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QUASI $b-$OPEN AND STRONGLY $b-$OPEN FUNCTIONS

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ABSTRACT. In this paper we introduce $b-open$, $b-closed$, $quasi\ b-open$, $quasi\ b-closed$, $strongly\ b-open$ and $strongly\ b-closed$ functions and investigate properties and characterizations of these new types of functions.

1. Introduction

In 1996, Andrijevic [1] introduced the notion of $b-open$ sets. This type of sets discussed by El-Atik [3] under the name of $\gamma-open$ sets. We continue to explore further properties and characterizations of $b-open$, $quasi\ b-open$ and $strongly\ b-open$ functions. We also introduce and study properties and characterizations of $b-closed$, $quasi\ b-closed$ and $strongly\ b-closed$ functions.

Let A be a subset of a space (X,τ). The closure (resp. interior) of A will be denoted by $Cl(A)$ (resp. $Int(A)$).

A subset A of a space (X,τ) is called $b-open$ [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. The complement of a $b-open$ set is called a $b-closed$ set. The union of all $b-open$ sets contained in A is called the $b-interior$ of A, denoted by $bInt(A)$ and the intersection of all $b-closed$ sets containing A is called the $b-closure$ of A, denoted by $bCl(A)$. The family of all $b-open$ (resp. $b-closed$) sets in (X,τ) is denoted by $BO(X)$ (resp. $BC(X)$).

A subset A of a space (X,τ) is called $semi-open$ [4] if $A \subseteq Cl(Int(A))$. The complement of a $semi-open$ set is called $semi-closed$ [2]. The family of all $semi-open$ (resp. $semi-closed$) sets in (X,τ) is denoted by $SO(X)$ (respectively $SC(X)$).

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2. $b-$Open and $b-$Closed Functions

In this section we define the concept of $b-open$ functions as a generalization of open functions and investigate some properties of such functions.

Definition 2.1. *A function $f:(X,\tau)\to(Y,\rho)$ is called $b-open$ if $f(U)\in BO(Y)$ for every open set U in X.*

The following theorem follows immediately from the above definition.

Theorem 2.2. *A function $f:(X,\tau)\to(Y,\rho)$ is $b-open$ if and only if for each $x\in X$, and each open set U in X with $x\in U$, there exists a set $V\in BO(Y)$ containing $f(x)$ such that $V\subseteq f(U)$.*

Theorem 2.3. *Let $f:(X,\tau)\to(Y,\rho)$ be $b-open$. If $V\subseteq Y$ and C is a closed subset of X containing $f^{-1}(V)$, then there exists a set $F\in BC(Y)$ containing V such that $f^{-1}(F)\subseteq C$.*

Proof. Let $F=Y-f(X-C)$. Then $F\in BC(Y)$. Since $f^{-1}(V)\subseteq C$, we have $f(X-C)\subseteq(Y-V)$ and so $V\subseteq F$.
Also $f^{-1}(F)=X-f^{-1}[f(X-C)]\subseteq X-(X-C)=C$. □

Theorem 2.4. *A function $f:(X,\tau)\to(Y,\rho)$ is $b-open$ if and only if $f[Int(A)]\subseteq bInt[f(A)]$, for every $A\subseteq X$.*

Proof. $\Rightarrow$). Let $A\subseteq X$ and $x\in Int(A)$. Then there exists an open set U_x in X such that $x\in U_x\subseteq A$. Now $f(x)\in f(U_x)\subseteq f(A)$. Since f is $b-open$, $f(U_x)\in BO(Y)$. Then $f(x)\in bInt[f(A)]$. Thus $f[Int(A)]\subseteq bInt[f(A)]$.

