APPROXIMATE SOLUTIONS TO NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN HEAT FLOW

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ABSTRACT. In this paper, two different methods based on variational iteration method (VIM) and on differential transform method (DTM) are developed to approximate solutions of some partial integro-differential equations with applications in heat flow. Approximate solutions are obtained for some important physical problems concerned with heat flow in materials with memory. The methods are capable of greatly reducing the size of computational domain. Some numerical examples are presented to show the performances and accuracy of the proposed methods.

1. Introduction

Many mathematical formulations of physical phenomena contain partial integro-differential equations of the form

$$u_t(x,t) = \int_0^t a(t-\tau)\frac{\partial}{\partial x}\sigma(u_x(x,\tau))d\tau + f(x,t), \ \ 0 < x < 1, \ 0 < t < T,$$

(1.1)

subject to the initial condition

$$u(x,0) = \phi(x), x \in (0,1)$$

(1.2)

where $\sigma(.)$ is a non-linear function, and $f(x,t)$ is the source term. These equations arise in problems that are concerned with heat flow in both materials with memory [14, 15] and linear viscoelasticity problems [6]. For example, in linear viscoelasticity, when we consider the propagation of some properties such as singularities in the boundary data into the medium. This equation is regarded as a model problem called the Rayleigh problem [21]

Problems of this kind, arising in hereditary mechanics, are indicated in the book [5] and this kind of problems are usually difficult to solve analytically, so that it requires an efficient approximate solution. Some attention has been paid in the literature to questions of existence and stability of solutions of equation (1.1) and its nonlinear counterparts, see [19], but it seems that less work has been devoted so far to numerical methods. On the other hand there is an extensive literature and various techniques for solving systems of integral or integro-differential equations, e.g. Adomian decomposition method [1, 11], Galerkin method [7, 8, 18], He's homotopy perturbation method [3]. Our particular concern in this paper will be in obtaining approximate numerical solutions by two different types of methods for equation (1.1), based on the DTM and VIM, respectively. The two methods, provide the solution in a rapidly convergent series with components that are elegantly computed. Moreover, the obtained solutions will be used to provide closed form solutions. The main advantage of the two methods is that they can be applied directly to all types of integro-differential equations without any need for restrictive assumptions.

The organization of this paper is as follows: In section 2, we describe the basic idea of VIM. In Section 3, the differential transform, which is based on one-dimensional differential transform and Taylors formula, will be introduced. In Section 4, the mentioned schemes in Sections 2 and 3 will be used to seek an approximate solution

of equation (1.1) with the given initial conditions (1.2). Also, the accuracy and efficiency of the schemes was investigated with three numerical illustrations in that section. Finally, Section 5 consists of some brief conclusions.

2. Basic Idea of VIM

Equation (1.1) can be easily integrated to yield

$$u(x,t) = \int_0^t \int_0^s a(s-\tau)\frac{\partial}{\partial x}\sigma(u_x(x,\tau))d\tau ds + \int_0^t f(x,s)ds + \phi(x)$$

(2.1)

which can be solved iteratively as

$$u^{k+1}(x,t) = \int_0^t \int_0^s a(s-\tau)\frac{\partial}{\partial x}\sigma(u_x^k(x,\tau))d\tau ds + \int_0^t f(x,s)ds + \phi(x)$$

(2.2)

where the superscript k denoted the kth iteration. In order to solve equation (1.1), He [9] employed restricted variations and a correction functional as follows. The following correction functional is introduced

$$u^{k+1}(x,t) = u^k(x,t) + \int_0^t \lambda \left\{ u_s^k(x,s) - \int_0^s a(s-\tau)\frac{\partial}{\partial x}\sigma(\tilde{u}_x^k(x,\tau))d\tau - f(x,s) \right\} ds$$

(2.3)

where λ is a Lagrange multiplier which can be identified optimally via variational theory [10], and $\tilde{u}$ denotes a restricted variation, i.e., $\delta\tilde{u} = 0$. By making the correction functional stationary with respect to u^k we obtain

$$\delta u^{k+1}(x,t) = \delta u^k(x,t) + \delta \left[\int_0^t \lambda \left\{ u_s^k(x,s) - \int_0^s a(s-\tau)\frac{\partial}{\partial x}\sigma(\tilde{u}_x^k(x,\tau))d\tau - f(x,s) \right\} ds \right]$$

(2.4)

keeping in mind that $\delta u^k(x,0) = 0$, one can easily obtain

$$\delta u^{k+1}(x,t) = \delta u^k(x,t) + \lambda\delta u^k(x,t) - \int_0^t \frac{\partial\lambda}{\partial s}\delta u^k ds = 0.$$

(2.5)

