ_{2\Gamma} z\} = \{z \mid x \in z \circ_{2\Gamma} z\}$ and $z \circ_{2\Gamma} z = \{u \mid \Gamma(z) = \Gamma(u)\}$ whence $x/y = \{z \mid \Gamma(x) = \Gamma(z)\}$.

4. On the Hypergroup $(H; \circ_{1\Gamma})$

Theorem 4. The hypergroup $(H, \circ_{3\Gamma})$ defined by $A_3(x)$ is a join space.

Proof. Indeed, if $a/b \cap c/d \neq \emptyset$, there exists p such that $a \in b \circ_{3\Gamma} p$, $c \in d \circ_{3\Gamma} p$. If $a \in b \circ_{3\Gamma} b$, then from $a \in b \circ_{3\Gamma} b \cap a \circ_{3\Gamma} a$, it follows that

$$a \circ_{3\Gamma} a = b \circ_{3\Gamma} b \subseteq b \circ_{3\Gamma} c \cap a \circ_{3\Gamma} d.$$

By the same way, we check the join space condition in the case $c \in d \circ_{3\Gamma} d$. Let us suppose now that $\{a, c\} \subseteq p \circ_{3\Gamma} p$. So, $\Gamma(a) \cap \Gamma(p) \neq \emptyset \neq \Gamma(c) \cap \Gamma(p)$. It follows that $p \in a \circ_{3\Gamma} a \cap c \circ_{3\Gamma} c$. By consequence, $a \circ_{3\Gamma} a = c \circ_{3\Gamma} c$, whence

$$a \circ_{3\Gamma} d \cap b \circ_{3\Gamma} c \supseteq a \circ_{3\Gamma} a = c \circ_{3\Gamma} c.$$

Then $(H, \circ_{3\Gamma})$ is a join space.

Let $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ be the classes of hypergroups of type $A_0(x)$, $A_1(x)$, $A_2(x)$, $A_3(x)$. It is clear that $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \subseteq \mathcal{A}_0$.

Let us consider the hypergroup $(H, \circ_{3\Gamma})$.

Let us suppose that $H = \{u_1, u_2, ..., u_n\}$ and let us denote $\forall i \leq n$, $u_i \circ_{3\Gamma} u_i = P_i$. Moreover, $\forall(i,j)$, $i \neq j$, $P_i \cap P_j = \emptyset$. Setting

$$_0H = \{u_1^1, u_2^1, ..., u_{k_1}^1, u_1^2, u_2^2, ..., u_{k_2}^2, ..., u_1^s, ..., u_{k_s}^s\}.$$

where $\sum_{i=1}^s = n$, for all q, $1 \leq q \leq k_r$, we obtain

$$u_q^r \circ_{3\Gamma} u_q^r = \{u_t^r \mid 1 \leq t \leq k_r\} = P_r.$$

Theorem 5. The hypergroup $_0H$ we described above has the structure represented by the following table:

$\circ_{3\Gamma}$	u_1^1	...	$u_{k_1}^1$	u_1^2	...	$u_{k_2}^2$	...	u_1^s	u_2^s	...	$u_{k_s}^s$
u_1^1	P_1	...	P_1	$P_1 \cup P_2$	...	$P_1 \cup P_2$	...	$P_1 \cup P_s$	$P_1 \cup P_s$	...	$P_1 \cup P_s$
...			...	...	...	...	...	...	...	...	...
$u_{k_1}^1$			P_1	$P_1 \cup P_2$	...	$P_1 \cup P_2$	...	$P_1 \cup P_s$	$P_1 \cup P_s$	...	$P_1 \cup P_s$
u_1^2				P_2	...	P_2	...	$P_2 \cup P_s$	$P_2 \cup P_s$	...	$P_2 \cup P_s$
...					...	...	...	...	...	...	...
$u_{k_2}^2$						P_2	...	$P_2 \cup P_s$	$P_2 \cup P_s$	...	$P_2 \cup P_s$
...								...	...	...	...
u_1^s								P_s	P_s	...	P_s
u_2^s									P_s	...	P_s
...										...	...
$u_{k_s}^s$											P_s