$\Leftarrow$). Let U be an open set in X. Then by assumption, $f[Int(U)]\subseteq bInt[f(U)]$. Since $bInt[f(U)]\subseteq f(U)$, $f(U)=bInt[f(U)]$. Thus $f(U)\in BO(Y)$.
So f is $b-open$. □

The equality in the last theorem need not be true as shown in the following example

Example **2.5.** *Let $X=Y=\{a,b\}$. Let τ be the indiscrete topology on X and ρ be the discrete topology on Y. Then $BO(X)=\{\phi,X,\{a\},\{b\}\}$ and $BO(Y)=\rho$. Let $f:(X,\tau)\to(Y,\rho)$ be the identity function and $A=\{a\}$. Then $f[Int(A)]=\phi$ and $bInt[f(A)]=\{a\}$.*

Theorem 2.6. *A function $f:(X,\tau)\to(Y,\rho)$ is $b-open$ if and only if $Int[f^{-1}(B)]\subseteq f^{-1}[bInt(B)]$, for every $B\subseteq Y$.*

Proof. $\Rightarrow$). Let $B\subseteq Y$. Then $f[Int(f^{-1}(B))]\subseteq f[f^{-1}(B)]\subseteq B$.
But $f[Int(f^{-1}(B))]\in BO(Y)$ since $Int[f^{-1}(B)]$ is open in X and f is $b-open$. Hence, $f[Int(f^{-1}(B))]\subseteq bInt(B)$. Therefore $Int[f^{-1}(B)]\subseteq f^{-1}[bInt(B)]$.

$\Leftarrow$). Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by assumption, we obtain, $Int(A) \subseteq Int[f^{-1}(f(A)] \subseteq f^{-1}[bInt(f(A))]$. Thus $f[Int(A)] \subseteq bInt[f(A)]$, for every $A \subseteq X$. Hence, by Theorem 2.4, f is $b-open$. $\square$

Theorem 2.7. *A function $f : (X, \tau) \rightarrow (Y, \rho)$ is $b-open$ if and only if $f^{-1}[bCl(B)] \subseteq Cl[f^{-1}(B)]$, for every $B \subseteq Y$.*

Proof. $\Rightarrow$). Assume that f is $b-open$ and $B \subseteq Y$. Let $x \in f^{-1}[bCl(B)]$. Then $f(x) \in bCl(B)$. Let U be an open set in X such that $x \in U$. Since f is $b-open$, then $f(U) \in BO(Y)$. Therefore, $B \cap f(U) \neq \phi$. Then $U \cap f^{-1}(B) \neq \phi$. Hence $x \in Cl[f^{-1}(B)]$. We conclude that $f^{-1}[bCl(B)] \subseteq Cl[f^{-1}(B)]$.

$\Leftarrow$). Let $B \subseteq Y$. Then $(Y - B) \subseteq Y$. By assumption,

$$f^{-1}[bCl(Y - B)] \subseteq Cl[f^{-1}(Y - B)].$$

This implies,

$$X - Cl[f^{-1}(Y - B)] \subseteq X - f^{-1}[bCl(Y - B)].$$

Hence

$$X - Cl[X - f^{-1}(B)] \subseteq f^{-1}[Y - bCl(Y - B)].$$

Now

$$X - Cl[X - f^{-1}(B)] = Int[X - (X - f^{-1}(B))] = Int[f^{-1}(B)]$$

then we have $Y - bCl(Y - B) = bInt[Y - (Y - B)] = bInt(B)$.
Then, $Int[f^{-1}(B)] \subseteq f^{-1}[bInt(B)]$. Now from Theorem 2.6, it follows that f is $b-open$. $\square$

Now we introduce $b-closed$ functions and study certain properties of this type of functions.

Definition 2.8. *A function $f : (X, \tau) \rightarrow (Y, \rho)$ is called $b-closed$ if $f(C) \in BC(Y)$ for every closed set C in X.*

Theorem 2.9. *A function $f : (X, \tau) \rightarrow (Y, \rho)$ is $b-closed$ if and only if $bCl[f(A)] \subseteq f[Cl(A)]$ for every $A \subseteq X$.*

Proof. $\Rightarrow$). Let f be $b-closed$ and let $A \subseteq X$. Then $f[Cl(A)] \in BC(Y)$. But $f(A) \subseteq f[Cl(A)]$. Then $bCl[f(A)] \subseteq f[Cl(A)]$.