Therefore, from equation (2.5) one obtains the following Euler-Lagrange equation, $\frac{\partial \lambda}{\partial s} = 0$ and the natural boundary condition $1 + \lambda = 0$, which imply that $\lambda = -1$ and therefore, the iteration formula in equation (2.4) can be written as

$$u^{k+1}(x,t) = u^k(x,t) - \left[\int_0^t \left\{ u_s^k(x,s) - \int_0^s a(s-\tau)\frac{\partial}{\partial x}\sigma(u_x^k(x,\tau))d\tau - f(x,s) \right\} ds \right]$$
(2.6)

which corresponds to He's variational iteration method. The successive approximation

$u^k(x,t), k \geq 0$, of the solution $u(x,t)$ will follow immediately. Consequently, the exact solution may be obtained by using $u(x,t) = \lim_{k\to\infty} u^k(x,t)$. As in [17], and by means of integration by parts, we next show that the variational iteration method is the well-known fixed-point theory for first-order (in time) nonlinear partial differential equations. For, using the identity

$$u(x,t) - \phi(x) = \int_0^t u_t dt$$

Equation (2.1) can be written as

$$u(x,t) = u_t(x,t) - \left[\int_0^t \left\{ u_s(x,s) - \int_0^s a(t-\tau)\frac{\partial}{\partial x}\sigma(u_x(x,\tau))d\tau - f(x,s) \right\} ds \right]$$

which can be solved iteratively as

$$u^{k+1}(x,t) = u^k(x,t) - \left[\int_0^t \left\{ u_s^k(x,s) - \int_0^s a(t-\tau)\frac{\partial}{\partial x}\sigma(u_x^k(x,\tau))d\tau - f(x,s) \right\} ds \right]$$

which is exactly the same expression as that of He's variational technique. This shows that we do not require at all the use of a correction functional and restricted variations. Furthermore, since the variational iteration method can be obtained directly from fixed-point theory, its convergence can be ensured if the resulting mapping is contractive according to the following Picard's fixed-point theorem.

Theorem 2.1. [22] *For a Banach space X, suppose the nonlinear mapping $T : X \to X$ satisfies*

$$\|T[u] - T[\bar{u}]\| \leq \gamma\|u - \bar{u}\|, \quad u, \bar{u} \in X,$$

for some constant $\gamma < 1$. Then T has a unique fixed point. Furthermore, the sequence $u_{n+1} = A[u_n]$ with arbitrary choice of $u_0 \in X$, converges to the fixed point of T, and

$$\|u_k - u_j\| \leq \|u_1 - u_0\| \sum_{\ell=j-1}^{k-2} \gamma^{\ell}.$$

The nonlinear mapping involved in the variational iteration method is

$$T[u] = u(x, t) - \left[\int_0^t \left\{u_s(x, s) - \int_0^s a(s - \tau)\frac{\partial}{\partial x}\sigma(u_x(x, \tau))d\tau - f(x, s)\right\} ds\right].$$

Therefore, according to the above theorem, a sufficient condition for the convergence of the variational iteration method is that the mapping T is a contraction map. Furthermore, the sequence (2.2) converges to the fixed point of T, which is also the solution of the differential equation in Equation (2.1).

3. Differential Transform Method (DTM)

Differential transform technique proposed by Zhou [24] in 1986 is a numerical method used to solve both ordinary and partial differential equations. It uses polynomial forms as approximation to exact solutions that are sufficiently differentiable. This technique provides an iterative procedure to obtain higher-order series. Therefore, it can be applied to equations of higher order. Basic definitions, operations of differential transformation, description of the procedure of DTM, and application of the method are introduced in the following subsections.

3.1. **One-Dimensional Differential Transform.** The differential transform of the $k-$th derivative of a function $u(x)$ is defined to be

$$U(k) = \frac{1}{k!} \left(\frac{d^k}{dx^k} u(x) \right) \Big|_{x=x_0} \qquad (3.1)$$

and the inverse transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k \qquad (3.2)$$

Equation (3.2) is known as the Taylor series expansion of $u(x)$ around $x = x_0$.

3.2. **Two-Dimensional Differential Transform.** We define the differential transform

of the (k, h)th derivative of $f(x, t)$ (see for example, [2]) in (x_0, y_0) as

$$F(k, h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} f(x, t) \right]_{x=x_0, t=t_0} \qquad (3.3)$$

and its inverse transform is defined as

$$f(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} f(x, t) \right]_{x=x_0, t=t_0} (x - x_0)^k (t - t_0)^h \qquad (3.4)$$

which is the Taylor series of $f(x, t)$. In the following theorems, we summarize fundamental properties of the differential transform needed in our work, for more details we refer the reader for example to, [2, 20, 16].

