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CHARACTERIZATION OF MULTIWAVELET PACKETS IN $L^2(\mathbb{R}^d)$

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ABSTRACT. Orthogonal multiwavelet packets are based on several scaling functions and several wavelets. They allow properties like regularity, orthogonality and symmetry being impossible in the single wavelet case. In this paper, we establish a complete characterization of multiwavelet packets for arbitrary dilation matrices by means of basic equations in Fourier domain.

1. **Introduction**

A finite set $\Psi = \{\psi_1, \psi_2, ..., \psi_L\} \subset L^2(\mathbb{R}^d)$ is called an orthonormal multiwavelet if and only if the system

(1.1) $\psi^{\ell}_{j,k}(x) = |detA|^{j/2}\psi^{\ell}\big(A^j x - k\big)\ \ell = 1, ..., L,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d,$

is an orthonormal basis for $L^2(\mathbb{R}^d)$.

For a given lattice $\Gamma = P.\mathbb{Z}^d$, P an invertible $d \times d$ real matrix with $|detP| > 0$, and dilation matrix A that preserves the lattice Γ, Calogero [6] characterized all orthonormal multiwavelets $\Psi = \{\psi_1, \psi_2, ..., \psi_L\}$ in $L^2(\mathbb{R}^d)$ in terms of two basic equations given by

(1.2) $\displaystyle\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^{\ell}\big(A^{*j}\xi\big)\right|^2 = |detP|$ for $a.e.\ \xi \in \mathbb{R}^d,$

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$$(1.3)\ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}^{\ell}\big(A^{*j}\xi\big)\overline{\hat{\psi}^{\ell}\big(A^{*j}(\xi+\gamma^{*})\big)}=0 \text{ for } a.e.\ \xi\in\mathbb{R}^{d},\gamma^{*}\in\Gamma^{*}\setminus A^{*}\Gamma^{*}$$

together with the assumptions $\|\psi^{\ell}\|_2 \geq 1$, for $1 \leq \ell \leq L$. This characterization follows the strategy of the proof of Frazier et al. [7] in the classical case. However, there are several differences. For example, in the general case A will not be isotropic, since there may be different eigenvalues. Moreover, A may contain a rotation, so that a neighbourhood U of the origin in general will not be contained in AU.

Rzeszotnik [12] presented the proof of the fact that the Calderón condition characterizes the completeness of an orthonormal multiwavelet basis associated with integer dilation matrix A in the following way:

Theorem 1.1. Suppose $\Psi = \{\psi_1, \psi_2, ..., \psi_L\} \subset L^2(\mathbb{R}^d)$ and the system $\{\psi^{\ell}_{j,k}\}$, defined by Equation (1.1), is orthonormal, then it is complete in $L^2(\mathbb{R}^d)$ (i.e., it is an orthonormal basis) if and only if

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^{\ell}\big(B^{j}\xi\big)\right|^{2}=1 \text{ for } a.e.\ \xi\in\mathbb{R}^{d},$$

where B denotes the transpose of A.

On the other hand Bownik [2, 3] presented a new approach to characterize multiwavelets by means of basic equations in the Fourier transform domain given by the following theorem:

Theorem 1.2. Suppose $\Psi = \{\psi_1, \psi_2, ..., \psi_L\} \subset L^2(\mathbb{R}^d)$ and the system $\{\psi^{\ell}_{j,k}\}$, given by (1.1), is a tight frame with constant 1 for $L^2(\mathbb{R}^d)$, i.e.,

$$\|f\|^2=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\psi^{\ell}_{j,k}\rangle\right|^{2} \text{ for all } f\in L^2(\mathbb{R}^d)$$

if and only if

(1.4) $$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^{\ell}\big(B^{j}\xi\big)\right|^{2}=1 \text{ for } a.e.\ \xi\in\mathbb{R}^{d},$$
and

(1.5) $$t_s(\xi)=\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}^{\ell}\big(B^{j}\xi\big)\,\overline{\hat{\psi}^{\ell}\big(B^{j}(\xi+s)\big)}=0$$
for $a.e.$ $\xi\in\mathbb{R}^d$, $s\in\mathbb{Z}^d\setminus B\mathbb{Z}^d$, where $B=A^t$. In particular, Ψ is a multiwavelet if and only if Equations (1.4) and (1.5) are satisfied and $\|\psi^{\ell}\|_2=1$ for $\ell=1,...,L$. In the present paper, we follow the construction of multiwavelet packets given by Behera [1] for arbitrary dilation matrices and study the characterization of multiwavelet packets based on the new approach given by Bownik for dilation matrices discussed above. Our results are the generalization of the results of Frazier et al. [7], Bownik [2, 3], Calogero [6], and Rzeszotnik [12].

2. **Preliminaries**

Assume that we have a dilation matrix A preserving $\mathbb{Z}^d$, i.e, A is a $d\times d$ matrix such that

(i) $A(\mathbb{Z}^d)\subset\mathbb{Z}^d$

(ii) all eigen values λ of A satisfy $|\lambda|>1$.

Property (i) implies that A has integer entries and hence $|detA|$ is an integer, and (ii) says that $|detA|$ is greater than 1. Let $B=A^T$, the transpose of A and $a=|detA|=|detB|$. Considering $\mathbb{Z}^d$ as an additive group, we see that $A\mathbb{Z}^d$ is a normal subgroup of $\mathbb{Z}^d$ so we can form the cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$. It is well known fact that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is equal to $a=|detA|$ (see [13]). For $j\geq 0$, let D_j denote a set of a^j representatives of distinct cosets of $\mathbb{Z}^d\setminus A\mathbb{Z}^d$ where $|detA|=a$. For $j<0$, we define $D_j=\{0\}$. We also take $\mathbb{T}^d=\mathbb{R}^d\setminus\mathbb{Z}^d$ with its fundamental domain $\mathbb{T}^d=[-1/2,1/2)^d$.