Let us calculate now μ_1 on $_0H$. We have $\forall r \leq s, \forall j \leq k_r$,

$$A_1(u_j^r) = k_r^2/k_r + 2\sum_{i \neq r} k_i k_r/(k_i + k_r)$$

$$q_1(u_j^r) = k_r^2 + 2\sum_{i \neq r} k_i k_r,$$

whence

$$\mu_1(u_j^r) = [1 + 2\sum_{i \neq r} k_i/(k_i + k_r)]/[k_r + 2\sum_{i \neq r} k_i].$$

Let us suppose that $\forall(i,j)$, $k_i = k_j = p$. Then we have $\forall r$, $\forall j : 1 \leq j \leq k_j$,

$$A_1(u_j^r) = p^2/p + (s-1)2p^2/2p = ps$$

$$q_1(u_j^r) = p^2 + 2p^2(s-1) = p^2(2s-1),$$

whence

$$\mu_1(u_j^r) = sp/p^2(2s-1) = s/p(2s-1).$$

So, we have clearly $_1H = T$, see [3] and $\partial(_0H) = 1$, see[6].

Let us suppose now that $s = 3$. So $\forall j$, $1 \leq j \leq k_1$ we have

$$\mu_1(u_j^1) = [1 + 2k_2/(k_1 + k_2) + 2k_3/(k_1 + k_3)] / (k_1 + 2k_2 + 2k_3)$$

and $\forall t$, $1 \leq t \leq k_2$, $\forall f$, $1 \leq f \leq k_3$, we have

$$\mu_1(u_t^2) = [1 + 2k_1/(k_2 + k_1) + 2k_3/(k_2 + k_3)] / (k_2 + 2k_1 + 2k_3),$$

$$\mu_1(u_f^3) = [1 + 2k_1/(k_1 + k_3) + 2k_2/(k_2 + k_3)] / (k_3 + 2k_1 + 2k_2).$$

Setting

$$I = [(k_1 + k_2)(k_1 + k_3) + 2k_2(k_1 + k_3) + 2k_3(k_1 + k_2)](k_3 + k_2)(k_3 + 2k_1 + 2k_2),$$

$$II = [(k_1 + k_2)(k_1 + k_3) + 2k_2(k_1 + k_3) + 2k_3(k_1 + k_2)](k_2 + k_3)(k_2 + 2k_1 + 2k_3),$$

$$III = [(k_1 + k_3)(k_2 + k_3) + 2k_1(k_2 + k_3) + 2k_2(k_1 + k_3)](k_1 + k_2)(k_1 + 2k_2 + 2k_3),$$

$$IV = [(k_1 + k_2)(k_2 + k_3) + 2k_1(k_2 + k_3) + 2k_3(k_1 + k_2)](k_1 + k_3)(k_1 + 2k_2 + 2k_3),$$

$$V = [(k_1 + k_2)(k_1 + k_3) + 2k_2(k_1 + k_3) + 2k_3(k_1 + k_2)](k_3 + k_2)(k_2 + 2k_1 + 2k_3),$$

we have

$$\mu_1(u_j^1) > \mu_1(u_f^3) \Leftrightarrow I > III,$$

$$\mu_1(u_t^2) > \mu_1(u_f^3) \Leftrightarrow II > IV,$$

$$\mu_1(u_j^1) > \mu_1(u_t^2) \Leftrightarrow V > IV.$$

Let us suppose now that $k_1 = p, \ k_2 = p + 1, \ k_3 = p + 2$. So we have

$$\mu_1(u_j^1) = [1 \ + \ 2(p+1)/(p+p+1) \ + \ 2(p+2)/(p+p+2)] \ / \ [p + 2(p+1) + 2(p+2)]$$

$$= [(2p + 1)(2p + 2) + (2p + 2)^2 + (2p + 4)(2p + 1)] \ / \ (2p + 1)(2p + 2)(5p + 6)$$

$$= (12p^2 + 24p + 10) \ / \ (2p + 1)(2p + 2)(5p + 6),$$

$$\mu_1(u_t^2) = [1 \ + \ 2p/(p+1+p) \ + \ 2(p+2)/(p+1+p+2)] \ / \ (p + 1 + 2p + 2p + 4)$$

$$= [1 + 2p/(2p + 1) + (2p + 4)/(2p + 3)] \ / \ (5p + 5)$$

$$= [(2p + 1)(2p + 3) + 2p(2p + 3) + (2p + 4)(2p + 1)] \ / \ (2p + 1)(2p + 3)(5p + 5)$$