Theorem 3.1. *If $F(k, h), U(k, h)$ and $V(k, h)$ are the differential transform of the functions $f(x, t)$,*
$u(x, t)$ and $v(x, t)$ at $(0, 0)$ respectively, then:

> (1) *If $f(x, t) = u(x, t) \pm v(x, t)$, then $F(k, h) = U(k, h) \pm V(k, h)$*
> (2) *If $f(x, t) = au(x, t)$, then $F(k, h) = aU(k, h)$*

(3) If $f(x,t) = u(x,t)v(x,t)$, then $F(k,h) = \sum_{j=0}^{h}\sum_{i=0}^{k} U(i,j)V(k-i,h-j)$

(4) If $f(x,t) = x^r t^s$, then $F(k,h) = \delta_{k,r}\delta_{h,s}$

(5) If $f(x,t) = x^r \sin(at+b)$, then $F(k,h) = \frac{a^h}{h!}\delta_{k,r}\sin(\frac{h\pi}{2}+b)$

(6) If $f(x,t) = x^r \cos(at+b)$, then $F(k,h) = \frac{a^h}{h!}\delta_{k,r}\cos(\frac{h\pi}{2}+b)$

(7) If $f(x,t) = x^r e^{at}$, then $F(k,h) = \frac{a^h}{h!}\delta_{k,r}$

Theorem 3.2. *If $G(k,h), H(k,h), U(k,h)$ and $V(k,h)$ are the differential transform of the functions $g(x,t), h(x,t), u(x,t)$ and $v(x,t)$ at $(0,0)$ respectively, then*

(1) *If $g(x,t) = \int_0^t \int_0^x u(y,s)v(y,s)dyds$, then*

$$G(k,0) = G(0,h) = 0, \qquad k,h = 0,1,... \tag{3.5}$$

$$G(k,h) = \frac{1}{kh}\sum_{j=0}^{h-1}\sum_{i=0}^{k-1} U(i,j)V(k-i-1,h-j-1), \qquad k,h = 1,2,... \tag{3.6}$$

(2) *If $g(x,t) = h(x,t)\int_0^t \int_0^x u(y,s)dyds$, then*

$$G(k,0) = G(0,h) = 0, \qquad k,h = 0,1,... \tag{3.7}$$

$$G(k,h) = \sum_{j=0}^{h-1}\sum_{i=0}^{k-1} H(i,j)\frac{V(k-i-1,h-j-1)}{(k-i)(h-j)}, \qquad k,h = 1,2,... \tag{3.8}$$

3.3. **Procedure of DTM.** In this part we consider a special case for the function $a(t-\tau)$ in equation (1.1) to have the property $a(t-\tau) = a_1(t)a_2(\tau)$, accordingly equation (1.1) can be written as

$$u_t(x,t) = a_1(t)\int_0^t a_2(\tau)\frac{\partial}{\partial x}\sigma(u_x(x,\tau))d\tau + f(x,t), x \in (0,1), t \in (0,T) \tag{3.9}$$

$$u(x,0) = \psi(x) \tag{3.10}$$

Now applying DTM to the above equations yields

$$(h+1)U(k,h+1) = \sum_{s=0}^{h} A_1(h-s)G(k,s) + F(k,h) \qquad (3.11)$$

where,

$$G(k,s) = \frac{1}{s}\sum_{j=0}^{s-1} A_2(j)B(k,s-j-1). \qquad k,s=1,2,... \qquad (3.12)$$

$$G(k,0) = 0. \qquad k=0,1,... \qquad (3.13)$$

and,

$$U(k,h), \quad A_1(h), \quad A_2(h), \quad B(k,h), \quad F(k,h)$$

are the differential transform of the functions

$$u(x,t), \quad a_1(t), \quad a_2(t), \quad \frac{\partial}{\partial x}\sigma(u_x(x,t)), \quad f(x,t)$$

The inverse of equation (3.11) is

$$u(x,t) = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} U(k,h)x^k t^h \qquad (3.14)$$

4. Numerical Applications

In this section we will apply both DTM and VIM to solve equation (1.1) for different forms of the kernel $a(.)$ and the nonlinear function $\sigma(.)$. We also choose $f(x,t)$ to be a specific function in order to hold an exact solution which we compare with the obtained approximate solutions derived by either DTM or VIM. To illustrate the strength of the methods, and to establish approximations of high accuracy, Figures and Tables will be given. In what follows, three examples will be investigated to show the reliability of the proposed schemes. All of these examples are chosen such that

there exist analytical solutions for them (see,[15]), to give an obvious overview of the methods. The computer application program *Mathematica* was used to execute the algorithms that were used to solve the given examples.

The solution process can be illustrated for various physical processes, such as glass-forming process, a nano-hydrodynamics, and drop wise condensation.

