

(Short Note)

REMOTAL SETS IN THE SPACE OF P-INTEGRABLE FUNCTIONS

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ABSTRACT. Let X be a Banach space and G be a closed bounded subset of X . For $x \in X$, we set $\rho(x, G) = \sup \{\|x - y\| : y \in G\}$. The set G is called remotal if for any $x \in X$, there exists $g \in G$ such that $\rho(x, G) = \|x - g\|$. In this paper we show that for a separable remotal set $G \subset X$, the set of p -Bochner integrable functions, $L^p(I, G)$ is in remotal $L^p(I, X)$. Some other results are presented.

1. Introduction

Let X be a Banach space and G be a bounded subset in X . For $x \in X$, set $\rho(x, G) = \sup\{\|x - y\| : y \in G\}$. A point $g_0 \in G$ is called a farthest point of G if there exists $x \in X$ such that $\|x - g_0\| = \rho(x, G)$. For $x \in X$, the farthest point map $F_G(x) = \{g \in G : \|x - g\| = \rho(x, G)\}$ i.e. the set of points of G farthest from x . Note that, this set may be empty. Let $R(G, X) = \{x \in X : F_G(x) \neq \emptyset\}$. Call a closed bounded set G remotal if $R(G, X) = X$ and densely remotal if $R(G, X)$ is norm dense in X . The concept of remotal sets in Banach spaces goes back to the sixties. Edelstein [8] showed that if X is uniformly convex Banach space, then the set $R(G, X)$ is dense in X . Asplund [1] showed that if X is both reflexive and locally uniformly rotund,

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then the set $R(G, X)$ is dense in X . In several years ago Lau [10] proved that if G is weakly compact subset of a Banach space X , then the set $R(G, X)$ contains G_δ subset of X . Deville and Zizler [6] characterizes weak compactness in terms of farthest points. The study of remotal sets is little more difficult and less developed than that of proximal sets. While best approximation has applications in many branches of mathematics, remotal sets and farthest points have applications in the study of geometry of Banach spaces, [3], [4], [12]. Remotal sets in vector valued continuous functions was considered in [5]. Related results on the spaces of Bochner integrable functions $L^1(I, X)$ are given in [9] and [2]. Remotal sets in the space of Bochner p -integrable functions $L^p(I, X)$, $1 < p \leq \infty$ never been considered. The object of this paper is extend some of the results in [9] and [2] to the space of Bochner p -integrable functions $L^p(I, X)$, $1 < p \leq \infty$.

Throughout this paper, I denotes the unit interval with the Lebesgue measure. For Banach space X , $L^p(I, X)$ denotes the Banach space of Bochner p -integrable functions, $1 \leq p < \infty$, (equivalence classes) (essentially bounded for $p = \infty$) on I with values in X . For $f \in L^p(I, X)$, we set

$$\|f\|_p = \begin{cases} (\int \|f(s)\|^p)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \text{ess sup } \{\|f(s)\| : s \in I\}, & p = \infty. \end{cases}$$

For $G \subset X$, we set $L^p(I, G) = \{f \in L^p(I, X) : f(s) \in G \text{ almost all } s \in I\}$.

2. Distance formula

Progress in the discussion of the farthest point when X does not possess pleasant properties is greatly facilitated by the fact that the p -farthest distance from an element $f \in L^p(I, X)$ to a bounded subset $L^p(I, G)$ is computed by the following theorem which is similar to that of best approximation.

Theorem 2.1. *Let X be a Banach space and G be a closed bounded subset of X . Then for $1 \leq p \leq \infty$ and $f \in L^p(I, X)$*

$$\rho(f, L^p(I, G)) = \|\rho(f(\cdot), G)\|_p.$$

Proof. Is similar to that of best approximation and it will be omitted. \square

Corollary 2.2. *Let G be a remotal subset of a Banach space X . In order that an element $g \in L^p(I, G)$ be a farthest point to an element f in $L^p(I, X)$, it is necessary and sufficient that $g(s)$ be a farthest point in G to $f(s)$ for almost all $s \in I$, $1 \leq p < \infty$.*