Definition 2.1[5, 9]. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

(2.1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(2.2) $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1},\ \forall j \in \mathbb{Z},\ x \in \mathbb{R}^d$;

(2.3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\},\ \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;

(2.4) There exist L functions $\varphi_1, ..., \varphi_L \in V_0$, such that the system of functions $\left\{\varphi_\ell(x-k) : k \in \mathbb{Z}^d, 1 \le \ell \le L\right\}$ forms an orthonormal basis of V_0.

The L functions whose existence is asserted in (2.4) are called scaling functions of the given MRA .

Given a multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$, we define another sequence $\{W_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ by $W_j = V_{j+1} \ominus V_j,\ j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

(2.5) $f \in W_j$ if and only if $f(A^{-j}.) \in W_{j+1}$.

Moreover, the subspaces $\{W_j\}$ are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{2.6}$$

(2.7) $= V_0 \bigoplus \Big(\bigoplus_{j\ge 0} W_j\Big)$.

A set of functions $\{\psi_\ell^r : 1 \le r \le a-1, 1 \le \ell \le L\}$ in $L^2(\mathbb{R}^d)$ is said to be a set of basic multiwavelets associated with the multiresolution analysis of multiplicity L if the collection $\left\{\psi_\ell^r(.-k) : 1 \le r \le a-1, 1 \le \ell \le L,\ k \in \mathbb{Z}^d\right\}$ forms an orthonormal basis for W_0.

Now, in view of (2.5) and (2.6), it is clear that if $\{\psi_\ell^r : 1 \le r \le a-1, 1 \le \ell \le L\}$ is a basic set of multiwavelets, then

$$\left\{a^{j/2}\psi_\ell^r(A^j. - k) : 1 \le r \le a-1, 1 \le \ell \le L,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\right\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$ (see [1], [5]).

Now, for any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the basic multiwavelet packets ω_ℓ^n; $1 \le \ell \le L$ recursively as follows. We denote $\omega_\ell^0 = \varphi_\ell$, $1 \le \ell \le L$, the scaling functions and $\omega_\ell^r = \psi_\ell^r, r \in \mathbb{N}_0, 1 \le \ell \le L$ as the possible candidates for basic multiwavelets. Then, define

$$\omega_\ell^{s+ar}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{\ell jk}^s\, a^{1/2}\, \omega_\ell^r(Ax-k),\ 0 \le s \le a-1\,, 1 \le \ell \le L \tag{2.8}$$

where $\left(h_{\ell jk}^s\right)$ is a unitary matrix (see [1]).

Taking Fourier transform in both sides of (2.8), we get

$$\left(\omega_\ell^{s+ar}\right)^\wedge(\xi) = \sum_{j=1}^{L} h_{\ell j}^s(B^{-1}\xi)\left(\omega_\ell^r\right)^\wedge(B^{-1}\xi). \tag{2.9}$$

Note that (2.8) defines ω_ℓ^n for every non-negative integer n and every ℓ such that $1 \le \ell \le L$. The set of functions $\{\omega_\ell^n : n \ge 0, 1 \le \ell \le L\}$ as defined above are called the basic multiwavelet packets corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A.

Definition 2.2[1]. Let $\{\omega_\ell^n : n \ge 0, 1 \le \ell \le L\}$ be the basic multiwavelet packets. The collection

$$P = \left\{|detA|^{j/2}\omega_\ell^n(A.-\ k) : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\right\}$$

is called the "general multiwavelet packets" associated with MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L over dilation matrix A.

Corresponding to some orthonormal scaling vector $\Phi = \omega_\ell^0$, the family of multiwavelet packets ω_ℓ^n defines a family of subspaces of $L^2(\mathbb{R}^d)$ as follows:

(2.10) $U_j^n = \overline{\text{span}}\Big\{a^{j/2}\omega_\ell^n(A^j x - k) : \ k \in \mathbb{Z}^d, 1 \leq \ell \leq L\Big\};\ j \in \mathbb{Z}\ n = 0, 1, ...$

Observe that

$$U_j^0 = V_j,\ U_j^1 = W_j = \bigoplus_{r=1}^{a-1} U_j^r,\ j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

(2.11) $U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r.$

A generalization of this result for other values of $n = 1, 2, ...$ can be written as

(2.12) $U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r},\ j \in \mathbb{Z}.$

Lemma 2.3[1]. If $j \geq 0$, then

$$W_j = \bigoplus_{r=0}^{a-1} U_j^r = \bigoplus_{r=a}^{a^2-1} U_{j-1}^r = ... = \bigoplus_{r=a^t}^{a^{t+1}-1} U_{j-t}^r,\ t \leq j$$

$$= \bigoplus_{r=a^j}^{a^{j+1}-1} U_0^r$$

where U_j^n is defined in (2.10). Using this decomposition, we get the multiwavelet packets decomposition of subspaces $W_j,\ j \geq 0$.

Let $\{\omega_\ell^n : n \geq 0, 1 \leq \ell \leq L\}$ be a family of functions in $L^2(\mathbb{R}^d)$. Then the system (resp. quasi) generated by ω_ℓ^n and associated with matrix A is the collection

(2.13) $F(\omega_\ell^n) = \Big\{\omega_{\ell,j,k}^n : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq \ell \leq L, a^j \leq n < a^{j+1}\Big\}$

(2.14) $F^q(\omega_\ell^n) = \left\{\tilde{\omega}_{\ell,j,k}^n : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq \ell \leq L, a^j \leq n < a^{j+1}\right\}$

where we use the convention

$$\omega_{\ell,j,k}^n(x) = D_{A^j} T_k \omega_\ell^n(x) = |detA|^{j/2} \omega_\ell^n(A^j x - k),\ j \in \mathbb{Z}, k \in \mathbb{Z}^d,$$

and

$$\tilde{\omega}_{\ell,j,k}^n(x) = \begin{cases} D_{A^j} T_k \omega_\ell^n(x) = |detA|^{j/2} \omega_\ell^n(A^j x - k),\ j \geq 0,\ k \in \mathbb{Z}^d, \\ \\ |detA|^{j/2} T_k D_{A^j} \omega_\ell^n(x) = |detA|^j \omega_\ell^n\big(A^j(x-k)\big),\ j < 0,\ k \in \mathbb{Z}^d, \end{cases}$$

where $T_y f(x) = f(x-y)$ is a translation by a vector $y \in \mathbb{R}^d$ and $D_{A^j} f(x) = |detA|^{j/2} f(Ax)$ is dilation by the matrix A.

