Jordan Journal of Mathematics and Statistics (JJMS) 3(2), 2010, pp. 127 - 131

(Short Note)

ON THE DIAMETER OF ZERO-DIVISOR GRAPHS OF IDEALIZATIONS WITH RESPECT TO INTEGRAL DOMAIN

MANAL AL-LABADI

ABSTRACT. Let R be a ring with unity and let M be an R - module. Let $R(+)M$ be the idealization of the ring R by the R - module M. In this paper, we give new results on the diameter of $\Gamma(R(+)M)$ when R is an integral domain.

1. **Introduction**

The zero divisor graph of a ring is the (simple) graph whose vertex set is the set of non-zero zero divisors, and an edge is drawn between two distinct vertices if their product is zero. The zero divisor graph of a commutative ring has been studied extensively by several authors, see [1, 2, 3 and 4]. Let R be a commutative ring with unity. We use the notation A^* to refer to the nonzero elements of A. For two distinct vertices a and b in a graph $\Gamma(R)$, the distance between a and b, denoted by $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists, otherwise, $d(a, b) = \infty$. The diameter of a graph $\Gamma(R)$ is $diam(\Gamma) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct verices of } \Gamma\}$. We will use the notation $diam(\Gamma(R))$ to denote the diameter of the θnonzero zero divisors of R). Let

2000 *Mathematics Subject Classification.* Primary: 13A99, Secondary: 13A15.

Key words and phrases. : Zero - divisor graphs, Idealization rings.

Received: Nov. 16,2009. Accepted : Feb. 8, 2010.

M be an R - module. Consider $R(+)M = \{(a,m) : a \in R, m \in M\}$ and let (a,m) and (b,n) be two elements of $R(+)M$. Define $(a,m)+(b,n) = (a+b, m+n)$ and $(a,m)(b,n) = (ab, an+bm)$. Under this definition $R(+)M$ becomes a commutative ring with unity. Call this ring the idealization ring of M in R. For more details, we refer the reader to [5]. The set of all nonzero zero divisors of a ring R is denoted by $Z^*(R)$.

$Diam(\Gamma(R(+)M))$ when R is an Integral Domain

Lemma 1.1. *Let R be an integral domain such that $\mathbb{Z}_2$ is an R-module with $ann(\mathbb{Z}_2) = \{0\}$. Then $R \cong \mathbb{Z}_2$.*

Proof. Now $1_R.1 = 1$ since $\mathbb{Z}_2$ is unitary R-module. So, $(1_R + 1_R).1 = 1_R.1 + 1_R.1 = 1+1 = 0$. Hence $2.1_R = 1_R + 1_R \in ann(\mathbb{Z}_2) = \{0\}$. Hence $ch(R) = 2$. Moreover, $r.1 = 1$ for all $r \in R^*$ because $ann(\mathbb{Z}_2) = \{0\}$. Assume that $r \in R - \{0_R,\ 1_R\}$. Then $(r+1_R).1 = 1+1 = 0$, hence $r = -1_R = 1_R$, a contradiction. Then $R \cong \mathbb{Z}_2$. □

Lemma 1.2. *Let R be an integral domain such that $\mathbb{Z}_3$ is an R-module with $ann(\mathbb{Z}_3) = \{0\}$. Then $R \cong \mathbb{Z}_3$.*

Proof. $0, 1_R \in R$ and $1_R \neq 0_R$. Then $2\ 1_R \in R$. If $2\ 1_R = 0_R$, then $3\ 1_R.1 = 0$ because $3\ 1_R.1 = (1_R + 1_R + 1_R).1 = 1+1+1 = 0$. So, $0 = 3\ 1_R.1 = (2\ 1_R + 1_R).1 = 2\ 1_R.1 + 1_R.1 = 0+1 = 1$, contradiction. So, $2\ 1_R \neq 0_R$. If $2\ 1_R = 1_R$, then $1_R = 0_R$ since $(R,+)$ is abelian group. So, R has at least 3 element $0_R, 1_R$ and $2\ 1_R$. Assume that there exist, $r \in R - \{0_R, 1_R, 21_R\}$, $r \in R$. If $r.1 = 0$, then $r \in ann(\mathbb{Z}_3)$. Hence $r = 0_R$, contradiction. If $r.1 = 1$, then