Example 4.1. *Consider Equation (1.1) with $a(\xi) = e^{-\xi}, \sigma(\xi) = \xi^2$ and $f(x,t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t})$. Also in equation (1.2) we take $\psi(x) = e^{-x}$. Equations (1.1)-(1.2) with the above conditions has the exact solution $u(x,t) = e^{-(x+t)}$.*

Based on the properties of DTM proposed in section 3 and the above procedure we have

$$U(k, h+1) = \frac{2}{h+1} \sum_{s=0}^{h} \frac{(-1)^{h-s}}{(h-s)!} G(k,s) + \frac{1}{h+1} F(k,h) \tag{4.1}$$

$$G(k,s) = \frac{1}{s} \sum_{j=0}^{s-1} \frac{1}{j!} B(k, s-j-1).k, s = 1, 2, ... \tag{4.2}$$

$$G(k,0) = 0, k = 0, 1, ... \tag{4.3}$$

and

$$B(k,\ell) = \sum_{i=0}^{k} \sum_{r=0}^{\ell} (i+1)(k-i+2)(k-i+1)U(i+1,r)U(k-i+2,\ell-r) \tag{4.4}$$

Also

$$U(k,0) = \frac{(-1)^k}{k!} \tag{4.5}$$

and

$$F(k, h) = -\frac{(-1)^k}{k!}\frac{(-1)^h}{h!} + 2\frac{(-2)^k}{k!}\left(\frac{(-1)^h}{h!} - \frac{(-2)^h}{h!}\right) \qquad (4.6)$$

By using the above formulas, the following coefficients are obtained

$$U(0,0) = 1, \quad U(1,0) = -1, \quad U(2,0) = \tfrac{1}{2}, \quad U(3,0) = -\tfrac{1}{6}, \quad U(4,0) = \tfrac{1}{24}$$

$$U(0,1) = -1, \quad U(1,1) = 1, \quad U(2,1) = -\tfrac{1}{2}, \quad U(3,1) = \tfrac{1}{6}, \quad U(4,1) = -\tfrac{1}{24}$$

$$U(0,2) = \tfrac{1}{2}, \quad U(1,2) = -\tfrac{1}{2}, \quad U(2,2) = \tfrac{1}{4}, \quad U(3,2) = -\tfrac{1}{12}, \quad U(4,2) = \tfrac{1}{48}$$

$$U(0,3) = -\tfrac{1}{6}, \quad U(1,3) = \tfrac{1}{6}, \quad U(2,3) = -\tfrac{1}{12}, \quad U(3,3) = \tfrac{1}{36}, \quad U(4,3) = -\tfrac{1}{144}$$

Consequently substituting all $U(k, h)$ into Eq. (4.1) and after some manipulations, we obtain the series form solutions of the model in this example as

$$u(x,t) = \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \mathcal{O}(t^6)\right)$$

$$+ \left(-1 + t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120} + \mathcal{O}(t^6)\right)x$$

$$+ \left(\frac{1}{2} - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{12} + \frac{t^4}{48} - \frac{t^5}{240} + \mathcal{O}(t^6)\right)x^2$$

$$+ \left(-\frac{1}{6} + \frac{t}{6} - \frac{t^2}{12} + \frac{t^3}{36} - \frac{t^4}{144} + \frac{t^5}{720} + \mathcal{O}(t^6)\right)x^3$$

$$+ \left(\frac{1}{24} - \frac{t}{24} + \frac{t^2}{48} - \frac{t^3}{144} + \frac{t^4}{576} - \frac{t^5}{2880} + \mathcal{O}(t^6)\right)x^4$$

$$+ \left(-\frac{1}{120} + \frac{t}{120} - \frac{t^2}{240} + \frac{t^3}{720} - \frac{t^4}{2880} + \frac{t^5}{14400} + \mathcal{O}(t^6)\right)x^5 + \mathcal{O}(x^6)$$

The above results matches the same coefficients in the Taylor expansion of the exact solution, and evaluating more terms of the above approximate solution will converges to the exact solution, known as $u(x,t) = e^{-(x+t)}$. For the solution of the problem in this example using VIM, the variational iteration formula (2.6) reads as

$$u^{k+1}(x,t) = u^k(x,t) - \left[\int_0^t \left\{u_s^k(x,s) - \int_0^s e^{(s-\tau)}\frac{\partial}{\partial x}(u_x^k(x,\tau))^2 d\tau - f(x,s)\right\} ds\right]$$
$$(4.7)$$

Using this iteration, we find that approximation of the exact solution can be obtained for sufficiently large values of k. Taking the initial iteration to be $u_0(x,t) = e^{-x}$, the first two iterations are computed as follows

$$u_1(x,t) = e^{-x} - e^{-2(x+t)}\{-1 + e^t(4 - e^x + e^t[-3 + e^x + 2t])\}$$

$$u_2(x,t) = e^{-x} + \frac{1}{3}e^{-4(t+x)}\left[-4 + 64e^t - 6e^{t+x} - 3e^{2(t+x)} + 72e^{2t+x} + ...\right]$$

In the same manner the rest of components of the iteration formula (4.7) can be obtained. We should pointout that for the solution using VIM, the selection of $u_0(x,t)$ is arbitrary, but a suitable selection is effective for fast convergence and fit accuracy. In this paper, we suggest the initial approximations to be selected well-set with $u_0(x,t) = \phi(x)$.