Definition 2.4. A countable family $F \subset L^2(\mathbb{R}^d)$ is a frame if there exist constants $a, b, 0 < a \leq b < \infty$ satisfying

(2.15) $a\|f\|^2 \leq \sum_{\eta \in \{F} |\langle f, \eta \rangle|^2 \leq b\|f\|^2$ for all $f \in L^2(\mathbb{R}^d)$.

A frame is a tight frame if a and b can be chosen so that $a = b$, and is a normalized tight frame if $a = b = 1$. The (quasi) system $F(\omega_\ell^n)$ (resp. $F^q(\omega_\ell^n)$) is a (quasi) frame if (2.15) holds for $F = F(\omega_\ell^n)$ ($F = F^q(\omega_\ell^n)$).

Given a second set of functions $\Theta \subset L^2(\mathbb{R}^d)$ with the same cardinality $L \in \mathbb{N}$, we let $\theta_{\ell,j,k}^n, \tilde{\theta}_{\ell,j,k}^n$ be defined as we define $\omega_{\ell,j,k}^n$, and $\tilde{\omega}_{\ell,j,k}^n$. We also define an operator

(2.16) $P_{\omega_\ell^n,\Theta}(f,g) := \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \langle f, \omega_{\ell,j,k}^n \rangle \langle \theta_{\ell,j,k}^n, g \rangle,\ f, g \in L^2(\mathbb{R}^d),$

which is obviously a bounded sesquilinear operator on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, if both $F(\omega_\ell^n)$ and $F(\Theta)$ are Bessel families. Similarly, we have another operator associated with the quasi system $F^q(\omega_\ell^n)$ given by

$$(2.17)\quad P^q_{\omega^n_\ell,\Theta}(f,g) := \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\omega}^n_{\ell,j,k}\rangle\,\langle\tilde{\theta}^n_{\ell,j,k},g\rangle,\ \ f,g\in L^2(\mathbb{R}^d).$$

Here we recall that $\tilde{\omega}^n_\ell$ is called a-dual of ω^n_ℓ, if

$$P_{\omega^n_\ell,\tilde{\omega}^n_\ell}(f,g) = \langle f,g\rangle,\ \ f,g\in L^2(\mathbb{R}^d)$$

and a-quasi dual if

$$P^q_{\omega^n_\ell,\tilde{\omega}^n_\ell}(f,g) = \langle f,g\rangle,\ \ f,g\in L^2(\mathbb{R}^d).$$

We further use the notations

$$(2.18)\quad \begin{cases} P_j(f,g) := \displaystyle\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\langle f,\omega^n_{\ell,j,k}\rangle\,\langle\theta^n_{\ell,j,k},g\rangle \\ \\ P^q_j(f,g) := \displaystyle\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\omega}^n_{\ell,j,k}\rangle\,\langle\tilde{\theta}^n_{\ell,j,k},g\rangle \end{cases}$$

for any $j\in\mathbb{Z}$. Note that ω^n_ℓ and Θ are usually defined by the context and, in particular, $P_j = P^q_j$ holds for all $j\geq 0$.

Lemma 2.5[8]. Suppose $\{e_j : j = 1,2,...\}$ be a family of elements in a Hilbert space $\mathbb{H}$ such that

(i) $\|f\|^2 = \displaystyle\sum_{j=1}^{\infty}|\langle f,e_j\rangle|^2$ holds for all $f\in\mathbb{H}$

(ii) $\|e_j\|\geq 1$ for $j = 1,2,...$

then $\{e_j : j = 1,2,...\}$ is an orthonormal basis for $\mathbb{H}$.

Definition 2.6[2]. For a given family of vectors $\{t_i : i\in\mathbb{N}\}\subset\ell^2(\mathbb{Z}^d)$, consider the operator $H : \ell^2(\mathbb{Z}^d)\longrightarrow\ell^2(\mathbb{N})$ defined by

(2.19) $H(v) = \Big(\langle v, t_i\rangle\Big)_{i\in\mathbb{N}}$, for $v = \Big(v(k)\Big)_{k\in\mathbb{Z}^d} \subset \ell^2(\mathbb{N})$.

If H is bounded, then the dual Gramian of $\{t_i : i \in \mathbb{N}\}$ is the operator $\widetilde{G} : \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$ given by $\widetilde{G} = H^*H$.

Note that $\widetilde{G}$ is a non-negative definite operator on $\ell^2(\mathbb{Z}^d)$. Also, for $k, p \in \mathbb{Z}^d$, we have

(2.20) $\langle \widetilde{G}e_k, e_p\rangle = \langle He_k, He_p\rangle = \sum_{i\in\mathbb{N}} t_i(k)\overline{t_i(p)}$

where $\{e_k : k \in \mathbb{Z}^d\}$ is the standard basis of $\ell^2(\mathbb{Z}^d)$. By the Cauchy-Schwarz inequality the entries of the matrix $\widetilde{G}$ in (2.20) are meaningfully defined if the series $\sum_{i\in\mathbb{N}} |t_i(k)|^2 < \infty$ for all $k \in \mathbb{Z}^d$. The operator H defined in (2.19) is bounded whenever the matrix $\Big(\sum_{i\in\mathbb{N}} t_i(k)\,\overline{t_i(p)}\Big)_{k,p\in\mathbb{N}}$ is bounded operator on $\ell^2(\mathbb{Z}^d)$. Hence, H is bounded iff $\widetilde{G}$ is bounded.

The following Lemma due to Ron and Shen [11] characterizes when the system of translates of a given family of functions is a frame in terms of dual Gramian.