$\Leftarrow$). Let $A \subseteq X$ be a closed set. Then by assumption, $bCl[f(A)] \subseteq f[Cl(A)] = f(A)$. This shows that $f(A) \in BC(Y)$. Hence f is $b-closed$. $\square$

Corollary 2.10. *Let $f : (X, \tau) \to (Y, \rho)$ be $b-closed$ and let $A \subseteq X$. Then $bInt[bCl(f(A))] \subseteq f[Cl(A)]$.*

Theorem 2.11. *Let $f : (X, \tau) \to (Y, \rho)$ be a surjective function. Then f is $b-closed$ if and only if for each subset B of Y and each open set U in X containing $f^{-1}(B)$, there exists a set $V \in BO(Y)$ containing B such that $f^{-1}(V) \subseteq U$.*

Proof. $\Rightarrow$). Let $V = Y - f(X - U)$. Then $V \in BO(Y)$. Since $f^{-1}(B) \subseteq U$, we have $f(X - U) \subseteq Y - B$ and so $B \subseteq V$. Also,

$$f^{-1}(V) = X - f^{-1}[f(X - U)] \subseteq X - (X - U) = U.$$

$\Leftarrow$). Let C be a closed set in X and $y \in Y - f(C)$. Then,
$f^{-1}(y) \subseteq X - f^{-1}(f(C)) \subseteq X - C$ and $X - C$ is open in X. Hence by assumption, there exists a set $V_y \in BO(Y)$ containing y such that $f^{-1}(V_y) \subseteq X - C$. This implies that $y \in V_y \subseteq Y - f(C)$. Thus $Y - f(C) = \cup\{V_y : y \in Y - f(C)\}$. Hence $Y - f(C) \in BO(Y)$. Thus $f(C) \in BC(Y)$. □

Definition 2.12. [3]**.** *A function $f : (X, \tau) \to (Y, \rho)$ is said to be $b-continuous$ if $f^{-1}(V) \in BO(X)$ for every open set V in Y.*

Theorem 2.13. *Let $f : (X, \tau) \to (Y, \rho)$ be a bijection. Then the following are equivalent:*

1) f is $b-closed$
2) f is $b-open$
3) f^{-1} is $b-continuous$

Proof. (1) $\to$ (2). Let U be an open subset of X. Then $X - U$ is closed in X. By (1), $f(X - U) \in BC(Y)$. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U) \in BO(Y)$.

(2) $\to$ (3). Let U be an open subset of X. Since f is $b-open$
$f(U) = (f^{-1})^{-1}(U) \in BO(Y)$. Hence f^{-1} is $b-continuous$.

(3) $\to$ (1). Let C be an arbitrary closed set in X. Then $X - C$ is open in X. Since f^{-1} is $b-continuous$, $(f^{-1})^{-1}(X - C) \in BO(Y)$. But,

$$(f^{-1})^{-1}(X - C) = f(X - C) = Y - f(C).$$

Thus, $f(C) \in BC(Y)$. □

Definition 2.14. [3]**.** *A space X is called:*

a) $b - T_1$ if for each pair of distinct points x and y in X, there exist $b-open$ sets U and V of X containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

b) $b - T_2$ if for each pair of distinct points x and y in X, there exist disjoint $b-open$ sets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Theorem 2.15. *Let $f : (X, \tau) \to (Y, \rho)$ be a $b-open$ bijection. Then the following hold*

a) If X is T_1 then Y is $b - T_1$.
b) If X is T_2 then Y is $b - T_2$.

Proof. (a) Let y_1 and y_2 be any distinct points in Y. Then there exist x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is T_1 there exist two open sets U and V in X with $x_1 \in U$, $x_2 \notin U$ and $x_2 \in V$, $x_1 \notin V$. Now $f(U)$ and $f(V)$ are $b - open$ in Y with $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$.

(b) Similar to (a). □

Definition 2.16. [3]**.** *A space X is said to be $b - compact$ (resp. $b - Lindelöf$) if every $b - open$ cover of X has a finite (resp. countable) subcover.*

Theorem 2.17. *Let $f : (X, \tau) \to (Y, \rho)$ be a $b - open$ bijection. Then the following hold*

a) If Y is $b - compact$, then X is compact.
b) If Y is $b - Lindelöf$, then X is $Lindelöf$.