In order to verify the efficiency of both methods (DTM, and VIM) in comparison with the exact solution, we report the maximum pointwise error for different values of $x \in [0, 1.0]$ and $t = 0.1$. Numerical results corresponding to Example 4.1 are given in Table 1. Figures 1, 2 show that the results of the VIM is in excellent agreement with the exact solution. Therefore, it is evident that the maximum pointwise error can be made smaller by computing more terms using both methods.
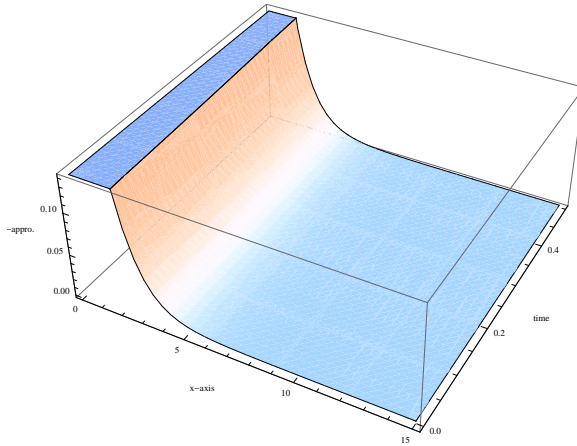

FIGURE 1. The approximate solution for Example 4.1 using VIM for $0 \leq t \leq 0.5$ and $0 \leq x \leq 15$

x_i	VIM Error	DTM Error
0.1	2.22518E-08	2.47797E-09
0.2	7.07825E-09	7.93093E-08
0.3	4.44105E-11	8.79324E-07
0.4	7.27520E-09	4.86837E-06
0.5	7.94931E-09	1.83175E-05
0.6	3.07950E-09	5.39617E-04
0.7	2.05634E-09	1.34267E-04
0.8	9.00444E-10	2.95247E-04
0.9	7.18900E-10	5.90778E-04
1.0	2.31384E-10	1.09736E-03

TABLE 1. Results for Example 4.1, when $t = 0.1$ and different values of x by VIM using $u_3(x, t)$ and using 5-terms of DTM.
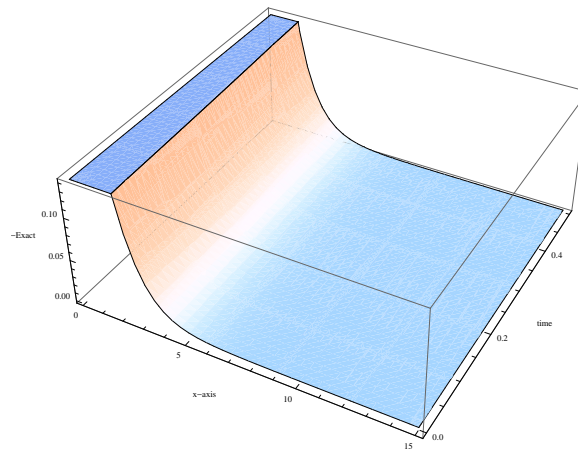

FIGURE 2. The Exact solution for Example 4.1 for $0 \leq t \leq 0.5$ and $0 \leq x \leq 15$

x_i	VIM Error	DTM Error
0.1	5.79925E-05	4.42648E-08
0.2	2.30424E-05	2.78627E-06
0.3	7.52382E-06	3.11476E-05
0.4	1.39419E-06	1.71788E-04
0.5	5.16161E-07	6.43416E-04
0.6	7.44490E-07	1.88677E-03
0.7	4.52347E-07	4.67349E-03
0.8	1.13642E-07	1.02313E-02
0.9	1.19931E-07	2.03837E-02
1.0	2.32594E-07	3.77016E-02

TABLE 2. Results for Example 4.2, when $t = 0.1$ and different values of x by VIM using $u_2(x, t)$ and using 7-terms of DTM.

Example 4.2. *In this example, we still choose $a(\xi) = e^{-\xi}$ and $\sigma(\xi) = \xi^2$, but we choose $f(x,t)$ as*

$$f(x,t) = -x(1-x)e^{-(x+t)} + (8 - 34x + 40x^2 - 16x^3 + 2x^4)(e^{-2x-t} - e^{-2x-2t}).$$

and the initial condition to be $\psi(x) = x(1-x)e^{-x}$, then the exact solution is given by $u(x,t) = x(1-x)e^{-(x+t)}$.