$$= (12p^2 + 24p + 7) \ / \ (2p + 1)(2p + 3)(5p + 5).$$

$$\mu_1(u_f^3) = [1 \ + \ 2p/(p+p+2) \ + \ (2p+2)/(p+1+p+2)] \ / \ (p + 2 + 2p + 2p + 2)$$

$$= [(2p + 2)(2p + 3) + 2p(2p + 3) + (2p + 2)(2p + 2)] \ / \ (2p + 2)(2p + 3)(5p + 4)$$

$$= (12p^2 + 24p + 10) \ / \ (2p + 2)(2p + 3)(5p + 4).$$

We have $\mu_1(u_j^1) > \mu_1(u_t^2) \Leftrightarrow$

$$\Leftrightarrow (12p^2 + 24p + 10)(2p + 3)(5p + 5) > (12p^2 + 24p + 7)(2p + 2)(5p + 6)$$

$$\Leftrightarrow (12p^2 + 24p + 10)(10p^2 + 25p + 15) > (12p^2 + 24p + 7)(10p^2 + 22p + 12).$$

On the other hand, $\mu_1(u_t^2) > \mu_1(u_f^3) \Leftrightarrow$

$$\Leftrightarrow (12p^2 + 24p + 7) \,/\, (2p + 1)(5p + 5) > (12p^2 + 24p + 10) \,/\, (2p + 2)(5p + 4)$$

$$\Leftrightarrow 36p^3 + 78p^2 + 48p + 6 > 0.$$

So for all p, we have

$$\mu_1(u_j^1) > \mu_1(u_t^2) > \mu_1(u_f^3).$$

We obtain the join space represented by the following table:

$_1H$	u_1^1	u_2^1	...	$u_{k_1}^1$	u_1^2	...	$u_{k_2}^2$	u_1^3	u_2^3	...	$u_{k_3}^3$
u_1^1	P_1	P_1	...	P_1	$P_1 \cup P_2$	...	$P_1 \cup P_2$	H	H	...	H
u_2^1		P_1	...	P_1	$P_1 \cup P_2$	...	$P_1 \cup P_2$	H	H	...	H
...			...	...	...	...	...	...	...	...	...
$u_{k_1}^1$				P_1	$P_1 \cup P_2$	...	$P_1 \cup P_2$	H	H	...	H
u_1^2					P_2	...	P_2	$P_2 \cup P_3$	$P_2 \cup P_3$	...	$P_2 \cup P_3$
...			...	...	...	...	....	...	...	...	...
$u_{k_2}^2$						...	P_2	$P_2 \cup P_3$	$P_2 \cup P_3$	...	$P_2 \cup P_3$
u_1^3						...		P_3	P_3	...	P_3
u_2^3						...			P_3	...	P_3
...			...	...	...	...	...	...	...	...	...
$u_{k_3}^3$						...				...	P_3

We have $A_2(u_j^1) = k_1^2/k_1 \; + \; 2k_1k_2/(k_1 + k_2) \; + \; 2k_1k_3/(k_1 + k_2 + k_3),$

$$q_2(u_j^1) = k_1^2 + 2k_1k_2 + 2k_1k_3,$$

$A_2(u_t^2) = k_2^2/k_2 \; + \; 2k_1k_2/(k_1 + k_2) \; + \; 2k_2k_3/(k_2 + k_3) \; + \; 2k_1k_3/(k_1 + k_2 + k_3),$

$$q_2(u_t^2) = k_2^2 + 2k_1k_2 + 2k_2k_3 + 2k_1k_3,$$

$A_2(u_f^3) = k_3^2/k_3 \; + \; 2k_2k_3/(k_2 + k_3) \; + \; 2k_1k_3/(k_1 + k_2 + k_3),$

$$q_2(u_f^3) = k_3^2 + 2k_2k_3 + 2k_1k_3.$$

We obtain

$$\mu_2(u_j^1) = [1 + 2(p+1)/(p+p+1) + 2(p+2)/(p+p+1+p+2)] \; / \; [p + 2(p+1) + 2(p+2)]$$