Proof. (a) Let $U = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Then $O = \{f(U_\alpha) : \alpha \in \Delta\}$ is a cover of Y by $b - open$ sets in Y. Since Y is $b - compact$, O has a finite subcover $O' = \{f(U_{\alpha_1}), f(U_{\alpha_2}), ..., f(U_{\alpha_n})\}$ for Y. Then $U' = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha n}\}$ is a finite subcover of U for X.

(b) Similar to (a). □

Definition 2.18. [3]**.** *A space X is said to be $b - connected$ if it cannot be written as a union of two non-empty disjoint $b - open$ sets.*

Theorem 2.19. *If $f : (X, \tau) \to (Y, \rho)$ is a $b-open$ surjection and Y is $b-connected$ then X is connected.*

Proof. Suppose that X is not *connected*. Then there exist two non-empty disjoint open sets U and V in X such that $X = U \cup V$. Then $f(U)$ and $f(V)$ are non-empty disjoint $b - open$ sets in Y with $Y = f(U) \cup f(V)$ which contradicts the fact that Y is $b - connected$. □

3. Quasi $b-$Open and Quasi $b-$Closed Functions

Definition 3.1. *A function $f : (X, \tau) \to (Y, \rho)$ is said to be quasi $b - open$ if $f(U)$ is open in Y for every $U \in BO(X)$.*

Clearly, every *quasi $b - open$* function is $b - open$.

Definition 3.2. *A subset A is called a $b-neighborhood$ of a point x in X if there exists a $b-open$ set U such that $x \in U \subseteq A$.*

Theorem 3.3. *Let $f : (X, \tau) \to (Y, \rho)$ be a function. then the following are equivalent:*

1) f is quasi $b-open$.
2) For any subset A of X we have $f[bInt(A)] \subseteq Int[f(A)]$.
3) For any $x \in X$ and any $b-neighborhood$ U of x, there exists a neighborhood V of $f(x)$ in Y such that $V \subseteq f(U)$.

Proof. (1) → (2). Let f be *quasi* $b-open$ and $A \subseteq X$. Now we have $Int(A) \subseteq A$ and $bInt(A) \in BO(X)$. Hence we obtain that $f[bInt(A)] \subseteq f(A)$. Since $f[bInt(A)]$ is open, $f[bInt(A)] \subseteq Int[f(A)]$.

(2) → (3). Let $x \in X$ and U be a $b-neighborhood$ of x in X. Then there exists $V \in BO(X)$ such that $x \in V \subseteq U$. Then by (2), we have,

$$f(V) = f[bInt(V)] \subseteq Int[f(V)]$$

and hence $f(V) = Int[f(V)]$. Therefore $f(V)$ is open in Y such that $f(x) \in f(V) \subseteq f(U)$.

(3) → (1). Let $U \in BO(X)$. Then for each $y \in f(U)$, there exists a *neighborhood* V_y of y in Y such that $V_y \subseteq f(U)$. Since V_y is a *neighborhood* of y, there exists an open set W_y in Y such that $y \in W_y \subseteq V_y$. Thus, $f(U) = \cup\{W_y : y \in f(U)\}$ which is an open set in Y. This implies that f is *quasi* $b-open$ function. □

Theorem 3.4. *A function $f : (X, \tau) \to (Y, \rho)$ is quasi $b-open$ if and only if $bInt[f^{-1}(B)] \subseteq f^{-1}[Int(B)]$ for every subset B of Y.*

Proof. ⇒). Let B be any subset of Y. Then, $bInt[f^{-1}(B)] \in BO(X)$ and f is *quasi* $b-open$, then $f[bInt(f^{-1}(B))] \subseteq Int[f(f^{-1}(B))] \subseteq Int(B)$. Thus , $bInt[f^{-1}(B)] \subseteq f^{-1}[Int(B)]$.