According to this Example, Equations (4.1)-(4.3) are still hold, but

$$U(k, 0) = \sum_{i=0}^{k} (\delta(i-1) - \delta(i-2)) \frac{(-1)^{k-i}}{(k-i)!} \tag{4.8}$$

and

$$
\begin{aligned}
F(k,h) &= \frac{(-1)^h}{h!} \sum_{i=0}^{k} (\delta(i-2) - \delta(i-1)) \frac{(-1)^{k-i}}{(k-i)!} \\
&+ \frac{(-1)^h - (-2)^h}{h!} \sum_{i=0}^{k} (2\delta(i-4) - 16\delta(i-3) + 40\delta(i-2) \\
&- 34\delta(i-1) + 8\delta(i)) \frac{(-2)^{k-i}}{(k-i)!}
\end{aligned}
\tag{4.9}
$$

By using the above formulas, the following coefficients are obtained

$$U(0,0) = 0, \quad U(1,0) = 1, \quad U(2,0) = -2, \quad U(3,0) = \tfrac{3}{2}, \quad U(4,0) = -\tfrac{2}{3}$$

$$U(0,1) = 0, \quad U(1,1) = -1, \quad U(2,1) = 2, \quad U(3,1) = -\tfrac{3}{2}, \quad U(4,1) = \tfrac{2}{3}$$

$$U(0,2) = 0, \quad U(1,2) = \tfrac{1}{2}, \quad U(2,2) = -1, \quad U(3,2) = \tfrac{3}{4}, \quad U(4,2) = -\tfrac{1}{3}$$

$$U(0,3) = 0, \quad U(1,3) = -\tfrac{1}{6}, \quad U(2,3) = \tfrac{1}{3}, \quad U(3,3) = -\tfrac{1}{4}, \quad U(4,3) = \tfrac{1}{9}.$$

The above results matches the same coefficients in the Taylor expansion of the exact solution, and our approximate solution is given by

$$u(x,t) = (x - 2x^2 + \frac{3}{2}x^3 - \frac{2}{3}x^4 + \ldots)(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \ldots)$$

For the solution using VIM, we first use $u_0(x,t)$ to be the given initial condition $\phi(x) = x(1-x)e^{-x}$, and then use the iterative scheme (2.6) to find $u_1(x,t)$ and $u_2(x,t)$. The numerical experiment is carried out for $t = 0.1$ and $x = 0.1, 0.2, \ldots, 1.0$.

Table 2 exhibits numerical results when $t = 0.1$ and different values of x by VIM using $u_2(x,t)$ and using 7-terms of DTM. Figures 3, 4 show some results using VIM.
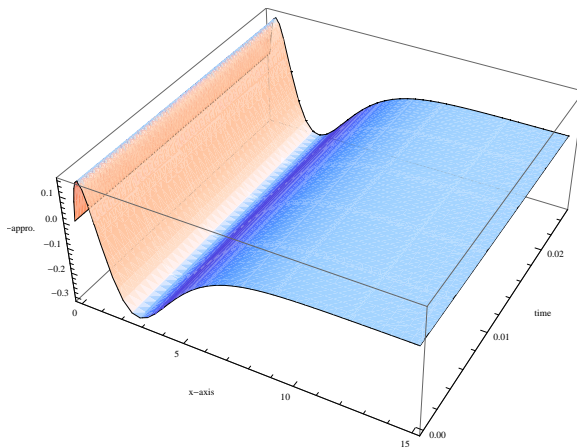

FIGURE 3. The approximate solution for Example 4.2 using VIM for $0 \leq t \leq 0.5$ and $0 \leq x \leq 15$
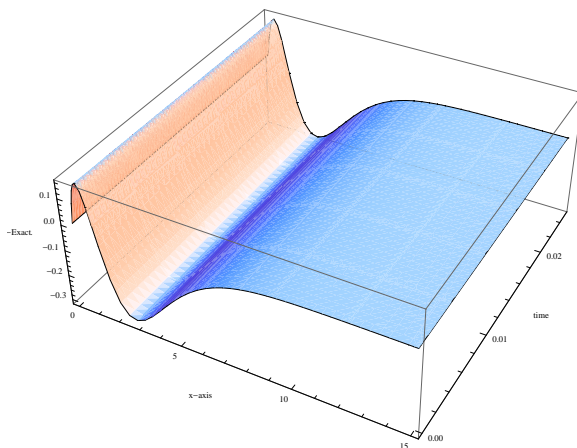

FIGURE 4. The Exact solution for Example 4.2 for $0 \leq t \leq 0.5$ and $0 \leq x \leq 15$

Example 4.3. *In this example, we choose* $a(\xi) = e^{-2\xi}$ *,* $\sigma(\xi) = \xi^2$*,* $\psi(x) = \sin x$*, and*

$$f(x,t) = \cos(x+t) + \frac{1}{4}\left[\sin 2(x+t) - \cos 2(x+t) - e^{-2t}\left(\sin 2x - \cos 2x\right)\right]$$

The exact solution is $\sin(x + t)$.