$$= [(2p+1)(3p+3) + (2p+2)(3p+3) + (2p+4)(2p+1)] \; / \; (2p+1)(3p+3)(5p+6)$$

$$= (16p^2 + 31p + 13)/(2p+1)(3p+3)(5p+6),$$

$$\mu_2(u_f^3) = [1 + 2(p+1)/(p+1+p+2) + 2p/(p+p+1+p+2)]/[p+2+2(p+1)+2p] =$$

$$= (16p^2 + 33p + 15)/(2p+3)(3p+3)(5p+4).$$

We have $A_2(u_t^2) = [(p+1)(2p+1)(2p+3)(3p+3) \; + \; 2p(p+1)(2p+3)(3p+3) +$

$+ \; 2(p+1)(p+2)(2p+1)(3p+3) \; + \; 2p(p+2)(2p+1)(2p+3)] \; / \; (2p+1)(2p+3)(3p+3) =$

$$= [(2p^2+3p+1)(6p^2+15p+9) + (2p^2+2p)(6p^2+15p+9) + (2p^2+6p+4)(6p^2+9p+3) +$$

$$+ \; (2p^2 + 4p)(4p^2 + 8p + 3)] \; / \; (2p+1)(2p+3)(3p+3).$$

From $q_2(u_t^2) = p^2 + 2p + 1 + 2p^2 + 2p + 2p^2 + 6p + 4 + 2p^2 + p = 7p^2 + 14p + 5$, it follows that

$$\mu_2(u_t^2) = (44p^4 + 176p^3 + 239p^2 + 126p + 21) \; / \; (2p+1)(2p+3)(3p+3)(7p^2+14p+5).$$

We have $\mu_2(u_j^1) > \mu_2(u_f^3) \;\Leftrightarrow\; (16p^2 + 31p + 13) \,/\, (2p+1)(3p+3)(5p+6) >$

$$> (16p^2 + 33p + 15) \,/\, (2p+3)(3p+3)(5p+4) \Leftrightarrow$$

$$\Leftrightarrow 160p^4 + 678p^3 + 1035p^2 + 671p + 156 > 160p^4 + 602p^3 + 302p^2 + 453p + 90.$$

Therefore for all p we have

$$\mu_2(u_j^1) > \mu_2(u_f^3).$$

On the other hand, we have $\mu_2(u_j^1) > \mu_2(u_t^2) \;\Leftrightarrow\;$

$$\Leftrightarrow \; (16p^2 + 31p + 13) \,/\, (2p+1)(3p+3)(5p+6) \; >$$

$$> \; (44p^4 + 176p^3 + 239p^2 + 126p + 21) \,/\, (2p+1)(2p+3)(3p+3)(7p^2 + 14p + 5)$$

$$\Leftrightarrow 224p^5 + 1218p^4 + 2533p^3 + 2489p^2 + 1141p + 195$$

$$> 220p^5 + 1144p^4 + 2251p^3 + 2064p^2 + 861p + 126,$$

whence

$$\mu_2(u_j^1) > \mu_2(u_t^2).$$

Let us see now the connection between $\mu_2(u_t^2)$ and $\mu_2(u_f^3)$. We have

$$\mu_2(u_t^2) > \mu_2(u_f^3)$$

$$\Leftrightarrow (44p^4 + 176p^3 + 239p^2 + 126p + 21)(5p+4) > (16p^2 + 33p + 15)(2p+1)(7p^2 + 14p + 5)$$

$$\Leftrightarrow \alpha(p) = 220p^5 + 1056p^4 + 1899p^3 + 1586p^2 + 609p + 84$$

$$> \beta(p) = 224p^5 + 1022p^4 + 1749p^3 + 1397p^2 + 525p + 75.$$

We find that for $p < 12,\ \alpha(p) > \beta(p)$, while for $p \geq 12,\ \alpha(p) < \beta(p)$.

More precisely, we find that

$$\alpha(11) = 53618374, \ \beta(11) = 53541332, \ \alpha(12) = 80157504, \ \beta(12) = 80160375.$$

On the other side, setting $\delta(p) = \beta(p) - \alpha(p)$, we obtain

$$\delta(p) = 4p^5 - 34p^4 - 150p^3 - 189p^2 - 84p - 9,$$

$$\delta'(p) = 20p^4 - 136p^3 - 450p^2 - 378p - 84,$$

which is positive for $p \geq 10$, so $\delta(p)$ increases for $p \geq 10$.

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(Corsini) Department of Biology and Agro-Industrial Economy, Udine University, Via delle Scienze 208, 33100 Udine, Italy

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E-mail address, Corsini: `corsini2002@yahoo.com`

(Davvaz) Department of Mathematics, Yazd University, Yazd, Iran

E-mail address: `davvaz@yazduni.ac.ir` `bdavvaz@yahoo.com`