Lemma 2.7. Suppose $\{\varphi_i : i \in \mathbb{N}\} \subset L^2(\mathbb{R}^d)$. For *a.e.* $\xi \in \mathbb{T}^d$, let $\widetilde{G}(\xi)$ be the dual Gramian of $\{t_i = (\hat{\varphi}_i(\xi + k))_{k\in\mathbb{Z}^d} : i \in \mathbb{N}\} \subset \ell^2(\mathbb{Z}^d)$. Then, the system of translates $\{T_k\varphi_i : k \in \mathbb{Z}^d, i \in \mathbb{N}\}$ is a frame for $L^2(\mathbb{R}^d)$ with constants b, b' if and only if $\widetilde{G}(\xi)$ is bounded for *a.e.* $\xi \in \mathbb{T}^d$ and

(2.21) $b'\|v\|^2 \le \langle \widetilde{G}(\xi)v, v\rangle \le b\|v\|^2$, for $v \in \ell^2(\mathbb{Z}^d)$, *a.e.* $\xi \in \mathbb{T}^d$.

This lemma still holds if $b' = 0$ and then it characterizes the system of translates $\{T_k\varphi_i : k \in \mathbb{Z}^d, i \in \mathbb{N}\}$ being a Bessel family with constant b.

Definition 2.8[10]. A quasi-norm associated with a dilation matrix B is a measurable mapping $\rho : \mathbb{R}^d \longrightarrow [0, \infty)$, so that

(i) $\rho(\xi) = 0 \Longleftrightarrow \xi = 0$,

(ii) $\rho(B\xi) = |detB|\, \rho(\xi)$, for all $\xi \in \mathbb{R}^d$,

(iii) there is $c > 0$ so that $\rho(\xi + \zeta) \leq c\big(\rho(\xi) + \rho(\zeta)\big)$ for all $\xi, \zeta \in \mathbb{R}^d$.

For a quasi-norm ρ we define its characteristic number $\kappa(\xi)$ by

$$(2.22)\ \kappa(\xi) = \int_{\mathbb{R}^d} \frac{I_d(\xi)}{\rho(\xi)} d\xi$$

where $D \subset \mathbb{R}^d$ is a measurable set such that $\big\{B^j D : j \in \mathbb{Z}\big\}$ partitions $\mathbb{R}^d$ (modulo sets of measure zero), i.e., $\bigcup_{j\in\mathbb{Z}} B^j D = \mathbb{R}^d$ and $B^i D \cap B^j D = \emptyset$ for $i \neq j,\ i, j \in \mathbb{Z}$. This number $\kappa(\xi)$ is independent upon the choice of D.

3. **Main Results**

Lemma 3.1. Let $\omega_\ell^n \in L^2(\mathbb{R}^d)$ and let $f \in L^2(\mathbb{R}^d)$ be a function with compact support. Then

$$(3.1)\ \lim_{N\to\infty} \sum_{j<0} P_j^q(D_N f, D_N f) = 0$$

and

$$(3.2)\ \lim_{N\to\infty} |detA|^{-N} \sum_{j<-N} \sum_{v\in D_N} P_j(D_N f, D_N f) = 0$$

for a special choice of $D_j = \mathbb{Z}^d \cap A^j([1, 2)^d)$.

Proof. Let Ω denote the support of f and let us first choose $N_0 > 0$ so that $D_{-N}\Omega$ is contained in a ball of radius $1/2$ around the origin for all $N \geq N_0$. Thus, in order to prove (3.1), we estimate

$$\sum_{j<0} P_j^q(D_N f, D_N f) = \sum_{j<0} |detA|^j \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \Big|\langle f, D_{-N} T_k D_j\, \omega_\ell^n\rangle\Big|^2$$

$$= \sum_{j<0} |detA|^j \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \Big|\langle\, T_{A^N k} f, D_{j-N}\, \omega_\ell^n\rangle\Big|^2$$

$$\leq \sum_{j<0} |detA|^j \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \|f\|^2 \int_{\Omega+A^N k} \Big|D_{j-N}\, \omega_\ell^n(x)\Big|^2 dx$$

$$(3.3) = \|f\|^2 \sum_{j<0} |detA|^j \sum_{k\in\mathbb{Z}^d} \int_{A^{j-N}\Omega+A^j k} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \Big|\,\omega_\ell^n(x)\Big|^2 dx.$$

By our choice of N_0, we obtain

$$\sum_{j<0} P_j^q(D_N f, D_N f) \leq \|f\|^2 \int_{\mathbb{R}^d} K_N(x) \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \Big|\,\omega_\ell^n(x)\Big|^2 dx$$

which holds for all $N \geq N_0$, where

$$(3.4)\ K_N(x) = \sum_{j<0} |detA|^j \chi_{A^j(\mathbb{Z}^d+A^{-N}\Omega)}(x),\ x \in \mathbb{R}^d.$$

Since

$$K_N(x) \leq \sum_{j<0} |detA|^j = |detA|/(|detA|-1),\ N \geq N_0,$$

and since $\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} |\,\omega_\ell^n(x)|^2 \in L^1(\mathbb{R}^d)$, the Dominated Convegence Theorem can be applied to the integral, or more precisely

$$\lim_{N\to\infty} K_N(x) = 0 \ \text{ for all } x \in U := \mathbb{R}^d \setminus \bigcup_{j<0} A^j\mathbb{Z}^d.$$

By the compactness of Ω, there exists a sequence of numbers $s_j := \sup\{\|A^j y\| : y \in \Omega\}$ which tends to zero as $j \to -\infty$. If we fix $x \in U$, then all the terms in (3.4) which satisfy

$$s_{j-N} < \operatorname{dist}\big(x, A^j\mathbb{Z}^d\big) = c_j(x),\ j < 0,$$

vanish. In other words,

$$K_N(x) \le \sum_{\substack{j<0 \\ s_{j-N} \ge c_j(x)}} |detA|^j \to 0 \text{ as } N \to \infty.$$