Corollary 2.3. *Let G be a remotal subset of a Banach space X . For $1 \leq p \leq \infty$, let $f \in L^p(I, X)$ and g be a strongly measurable function such that $g(s)$ is a farthest point to $f(s)$ from G for almost all $s \in I$. Then g is a farthest point to f from $L^p(I, G)$.*

Proof. Since G is bounded, the inequality

$$||f(s)| - |g(s)|| \leq \|f(s) - g(s)\| \leq \|f(s)\| + \|g(s)\| \leq \|f(s)\| + M$$

implies that

$$\|g(s)\| \leq 2\|f(s)\| + M.$$

Consequently $g \in L^p(I, G)$ and by using the assumption and corollary 2.1 we get the result. \square

3. Remotal Sets in $L^p(I, X)$, $1 \leq p \leq \infty$.

Let G be a closed bounded subset of a Banach space X . In [9], it has been proved that if G is remotal in X and $\overline{\text{span}G}$ is finite dimensional, then $L^1(I, G)$ is remotal

in $L^1(I, X)$. In [2], this result was extended to the case of separable set. It has been proved that if G is separable and remotal subset of X , then $L^1(I, G)$ is remotal in $L^1(I, X)$. In this section, we prove that if G is separable and remotal subset of X , then $L^p(I, G)$ is remotal in $L^p(I, X)$, $1 < p \leq \infty$.

Theorem 3.1. *For any densely remotal set $G \subset X$, $L^p(I, G)$ is densely remotal in $L^p(I, X)$.*

Proof. Let $f = \sum_{i=1}^n \chi_{A_i} x_i$ be a simple function in $L^p(I, X)$. Since $R(G)$ is dense, given $\epsilon > 0$, we get a simple function $\varphi = \sum_{i=1}^n \chi_{A_i} y_i$ with $y_i \in R(G)$ and $\|f - \varphi\|_p < \epsilon$. For each i let $e_i \in G$ be the farthest point from y_i , then $f_0 = \sum_{i=1}^n \chi_{A_i} e_i$ is the point wise farthest point from φ in $L^p(I, G)$. \square

Theorem 3.2. *For a bounded closed set $G \subset X$, if $L^p(I, G)$ is remotal in $L^p(I, X)$, then G is remotal in X .*

Proof. Let $x \in X$. Set $f = 1 \otimes x$, where 1 is the constant function 1. Clearly $f \in L^p(I, X)$. By assumption there exists $g \in L^p(I, G)$ such that for any $h \in L^p(I, G)$

$$\|f - g\|_p \geq \|f - h\|_p.$$

By Theorem 2.1

$$\|f(s) - g(s)\| \geq \|f(s) - h(s)\|$$

for almost $t \in I$. Equivalently

$$\|x - g(s)\| \geq \|x - h(s)\|.$$

Let h run over all functions $1 \otimes z$, for $z \in G$, we get

$$\|x - g(s)\| \geq \|x - z\|.$$

□

Theorem 3.3. *Let X be a Banach space and G be a closed bounded subset of X . If $L^1(I, G)$ is remotal in $L^1(I, X)$, then $L^\infty(I, G)$ is remotal in $L^\infty(I, X)$.*

Proof. Let $f \in L^\infty(I, X)$. Then $f \in L^1(I, X)$ and $\|f\|_1 \leq \|f\|_\infty$. By assumption, there exists $g \in L^1(I, G)$ such that

$$\|f - g\|_1 = \rho(f, L^1(I, G)).$$

Corollary 2.1, implies that

$$\|f(s) - g(s)\| = \rho(f(s), G)$$

for almost all $s \in I$. Hence

$$\|f(s) - g(s)\| \geq \|f(s) - y\|$$

for all $y \in G$. In particular

$$\|f(s) - g(s)\| \geq \|f(s) - h(s)\|$$

for almost $s \in I$ for all $h \in L^1(I, G)$. Since G is bounded and $L^\infty(I, G) \subseteq L^1(I, G)$, it follows that $g \in L^\infty(I, G)$ and

$$\|f - g\|_\infty \geq \|f - h\|_\infty$$

for every $h \in L^\infty(I, G)$. □

Theorem 3.4. *Let X be a Banach space and G be a closed bounded subset of X . For $1 \leq p < \infty$, $L^p(I, G)$ is remotal in $L^p(I, X)$ if and only if $L^1(I, G)$ is remotal in $L^1(I, X)$.*