$(r+21_R).1 = r.1 + 21_R.1 = r.1 + 1 + 1 = 1 + 1 + 1 = 0$. So, $r + 21_R = 0_R$ since $ann(\mathbb{Z}_3) = \{0\}$. Then $r = -21_R = 1_R$, contradiction. If $r.1 = 2$, then $(r + 1_R).1 = 0$. So, $r + 1_R = 0_R$ i.e $r = -1_R = 21_R$, contradiction. Thus $R = \{0_R, 1_R, 21_R\}$ and $31_R = 0_R$. Then $(R, +) \cong (\mathbb{Z}_3, +)$. $(R, .) =$

.	0_R	1_R	21_R
0_R	0_R	0_R	0_R
1_R	0_R	1_R	21_R
21_R	0_R	21_R	1_R

.

Then $R \cong \mathbb{Z}_3$ (as a ring). □

Theorem 1.3. *Let R be an integral domain and $M \cong \mathbb{Z}_2$ be an R-module.*

(i): *If $ann(\mathbb{Z}_2) = \{0\}$, then $diam(\Gamma(R(+)\mathbb{Z}_2)) = 0$.*

(ii): *If $ann(\mathbb{Z}_2) \neq \{0\}$, then $diam(\Gamma(R(+)\mathbb{Z}_2)) = 2$.*

Proof. (i) $M \cong \mathbb{Z}_2$, and $ann(\mathbb{Z}_2) = \{0\}$. Then, by Lemma 1.1, $R \cong \mathbb{Z}_2$. Then $Z^*(\mathbb{Z}_2(+)\mathbb{Z}_2) = \{(0,1)\}$, so $diam(\Gamma(R(+)\mathbb{Z}_2)) = 0$.

(ii) $M \cong \mathbb{Z}_2$ and $ann(\mathbb{Z}_2) \neq \{0\}$. Then there exists at least one element in R^* such that $r.1 = 0$. $Z^*(R(+)\mathbb{Z}_2) = \{(0,1)\} \cup \{(r,0), (r,1), ...\}$. Any two elements in $\{(r,0), (r,1), ...\}$ are non adjacent, but $(r,0).(0,1) = (0,0)$ and $(0,1).(r,1) = (0,0)$. $(r,0)$—$(0,1)$—$(r,1)$ so, $diam(\Gamma(R(+)Z_2)) = 2$. □

Theorem 1.4. *Let R be an integral domain and $M \cong Z_3$ be an R-module.*

(i): *If $ann(\mathbb{Z}_3) = \{0\}$, then $diam(\Gamma(R(+)\mathbb{Z}_3)) = 1$.*

(ii): *If $ann(\mathbb{Z}_3) \neq \{0\}$, then $diam(\Gamma(R(+)\mathbb{Z}_3)) = 2$.*

Proof. (i) $M \cong \mathbb{Z}_3$, $ann(\mathbb{Z}_3) = \{0\}$. Then, by Lemma 1.2, $R \cong \mathbb{Z}_3$. So,
$Z^*(\mathbb{Z}_3(+)\mathbb{Z}_3) = \{(0,1), (0,2)\}$.

Thus, $diam(\Gamma(R(+)\mathbb{Z}_3) = 1$. $\Gamma(R(+)\mathbb{Z}_3)$ is $(0,1)$—$(0,2)$.

(ii) $M \cong \mathbb{Z}_3$ and $ann(\mathbb{Z}_3) \neq \{0\}$. Then there exists at least one element in R^* such that $r.Z_{\ 3} = 0$. So, $Z^*(R(+)\mathbb{Z}_3) = \{(0,1),(0,2)\} \cup \{(r,0),(r,1),(r,2),...\}$. Any two elements in $\{(r,0),(r,1),(r,2),...\}$ are non adjacent since $r.s \neq 0$ for any $r,s \in R^*$. But $(r,0)$—$(0,1)$—$(r,2)$. Hence, $diam(\Gamma(R(+)\mathbb{Z}_2)) = 2$. □

Theorem 1.5. *Let R be an itegral domain and $|M| \geq 4$, be an R-module.*

- **(i):** *If $r.m \neq 0$, for any $r \in R^*$ and $m\epsilon M^*$, then $diam(\Gamma(R(+)M)) = 1$.*
- **(ii):** *If there exists at least one element $m \in M^*$ such that $r.m = 0$,for any $r\epsilon R$, then $diam(\Gamma(R(+)M)) = 2$.*
- **(iii):** *If there exists at least two elements in R^* such that $r_1.m = 0$, $r_2.n = 0$, $r_1.n \neq 0$, $r_2.m \neq 0$, $m \neq n$, for $m,n\epsilon M^*$, then $diam(\Gamma(R(+)M)) = 3$.*

Proof. (i) If there is no element $r\epsilon R^*$ such that $r.m = 0$, for any $m\epsilon M^*$, then $Z^*(R(+)M) = \{(0,m) : m\epsilon M^*\}$. Then any two elements in $\{(0,m) : m\epsilon M^*\}$ are at distance 1.