In order to prove the second relation (3.2) of our lemma, we let $N \ge N_0$ and use similar transformations as in (3.3) to obtain

$$|detA|^{-N} \sum_{j<-N} \sum_{v \in D_N} P_j(T_v f, T_v f)$$
$$\le \|f\|^2 |detA|^{-N}| \sum_{j<-N} \sum_{v \in D_N} \sum_{k \in \mathbb{Z}^d} \int_{A^j(\Omega+v)+k} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \left| \omega_\ell^n(x) \right|^2 dx.$$

We now define

$$\text{(3.5)}\ K'_N(x) = |detA|^{-N} \sum_{j<-N} \sum_{v \in D_N} \sum_{k \in \mathbb{Z}^d} \chi_{A^j(\Omega+v)+k}(x),\ x \in \mathbb{R}^d.$$

Now, we show that K'_N is uniformly bounded in N, as $N \to \infty$, and converges to zero pointwise $a.e.$ Since D_N is assumed to be of the form as given in (3.2), and hence

$$A^{-N}(\Omega + v) \subset I = (1/2, 3/2)^d, \text{ for all } N \ge N_0,\ v \in D_N.$$

In order to resolve the various summands in (3.5), we fix j and k and observe that

$$\sum_{v \in D_N} \chi_{A^j(\Omega+v)+k} \le c_1 \chi_{A^{j+N} I+k}$$

holds for all $N \ge N_0,\ j < -N,$ and $k \in \mathbb{Z}^d$, and the constant $c_1 > 0$ depends on the compact set Ω. By the properties of dilation matrix A, we have

$$\sum_{j<-N} \chi_{A^{j+N} I+k} \le c_2 \chi_{B_s(0)+k}$$

for all $N \ge N_0, k \in \mathbb{Z}^d$, where the constant $c_2,\ s > 0$ depends on A. Finally, summation over $k \in \mathbb{Z}^d$, gives another constant $c_3 > 0$ which depends on s (hence on A) such that

$$\sum_{k\in\mathbb{Z}^d}\chi_{Bs(0)+k}\leq c_3.$$

By combining all these relations, we get

$$K'_N(x)\leq c_1c_2c_3|detA|^{-N},\ x\in\mathbb{R}^d,$$

and hence K'_N converges to zero uniformly. This completes the proof of the lemma. □

Theorem 3.2. Suppose $\omega_\ell^n\in L^2(\mathbb{R}^d)$. Then:

(i) $F(\omega_\ell^n)$ is a Bessel family if and only if $F^q(\omega_\ell^n)$ is a Bessel family. Furthermore, their exact upper bounds are equal.

(ii) $F(\omega_\ell^n)$ is a frame if and only if $F^q(\omega_\ell^n)$ is a quasi frame. Furthermore, the constants $B=B_q$ and $B'=B'_q$ can be chosen as their exact upper and lower bounds.

Proof. (i) Let $\Theta=\omega_\ell^n$ in (2.16) and (2.17). First we note that all summands of $P_{\omega_\ell^n,\omega_\ell^n}$ and $P^q_{\omega_\ell^n,\omega_\ell^n}$ are non-negative. If ω_ℓ^n generates a Bessel family with upper bound $B\geq 0$, then

$$P^q_{\omega_\ell^n,\omega_\ell^n}(f,f)=\lim_{m\to\infty}\sum_{j\geq -m}P_j^q(f,f)$$

$$=\lim_{m\to\infty}|detA|^{-m}\sum_{v\in D_j}\sum_{j\geq -m}P_j(T_vf,T_vf)$$

$$\leq\lim_{m\to\infty}|detA|^{-m}\sum_{v\in D_j}P_{\omega_\ell^n,\omega_\ell^n}(T_vf,T_vf)$$

$$\leq\lim_{m\to\infty}|detA|^{-m}\sum_{v\in D_j}B\|T_vf\|^2=B\|f\|^2$$

holds for all $f\in L^2(\mathbb{R}^d)$. Here we have used the fact that

$$P_j^q(f,g)=|detA|^{-m}\sum_{v\in D_j}P_j(T_vf,T_vg),$$

for all $j \geq -m$ and $f, g \in L^2(\mathbb{R}^d)$. Thus, we have shown that quasi frame $F^q(\omega_\ell^n)$ is also a Bessel family with the same upper bound B.

Conversely, let us assume that $F^q(\omega_\ell^n)$ is a Bessel family with upper bound $B_q \geq 0$. Let us further assume that there exists $f \in L^2(\mathbb{R}^d)$ such that

$\|f\| = 1$ and $P_{\omega_\ell^n, \omega_\ell^n}(f, f) > B_q$.

Then, by the dilation invariance of $F(\omega_\ell^n)$, we can find $N \in \mathbb{N}$ such that

$$\sum_{j=-N}^{\infty} P_j(f, f) = \sum_{j=0}^{\infty} P_j(D_N f, D_N f) > B_q.$$

But this contradicts with the definition of B_q, since

$$P^q_{\omega_\ell^n, \omega_\ell^n}(D_N f, D_N f) \geq \sum_{j=0}^{\infty} P_j^q(D_N f, D_N f) = \sum_{j=0}^{\infty} P_j(D_N f, D_N f),$$

and the dilation D is an isometry. Thus, we conclude that $F(\omega_\ell^n)$ must be Bessel family with upper bound B_q.

(ii) The proof follows with the same argument as in the proof of (i). The only differences are the use of Lemma 3.1 at certain places. We have

$$P^q_{\omega_\ell^n, \omega_\ell^n}(f, f) = \lim_{m \to \infty} \sum_{j \geq -m} P_j^q(f, f) \;= \lim_{m \to \infty} |detA|^{-m} \sum_{v \in D_j} \sum_{j \geq -m} P_j(T_v f, T_v f)$$

$$= \lim_{m \to \infty} |detA|^{-m} \sum_{v \in D_j} \sum_{j \in \mathbb{Z}} P_j(T_v f, T_v f)$$

$$= \lim_{m \to \infty} |detA|^{-m} \sum_{v \in D_j} P_{\omega_\ell^n, \omega_\ell^n}(T_v f, T_v f)$$

$$\geq \lim_{m \to \infty} |detA|^{-m} \sum_{v \in D_j} B' \|T_v f\|^2 = B' \|f\|^2$$