Proof. Assume $L^1(I, G)$ is remotal in $L^1(I, X)$. Let $f \in L^p(I, X)$. Let (B_m) be a countable partition of I in measurable sets of finite measure. For each natural number n let $B_{nm} = \{s \in B_m : n - 1 \leq \|f(s)\| < n\}$ and let $f_{nm} = \chi_{B_{nm}} f$. Of course $f_{nm} \in L^1(I, X)$ and it has a farthest point g_{nm} in $L^1(\mu, G)$. This means by Theorem 1.1, [9] $g_{nm}(s)$ is a farthest point of $f_{nm}(s)$. Define $g = \sum_{nm} g_{nm}$. It is clear that g is measurable and $g(s)$ is a farthest point of $f(s) = \sum_{nm} f_{nm}$ for almost s . By Corollary 2.2 g is the farthest point of f in $L^p(I, G)$. The converse can be proved in analogous way. \square

Theorem 3.5. *Let G be a separable remotal subset of X . Then for $1 < p \leq \infty$, $L^p(I, G)$ is remotal in $L^p(I, X)$.*

Proof. Let $f \in L^p(I, X)$. Since $L^p(I, X) \subseteq L^1(I, X)$, $p > 1$, we have $f \in L^1(I, X)$. Theorem 3.10, [2] implies that $L^1(I, G)$ is remotal in $L^1(I, X)$. Hence by Theorem 3.3 and Theorem 3.4 we get the result. \square

Finally we give another proof for Theorem 3.5 based on the technique used by Mendoza, [11], for best approximation.

Lemma 3.6. *[Lemma 2.9 of [11]] Assume $\mu(I) < +\infty$. Suppose (M, d) is a metric space and A is a subset of I such that $\mu^*(A) = \mu(I)$, where μ^* denotes the outer measure associated to μ . If g is a mapping from I to M with separable range, then for any $\epsilon > 0$ there exists a countable partition $\{E_n\}$ of I in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and $\text{diam}(g(A_n)) < \epsilon$ for all n .*

Theorem 3.7. *Let G be a separable remotal subset of X . Then $L^p(I, G)$ is remotal in $L^p(I, X)$ if and only if G is remotal in X .*

Proof. Necessity is in Theorem 3.2. Let us show sufficiency. Suppose that G is remotal in X , and let f be a function in $L^p(I, X)$. Since f is measurable, we may assume that $f(I)$ is a separable set in X . Using the fact that μ is σ -finite we can find countable partitions $\{I_n\}_{n=1}^\infty$ of I in measurable sets such that $\text{diam}(f(I_n)) < \frac{1}{2}$ and $\mu(I_n) < \infty$, for all n , where

$$\text{diam}A = \sup \{ \|x - y\| : x, y \in A \}.$$

For each $t \in I$, let $g_0(t) \in G$ be a farthest point of $f(t)$ in G . Define g_0 from I into G such that $g_0(t)$ a farthest point of $f(t)$ in G . Applying Lemma 3.1 to the mapping g_0 in each I_n taking $\epsilon = \frac{1}{2}$ and $I = A = I_n$. We get a countable partition in each I_n and therefore a countable partition in the whole of I . Thus we get a countable partition $\{E_n\}_{n=1}^\infty$ of I in measurable sets and a sequence of subsets $\{A_n\}_{n=1}^\infty$ of I such that

$$\begin{aligned} A_n &\subseteq E_n, \mu^*(A_n) = \mu(E_n) < +\infty, \\ \text{diam}(g_0(A_n)) &< \frac{1}{2}, \text{diam}(f(E_n)) < \frac{1}{2}. \end{aligned}$$