Using DTM, equations (4.1)-(4.2) will be

$$U(k, h + 1) = \frac{2}{h + 1} \sum_{s=0}^{h} \frac{(-2)^{h-s}}{(h - s)!} G(k, s) + \frac{1}{h + 1} F(k, h) \tag{4.10}$$

$$G(k, s) = \frac{1}{s} \sum_{j=0}^{s-1} \frac{2^j}{j!} B(k, s - j - 1). \qquad k, s = 1, 2, ... \tag{4.11}$$

and equations (4.3)-(4.4) are still hold. Also

$$U(k, 0) = \frac{\sin(\frac{k\pi}{2})}{k!} \tag{4.12}$$

and

$$\begin{aligned}
F(k, h) &= \frac{\cos(\frac{k\pi}{2})}{k!} \frac{\cos(\frac{h\pi}{2})}{h!} - \frac{\sin(\frac{k\pi}{2})}{k!} \frac{\sin(\frac{h\pi}{2})}{h!} \\
&+ \frac{1}{4} 2^h 2^k \frac{\sin(\frac{k\pi}{2})}{k!h!} \left(\cos(\frac{h\pi}{2}) + \sin(\frac{h\pi}{2}) \right) \\
&+ \frac{1}{4} 2^h 2^k \frac{\cos(\frac{k\pi}{2})}{k!h!} \left(\sin(\frac{h\pi}{2}) - \cos(\frac{h\pi}{2}) \right) \\
&- \frac{1}{4} \frac{(-2)^h 2^k}{k!h!} \left(\sin(\frac{k\pi}{2}) - \cos(\frac{k\pi}{2}) \right)
\end{aligned} \tag{4.13}$$

By using the above formulas the following coefficients are obtained

$U(0, 0) = 0$, $U(1, 0) = 1$, $U(2, 0) = 0$, $U(3, 0) = -\frac{1}{6}$, $U(4, 0) = 0$

$U(0, 1) = 1$, $U(1, 1) = 0$, $U(2, 1) = -\frac{1}{2}$, $U(3, 1) = 0$, $U(4, 1) = \frac{1}{24}$

$U(0, 2) = 0$, $U(1, 2) = -\frac{1}{2}$, $U(2, 2) = 0$, $U(3, 2) = \frac{1}{12}$, $U(4, 2) = 0$

$U(0,3) = -\frac{1}{6}, \quad U(1,3) = 0, \quad U(2,3) = \frac{1}{12}, \quad U(3,3) = 0, \quad U(4,3) = -\frac{1}{144}$

The above results matches the same coefficients in the Taylor expansion of the exact solution known as

$$
\begin{aligned}
u(x,t) &= \sin(x+t) \\
&= (x+t) - \frac{1}{3!}(x+t)^3 + \frac{1}{5!}(x+t)^5 - ... \\
&= x + t - \frac{1}{6}(x^3 + 3x^2t + 3xt^3 + t^3) \\
&+ \frac{1}{120}(x^5 + 5x^4t + 10x^3t^2 + 10x^2t^3 + 5xt^4 + t^5) - ...
\end{aligned}
\tag{4.14}
$$

Table 3, exhibits numerical results using both methods. While Figure 5, 6 shows the approximate solution obtained by VIM and the exact solution of the problem.
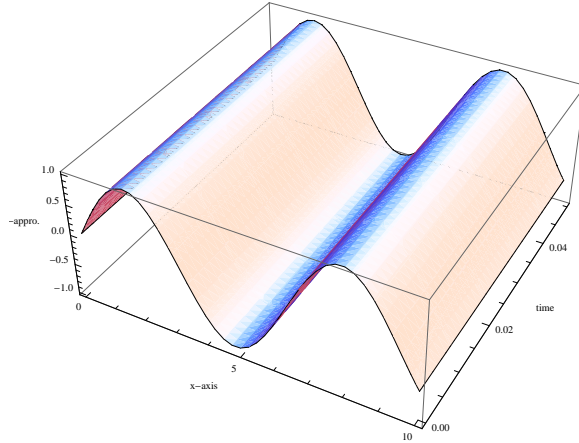

FIGURE 5. The approximate solution for Example 4.3 using VIM for $0 \leq t \leq 0.05$ and $0 \leq x \leq 15$

The series solution using regular VIM or DTM, has slow convergence rate over the wider regions. Furthermore VIM (or, DTM) needs to be modified in order to work for integral equations where their solutions consists of a rapidly and slowly

x_i	VIM Error	DTM Error
0.1	2.22518E-08	2.47797E-09
0.2	7.07825E-09	7.93093E-08
0.3	4.44105E-11	8.79324E-07
0.4	7.27520E-09	4.86837E-06
0.5	7.94931E-09	1.83175E-05
0.6	3.07950E-09	5.39617E-04
0.7	2.05634E-09	1.34267E-04
0.8	9.00444E-10	2.95247E-04
0.9	7.18900E-10	5.90778E-04
1.0	2.31384E-10	1.09736E-03