which holds for all $f \in L^2(\mathbb{R}^d)$ and has a compact support. Since this is a dense subset of $L^2(\mathbb{R}^d)$, the relation $P^q_{\omega^n_\ell,\omega^n_\ell}(f,f) \geq B'\|f\|^2$ holds for all $f \in L^2(\mathbb{R}^d)$. The opposite relation $B'_q \leq B'$ is shown by assuming the contrary, so that

$$P_{\omega^n_\ell,\omega^n_\ell}(f,f) \leq B'_q - \varepsilon \text{ for some } f \in L^2(\mathbb{R}^d),\ \|f\|^2 = 1,$$

and some $\varepsilon > 0$. Suppose that the function f has a compact support. Then dilation invariance of the operator $P_{\omega^n_\ell,\omega^n_\ell}$ gives

$$P_{\omega^n_\ell,\omega^n_\ell}(D_N f, D_N f) \leq B'_q - \varepsilon \text{ for all } N \in \mathbb{N}.$$

By relation (3.1) of Lemma 3.1, there exists $N \in \mathbb{N}$ such that

$$P^q_{\omega^n_\ell,\omega^n_\ell}(D_N f, D_N f) < \sum_{j=0}^{\infty} P^q_j(D_N f, D_N f) + \frac{\varepsilon}{2}$$

$$= \sum_{j=0}^{\infty} P_j(D_N f, D_N f) + \frac{\varepsilon}{2}$$

$$\leq P_{\omega^n_\ell,\omega^n_\ell}(D_N f, D_N f) + \frac{\varepsilon}{2} \leq B'_q - \frac{\varepsilon}{2}$$

which contradicts with the definition of the lower frame bound of B'_q of $F^q(\omega^n_\ell)$. Hence we get the desired result. □

Theorem 3.3. Suppose $\omega^n_\ell \in L^2(\mathbb{R}^d)$. Then, the system $F(\omega^n_\ell)$ associated with the dilation matrix A is orthonormal in $L^2(\mathbb{R}^d)$ if and only if

$$\sum_{k\in\mathbb{Z}^d} \hat{\omega}^n_\ell(\xi+k)\,\hat{\omega}^{n'}_{\ell'}\big(B^j(\xi+k)\big) = \delta_{n,n'}\delta_{\ell,\ell'}\delta_{j,0} \tag{3.6}$$

a.e. $\xi \in \mathbb{R}^d$ for $j \geq 0,\ a^j \leq n, n' < a^{j+1},\ \ell, \ell' = 1, ..., L$, where $B = A^T$.

Proof. By performing a change of variables, we see that

$$\langle \omega^n_{\ell,j,k}, \omega^{n'}_{\ell',j',k'} \rangle = \delta_{n,n'}\delta_{\ell,\ell'}\delta_{j,j'}\delta_{k,k'}$$

for $j, j' \in \mathbb{Z},\ k, k' \in \mathbb{Z}^d,\ a^j \le n, n' < a^{j+1},\ \ell, \ell' = 1, ..., L$, is equivalent to

$\langle \omega^n_{\ell,j,k}, \omega^{n'}_{\ell',0,0} \rangle = \delta_{n,n'}\delta_{\ell,\ell'}\delta_{j,0}\delta_{k,0}$

for $j \ge 0,\ k \in \mathbb{Z}^d,\ a^j \le n, n' < a^{j+1},\ \ell, \ell' = 1, ..., L.$

Take any $j \ge 0,\ k \in \mathbb{Z}^d,\ a^j \le n, n' < a^{j+1},\ \ell, \ell' = 1, ..., L$. By Plancherel formula, we have

$$\begin{aligned}
&\delta_{n,n'}\delta_{\ell,\ell'}\delta_{j,0}\delta_{k,0} = \langle \hat{\omega}^n_{\ell,j,k}\,, \hat{\omega}^{n'}_{\ell',0,0} \rangle \\
&= \int_{\mathbb{R}^d} |detA|^{-j/2} \hat{\omega}^n_\ell\left(B^{-j}\xi\right) e^{-2\pi i \langle k, B^{-j}\xi\rangle}\, \overline{\hat{\omega}^{n'}_{\ell'}(\xi)}\, d\xi \\
&= \int_{\mathbb{R}^d} |detA|^{j/2} \hat{\omega}^n_\ell(\xi)\, e^{-2\pi i \langle k, \xi\rangle}\, \overline{\hat{\omega}^{n'}_{\ell'}\left(B^j\xi\right)}\, d\xi \\
&= \sum_{\ell \in \mathbb{Z}^d} |detA|^{j/2} \int_{\ell + \mathbb{T}^d} \hat{\omega}^n_\ell(\xi)\, \overline{\hat{\omega}^{n'}_{\ell'}\left(B^j\xi\right)}\, e^{-2\pi i \langle k, \xi\rangle}\, d\xi \\
&= |detA|^{j/2} \int_{\mathbb{T}^d} \Big[\sum_{\ell \in \mathbb{Z}^d} \hat{\omega}^n_\ell(\xi + \ell)\, \overline{\hat{\omega}^{n'}_{\ell'}\left(B^j(\xi + \ell)\right)} \Big]\, e^{-2\pi i \langle k, \xi\rangle}\, d\xi \\
&= |detA|^{j/2} \int_{\mathbb{T}^d} K(\xi)\, e^{-2\pi i \langle k, \xi\rangle}\, d\xi,
\end{aligned}$$

where $K(\xi) = \sum_{\ell \in \mathbb{Z}^d} \hat{\omega}^n_\ell(\xi + \ell)\, \overline{\hat{\omega}^{n'}_{\ell'}\left(B^j(\xi + \ell)\right)}$.