Let us apply again the same argument in each E_n with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each n we get a countable partition $\{E_{n_k} : 1 \leq k < \infty\}$ of E_n in measurable sets and a sequence $\{A_{n_k} : 1 \leq k < \infty\}$ of subsets of I such that

$$\begin{aligned} A_{n_k} &\subseteq E_{n_k} \cap A_n, \mu^*(A_{n_k}) = \mu(E_{n_k}), \\ \text{diam}(g_0(A_{n_k})) &< \frac{1}{2^2} \text{ and } \text{diam}(f(E_{n_k})) < \frac{1}{2^2}, \end{aligned}$$

for all n and k . Let us proceed by induction. Now for each natural number k , let Δ_k be the set of k -tuples of natural numbers and let $\Delta = \bigcup_{k=1}^\infty \Delta_k$. On this Δ consider the partial order defined by $(m_1, m_2, \dots, m_i) \leq (n_1, n_2, \dots, n_j)$ if and only if $i \leq j$ and $m_k = n_k$ for $k = 1, 2, \dots, i$. Then by induction for each natural number k , we can take a partition $\{E_\alpha : \alpha \in \Delta_k\}$ of subsets of I and a collection $\{A_\alpha\}_{\alpha \in \Delta_k}$ such that

- (1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$ for each α .
- (2) $A_\alpha \subseteq A_\beta$ and $E_\alpha \subseteq E_\beta$ if $\beta \leq \alpha$.
- (3) $\text{diam}(f_i(E_\alpha)) < \frac{1}{2^k}$ for $i = 1, 2, \dots, m$ and $\text{diam}(g_0(A_\alpha)) < \frac{1}{2^k}$ if $\alpha \in \Delta_k$.

We may assume that $A_\alpha \neq \emptyset$ for all α (forget the α 's for which $A_\alpha = \emptyset$). For each $\alpha \in \Delta$ take $t_\alpha \in A_\alpha$ and define g_k from I into G by $g_k(\cdot) = \sum_{\alpha \in \Delta_k} \chi_{E_\alpha}(\cdot) g_0(t_\alpha)$. Then for each $t \in I$ and $n \leq k$ we have

$$\|g_n(t) - g_k(t)\| = \left\| \sum_{\alpha \in \Delta_n} \chi_{E_\alpha}(t) g_0(t_\alpha) - \sum_{\beta \in \Delta_k} \chi_{E_\beta}(t) g_0(t_\beta) \right\|$$

But since $n \leq k$ by 1 and 2 we have:

$$\begin{aligned} \|g_n(t) - g_k(t)\| &\leq \left\| \sum_{\beta \in \Delta_k} \chi_{E_\beta}(t) (g_0(t_\alpha) - g_0(t_\beta)) \right\| \\ &\leq \sum_{\beta \in \Delta_k} \|g_0(t_\alpha) - g_0(t_\beta)\| \mu(E_\beta) \\ &\leq \frac{1}{2^n} \end{aligned}$$

Therefore $(g_k(t))$ is a Cauchy sequence in X for every $t \in I$. Consequently $(g_k(t))$ is a convergent sequence for every $t \in I$. Let $g : I \rightarrow G$ be the point wise limit of (g_k) . Since g_k is measurable for each k , g is measurable. Let $t \in I$ and let n be a

natural number. Suppose $t \in E_\alpha$. We have:

$$\begin{aligned}
 \|f(t) - g_n(t)\| &= \|f(t) - g_0(t_\alpha)\| \\
 &\geq \|f(t_\alpha) - g_0(t_\alpha)\| - \|f(t) - f(t_\alpha)\| \\
 &\geq \left| \|f(t_\alpha) - g_0(t_\alpha)\| - \frac{1}{2^n} \right| \\
 &= \left| \rho((f(t_\alpha), G) - \frac{1}{2^n} \right| \\
 &\geq \rho((f(t), G) - \|f_i(t) - f_i(t_\alpha)\| - \frac{1}{2^n} \\
 &\geq \rho((f(t), G) - \frac{1}{2^{n-1}}
 \end{aligned}$$

Letting $n \rightarrow \infty$ we get

$$\|f(t) - g(t)\| = \lim_{n \rightarrow \infty} \|f(t) - g_n(t)\| = \rho((f(t), G).$$

and so $g(t)$ is a farthest point of $f(t)$ in G . By Corollary 2.2, it follows that g a farthest point of f in $L^p(I, G)$. \square

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