TABLE 3. Results for Example 4.3, when $t = 0.1$ and different values of x by VIM using $u_3(x, t)$ and using 5-terms of DTM.
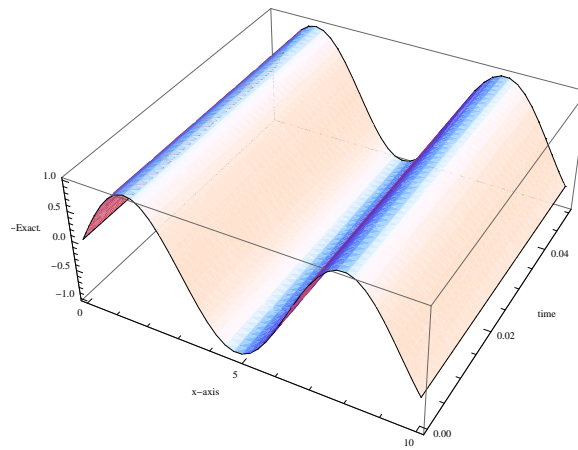

FIGURE 6. The Exact solution for Example 4.3 for $0 \leq t \leq 0.05$ and $0 \leq x \leq 15$

oscillating function (as in this example). In [13] the authors presented an alternative technique, which modified VIM series solution and makes it periodic for nonlinear oscillatory systems. Both Laplace transform and Pade approximant are used to deal with the truncated series. Pade approximant approximates a function by the ratio of two polynomials [4]. The coefficients of the powers occurring in the polynomials are determined by the coefficients in the VIM series solution of the function. Generally, the Pade approximant can enlarge the convergence domain of the series. In this case part of the VIM truncated series is the partial sum of the Taylor series of the true solution, and the lower-order Pade approximant is used to get the true solution. Therefore, we follow the same technique proposed by [12] which modifies the series solution obtained by regular VIM.

To solve the equation in this Example using the modified version of VIM approach, the first-order approximate solution can be expressed as

$$u_1(x,t) = \left(0.125 - 0.125e^{-2t}\right)\cos(2x) - 0.25\cos(t+2x)\sin(t) - 0.25e^{-2t}\cos(x)\sin(x)$$

$$- \; t\cos(x)\sin(x) + 0.25\cos(x)\sin(x) + \sin(t+x) + 0.25\sin(t)\sin(t+2x)$$

Applying a Laplace transformation to $u_1(x,t)$, we get the following result:

$$\mathcal{L}u_1(x,t) = \frac{0.125\cos(2x)}{s} - \frac{0.125\cos(2x)}{s+2} + \frac{0.25\cos(x)\sin(x)}{s}$$

$$- \frac{0.25\cos(x)\sin(x)}{s+2} - \frac{\cos(x)\sin(x)}{s^2} - \frac{0.25(s\cos(2x) - 4\cos(x)\sin(x))}{s\left(s^2+4\right)}$$

$$+ \frac{0.25(2\cos(2x) + s\sin(2x))}{s\left(s^2+4\right)} + \frac{\sin\left(x + \tan^{-1}\left(\frac{1}{s}\right)\right)}{\sqrt{s^2+1}}$$

For the sake of simplicity, let $s = 1/t$, to obtain the $[L/M]$ Pade approximant with $L \geq 4, M \geq 4$, and upon using the inverse Laplace transform to $[L/M]$ Pade, the

exact solution is obtained as $sin(x+t)$. For the solution using the modified version of DTM approach, Figure 7 shows the error between the exact solution and the solution obtained using Laplace-Pade method.
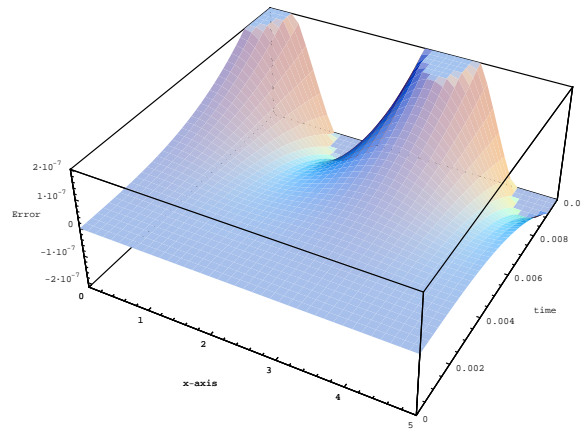

FIGURE 7. The error using Laplace-Pade in Example 4.3 for DTM with only three terms

Common to the above three examples is a justification for using DTM and VIM to solve the nonlinear problem (1.1). Indeed, the examples show that the error between the exact solutions and the approximate solutions obtained by the two methods with only 3 iterations is very small.

5. **Conclusions**

In this paper, the variational iteration method and differential transform method have been applied to solve partial integro-differential equations. Numerical results have been presented to show the efficiencies of both methods. We can conclude from the numerical results that the methods provide high accuracy for the partial integro-differential equation.

The comparison revealed that, although both methods can be seen as efficient methods for solving partial integro-differential equations, VIM is much easier, more convenient and more efficient

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