The interchange of summation and integration is justified by

$$\begin{aligned}
&\int_{\mathbb{T}^d} \sum_{\ell \in \mathbb{Z}^d} \left| \hat{\omega}^n_\ell(\xi + \ell)\, \hat{\omega}^{n'}_{\ell'}\left(B^j(\xi + \ell)\right) \right| d\xi = \int_{\mathbb{R}^d} \left| \hat{\omega}^n_\ell(\xi) \right| \left| \hat{\omega}^{n'}_{\ell'}\left(B^j(\xi)\right) \right| d\xi \\
&\le |detA|^{-j/2} \|\, \omega^n_\ell(\xi)\, \|^2\, \|\, \omega^{n'}_{\ell'}(\xi)\, \|^2 < \infty.
\end{aligned}$$

It is clear from the above computation that the Fourier coefficients of $K(\xi) \in L^1(\mathbb{T}^d)$ are zero except for the coefficients corresponding to $k = 0$ which is 1 if $j = 0$ and $\ell = \ell'$. Therefore, $K(\xi) = \delta_{j,0}\delta_{\ell,\ell'}\delta_{n,n'}$ for $a.e.$ $\xi \in \mathbb{T}^d$. □

Theorem 3.4. Suppose $\omega_\ell^n \in L^2(\mathbb{R}^d)$. The dual Gramian $\widetilde{G}(\xi)$ of $F^q(\omega_\ell^n)$ at $\xi \in \mathbb{T}^d$ is equal to

(3.7) $\langle \widetilde{G}(\xi)e_k,\, e_k\rangle = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \Big| \hat{\omega}_\ell^n\big(B^j(\xi+k)\big)\Big|^2$ for $k \in \mathbb{Z}^d$,

(3.8) $\langle \widetilde{G}(\xi)e_k,\, e_p\rangle = t_{B^{-m(p-k)}}\big(B^{-m}\xi + B^{-m}k\big)$ for $k \neq p \in \mathbb{Z}^d$,

where $B = A^T, m = \max\Big\{j \in \mathbb{Z} : B^{-j}(p-k) \in \mathbb{Z}^d\Big\}$ and the functions t_s, $s \in \mathbb{Z}^d \setminus B\mathbb{Z}^d$, are given by

(3.9) $t_s(\xi) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\omega}_\ell^n\big(B^j\xi\big)\, \overline{\hat{\omega}_\ell^n\big(B^j(\xi+s)\big)} = 0$

for $a.e.$ $\xi \in \mathbb{R}^d,\ s \in \mathbb{Z}^d \setminus B\mathbb{Z}^d$.

Proof. By applying (2.20) and the fact that $D_j = \{0\}$, $\forall\ j < 0$, we see that for $k, p \in \mathbb{Z}^d$

$$\langle \widetilde{G}(\xi)e_k,\, e_k\rangle = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j<0} \hat{\omega}_\ell^n\big(B^{-j}(\xi+k)\big)\, \overline{\hat{\omega}_\ell^n\big(B^{-j}(\xi+p)\big)}$$

$$+ \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\geq 0} \hat{\omega}_\ell^n\big(B^{-j}(\xi+k)\big)\, \overline{\hat{\omega}_\ell^n\big(B^{-j}(\xi+p)\big)}$$

$$\times\Big[\sum_{d\in D_j} |detA|^{-j} e^{-2\pi i\langle d\ ,B^{-j}(k-p)\rangle}\Big]$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{j=-\infty}^{m} \hat{\omega}_\ell^n\big(B^{-j}(\xi+k)\big)\, \overline{\hat{\omega}_\ell^n\big(B^{-j}(\xi+p)\big)}$$

where $m = \max\left\{ j \in \mathbb{Z} : k - p \in B^j\mathbb{Z}^d \right\}$, i.e., m is the unique integer so that $B^{-m}(k-p) \in \mathbb{Z}^d \setminus B\mathbb{Z}^d$, and $m = \infty$ when $k = p$. The expression in the bracket is equal to 1 if $(k-p) \in B^j\mathbb{Z}^d$ and is 0 otherwise (see [13]). Thus:

Case I. If $k = p$, then

$$\langle \widetilde{G}(\xi)e_k,\, e_k\rangle = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left| \hat{\omega}_\ell^n\big(B^j(\xi+k)\big)\right|^2.$$

Case II. If $k \neq p$, then

$$\langle \widetilde{G}(\xi)e_k,\, e_p\rangle = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\geq 0} \hat{\omega}_\ell^n\big(B^{j-m}(\xi+k)\big)\,\overline{\hat{\omega}_\ell^n\big(B^{j-m}(\xi+p)\big)}$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\geq 0} \hat{\omega}_\ell^n\big(B^j(B^{-m}\xi + B^{-m}k)\big)$$

$$\times\, \overline{\hat{\omega}_\ell^n\big(B^j(B^{-m}\xi + B^{-m}k + B^{-m}(p-k)\big)}$$

$$= t_{B^{-m}(p-k)}\big(B^{-m}\xi + B^{-m}k\big)$$

which is the desired result. □

Theorem 3.5. Suppose $\omega_\ell^n \in L^2(\mathbb{R}^d)$. Then, the system $F(\omega_\ell^n)$ associated with the dilation matrix A is a tight frame with constant 1 for $L^2(\mathbb{R}^d)$, i.e.,

$$\|f\|^2 = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle f\,, \omega_{\ell,j,k}^n\rangle\right|^2 \text{ for all } f \in L^2(\mathbb{R}^d)$$

if and only if

$$\text{(3.10)}\quad \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left| \hat{\omega}_\ell^n\big(B^j\xi\big)\right|^2 = 1 \text{ for } a.e.\ \xi \in \mathbb{R}^d,$$

and

$$\text{(3.9)}\quad t_s(\xi) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\omega}_\ell^n\big(B^j\xi\big)\,\overline{\hat{\omega}_\ell^n\big(B^j(\xi+s)\big)} = 0,$$

for a.e. $\xi \in \mathbb{R}^d,\ s \in \mathbb{Z}^d \setminus B\mathbb{Z}^d$, where $B = A^T$. In particular, ω_ℓ^n are the multiwavelet packets associated with the dilation matrix A if and only if (3.9), (3.10) and $\|\omega_\ell^n\|_2 = 1$ for $n \geq 0,\ \ell = 1, ..., L$.