(ii) If there exists at least one element $m \in M^*$ such that $r.m = 0$,for any $r\epsilon R$, then $Z^*(R(+)M) = \{(0,m) : m\epsilon M^*\} \cup \{(r,m),... : m\epsilon M\}$. Then any two elements in $\{(r,m),... : m\epsilon M\}$ are non adjacent. But $(r,n).(0,m) = (0,0)$, and $(0,m).(0,n) = (0,0)$. Then (r,n)—$(0,m)$—$(0,n)$, $m \neq n$, and m, $n\epsilon M^*$. Then $diam(\Gamma(R(+)M)) = 2$.

(iii) If there exists two elements r_1, $r_2 \in R^*$ such that $r_1.m = 0$, $r_2.n = 0$, $m \neq n$, and $m,n\epsilon M^*$, then $Z^*(R(+)M) = \{(0,m) : m\epsilon M^*\} \cup \{(r_1,m),(r_2,n),...: m \neq n, m,n\epsilon M^*\}$. Any two elements in $\{(r_1,n),(r_2,m),... : m \neq n, m,n\epsilon M^*\}$ are non adjacent . But $(r_1,0)$—$(0,m)$—$(0,n)$—$(r_2,0)$. Then $diam(\Gamma(R(+)M)) = 3$. □

Example **1.6.** *Consider the ring* $\mathbb{Z}_5(+)\mathbb{Z}_5$*. Then* $diam(\Gamma(R(+)M)) = 1$*. Since* $\mathbb{Z}_5$ *is an integral domain and* $\mathbb{Z}_5$ *is an* $\mathbb{Z}_5$*-module , and there is no element in* Z_5^* *such that* $r.m = 0$*, for any* $m \epsilon Z_5^*$*.*

Example **1.7.** *Consider the ring* $\mathbb{Z}(+)\mathbb{Z}_{18}$*. Then* $diam(\Gamma(R(+)M)) = 3$*. Since* $\mathbb{Z}$ *is an integral domain and* $\mathbb{Z}_{18}$ *is an* $\mathbb{Z}$*-module, and there exists two elements* $(r_1 = 2, r_2 = 9)$ *in* $\mathbb{Z}^*$ *such that* $2.9 = 0$*,* $9.2 = 0$*,* $2.2 \neq 0$*,* $9.9 \neq 0$*,* $9, 2 \epsilon Z_{18}^*$*, then* $(2,0)$—$(0,9)$—$(0,2)$—$(9,0)$*.*

Example **1.8.** *Consider the ring* $\mathbb{Z}(+)\mathbb{Z}_5$*. Then* $diam(\Gamma(R(+)M)) = 2$*. Snce* $\mathbb{Z}$ *is an integral domain and* $\mathbb{Z}_5$ *is an* $\mathbb{Z}$*-module, and there exists* $5 \in \mathbb{Z}$ *and* $5.3 = 0$*,* $3 \in \mathbb{Z}_5$*. And for all* $r \in \mathbb{Z}$*,* $r.3 = 0$*.*

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative rings, J. Algebra, 217 (1999), 434 - 447.

[2] M. Axtell and J. Stickles, Zero-divisor graphs of idealizations, Journal of Pure and Applied Algebra, 204 (2006), 235 - 243.

[3] I. Beck, Coloring of a commutative ring, J. Algebra, 116 (1988), 208 - 226.

[4] S. Ebrahimi Atani and Z. Ebrahimi Sarvandi, Zero-divisor graphs of idealizations with respect to prime modules, Int. J. Contemp. Math. Sciences, 2 (26) (2007), 1279 - 1283.

[5] J. Huckaba (1988), Commutative rings with zero divisors, in Monographs Pure Applied Mathematics, Marcel Dekker, Basel, New York.

Department of Mathematics, Philadelphia University, Amman, Jordan

E-mail address: `mlabadi@philadelphia.edu.jo`