Proof. By Theorem 3.2, $F(\omega_\ell^n)$ is a tight frame with constant 1 if and only if $F^q(\omega_\ell^n)$ is so. This is equivalent to the spectrum of dual Gramian $\widetilde{G}(\xi)$ consisting of a single point by Lemma 2.7. This in turn is equivalent to (3.9) and (3.10) by Theorem 3.4. By Lemma 2.5, $F(\omega_\ell^n)$ is an orthonormal basis if and only if $\|\omega_\ell^n\|_2 = 1$ for $\ell = 1, ..., L,\ n \geq 0.$ □

Theorem 3.6. Suppose $\omega_\ell^n \in L^2(\mathbb{R}^d)$. Assume that $F(\omega_\ell^n)$ is a Bessel family with constant 1. Then, the following are equivalent:

(i) $F(\omega_\ell^n)$ is a tight frame with constant 1,

(ii) ω_ℓ^n satisfy (3.10),

(iii) ω_ℓ^n satisfy

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \left| \hat{\omega}_\ell^n(\xi) \right|^2 \frac{d\xi}{\rho(\xi)} = \kappa(\rho), \tag{3.11}$$

for some quasi- norm ρ associated with B.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate. If $F(\omega_\ell^n)$ is a tight frame with constant 1, then (3.10) holds by Theorem 3.5. If in turn (3.10) holds, then

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \left| \hat{\omega}_\ell^n(\xi) \right|^2 \frac{d\xi}{\rho(\xi)} = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{B^jD} \left| \hat{\omega}_\ell^n(\xi) \right|^2 \frac{d\xi}{\rho(\xi)}$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_D \sum_{j\in\mathbb{Z}} \left| \hat{\omega}_\ell^n\left(B^j\xi\right) \right|^2 = \kappa(\rho),$$

where $D \subset \mathbb{R}^d$ is such that $\{B^j D : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^d$ (modulo sets of measure zero).

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Since $F(\omega_\ell^n)$ is a Bessel family with constant 1, by Theorem 3.2, $F^q(\omega_\ell^n)$ is also a Bessel family with constant 1. Let $\widetilde{G}(\xi)$ be the dual Gramian of $F^q(\omega_\ell^n)$ at $\xi \in \mathbb{T}^d$. Since $F^q(\omega_\ell^n)$ is a Bessel family with constant 1, we have $\|\widetilde{G}(\xi)\| \leq 1$ for *a.e.* $\xi \in \mathbb{T}^d$ by Lemma 2.7. In particular, $\| \widetilde{G}(\xi) e_k \| \leq 1$. Hence

$$(3.12)\ 1 \geq \| \widetilde{G}(\xi) e_k \|_2 = \sum_{p \in \mathbb{Z}^d} \left| \langle \widetilde{G}(\xi) e_k \,, e_p \rangle \right|^2$$

$$= \left| \langle \widetilde{G}(\xi) e_k \,, e_p \rangle \right|^2 + \sum_{p \in \mathbb{Z}^d, p \neq k} \left| \langle \widetilde{G}(\xi) e_k \,, e_p \rangle \right|^2.$$

By applying Theorem 3.4, we obtain

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\omega}_\ell^n \big(B^j (\xi + k) \big) \right|^2 \leq 1 \text{ for } k \in \mathbb{Z}^d,\ a.e.\ \xi \in \mathbb{T}^d.$$

Further

$$\kappa(\rho) = \int_D \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\omega}_\ell^n \big(B^j \xi \big) \right|^2 \frac{d\xi}{\rho(\xi)} \leq \int_D \frac{d\xi}{\rho(\xi)} = \kappa(\rho).$$

We have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\omega}_\ell^n \big(B^j \xi \big) \right|^2 = 1 \text{ for } a.e.\ \xi \in D,$$

and hence for a.e. $\xi \in \mathbb{R}^d$, i.e., (3.10) holds.

By using Theorem 3.4, $|\langle \widetilde{G}(\xi) e_k \,, e_p \rangle| = 1$ for all $k \in \mathbb{Z}^d$, *a.e.* $\xi \in \mathbb{T}^d$ and by (3.12), $\langle \widetilde{G}(\xi) e_k \,, e_p \rangle = 0$ for $k \neq p$, and $\widetilde{G}(\xi)$ is an identity on $\ell^2(\mathbb{Z}^d)$ for *a.e.* $\xi \in \mathbb{T}^d$. Therefore, by Lemma 2.7, $F^q(\omega_\ell^n)$ is a tight frame with constant 1 so is $F(\omega_\ell^n)$ by Theorem 3.2. This completes the proof. □

Theorem 3.7. Suppose $\omega_\ell^n \in L^2(\mathbb{R}^d)$. Then, the following are equivalent :

(i) ω_ℓ^n are the multiwavelet packets associated with the dilation matrix A.

(ii) ω_ℓ^n satisfy (3.6) and (3.10).

(iii) ω_ℓ^n satisfy (3.6) and (3.11).

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) follow immediately by Theorem 3.3 and Theorem 3.6. To prove (iii)$\Rightarrow$ (i), we assume (3.6) and (3.11). Then, (3.6) implies that $F(\omega_\ell^n)$ is a Bessel family with constant 1. Also, by Theorem 3.6, and (3.11) $\Rightarrow F(\omega_\ell^n)$ is a tight frame with constant 1. Since $\|F(\omega_\ell^n)\|_2 = 1,$ for $n \geq 0, \ell = 1, ...L,$ $F(\omega_\ell^n)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, i.e., ω_ℓ^n are multiwavelet packets in $L^2(\mathbb{R}^d)$. $\square$

Remark 3.8. Although the present results work nominally for the standard lattice $\mathbb{Z}^d$, they can be easily extended to the general lattice $\Gamma = P\mathbb{Z}^d$, where P is a $d \times d$ non-degenerate real matrix. In this set up the dilation is a $d \times d$ real matrix A such that all eigenvalues λ satisfy $|\lambda| > 1$ and $A\Gamma \subset \Gamma$.

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