⇐). Let $U \in BO(X)$. Then by assumption $bInt[f^{-1}(f(U))] \subseteq f^{-1}[Int(f(U))]$ then $bInt(U) \subseteq f^{-1}[Int(f(U))]$, but $bInt(U) = U$ so $U \subseteq f^{-1}[Int(f(U))]$ and hence $f(U) \subseteq Int(f(U)$ so f is *quasi* $b-open$. □

Theorem 3.5. *A function $f : (X, \tau) \to (Y, \rho)$ is quasi $b-open$ if and only if for any subset B of Y and for any set $C \in BC(X)$ containing $f^{-1}(B)$, there exists a closed subset F of Y containing B such that $f^{-1}(F) \subseteq C$.*

Proof. ⇒). Let f be *quasi* $b-open$ and $B \subseteq Y$. Let $C \in BC(X)$ with $f^{-1}(B) \subseteq C$. Now, put $F = Y - f(X - C)$. It is clear that since $f^{-1}(B) \subseteq C$, $B \subseteq F$. Since f is *quasi* $b-open$, F is a closed subset of Y. Also, we have $f^{-1}(F) \subseteq C$.

⇐). Let $U \in BO(X)$ and put $B = Y - f(U)$. Then $X - U \in BC(X)$ with $f^{-1}(B) \subseteq X - U$. By assumption, there exists a closed set F of Y such that $B \subseteq F$

and $f^{-1}(F) \subseteq X - U$. Hence, we obtain $f(U) \subseteq Y - F$. On the other hand, it follows that $B \subseteq F$, $Y - F \subseteq Y - B = f(U)$. Thus, we have $f(U) = Y - F$ which is open and hence f is a *quasi* $b-$*open* function. □

Theorem 3.6. *A function* $f : (X, \tau) \to (Y, \rho)$ *is quasi* $b-$*open if and only if* $f^{-1}[Cl(B)] \subseteq bCl[f^{-1}(B)]$ *for any subset* B *of* Y*.*

Proof. $\Rightarrow$). Suppose that f is *quasi* $b-$*open*. For any subset B of Y,

$f^{-1}(B) \subseteq bCl[f^{-1}(B)]$. Therefore by Theorem 3.5, there exists a closed set F in Y such that $B \subseteq F$ and $f^{-1}(F) \subseteq bCl[f^{-1}(B)]$. Therefore, we obtain ,

$$f^{-1}[Cl(B)] \subseteq f^{-1}(F) \subseteq bCl[f^{-1}(B)].$$

$\Leftarrow$). Let $B \subseteq Y$ and $C \in BC(X)$ with $f^{-1}(B) \subseteq C$. Put $F = Cl(B)$, then we have $B \subseteq F$ and F is closed and $f^{-1}(F) \subseteq bCl[f^{-1}(B)] \subseteq C$. Then by Theorem 3.5, the function f is *quasi* $b-$*open*. □

Definition 3.7. *A function* $f : (X, \tau) \to (Y, \rho)$ *is said to be quasi* $b-$*closed if* $f(C)$ *is closed in* Y *for every* $C \in BC(X)$ *.*

Clearly, every *quasi* $b-$*closed* function is $b-$*closed*.

Theorem 3.8. *If a function* $f : (X, \tau) \to (Y, \rho)$ *is quasi* $b-$*closed then* $f^{-1}[Int(B)] \subseteq bInt[f^{-1}(B)]$ *for every subset* B *of* Y*.*

Proof. Similar to the proof of Theorem 3.4. □

Theorem 3.9. *A function* $f : (X, \tau) \to (Y, \rho)$ *is quasi* $b-$*closed if and only if for any subset* B *of* Y *and for any* $U \in BO(X)$ *containing* $f^{-1}(B)$*, there exists an open subset* V *of* Y *containing* B *such that* $f^{-1}(V) \subseteq U$*.*

Proof. Similar to the proof of Theorem 3.5. □

In a similar way used in proving Theorem 2.15, Theorem 2.17 and Theorem 2.19, we can prove the following three theorems

Theorem 3.10. *Let* $f : (X, \tau) \to (Y, \rho)$ *be a quasi* $b-$*open bijection. Then the following hold*

a) If X *is* $b - T_1$ *then* Y *is* T_1*.*
b) If X *is* $b - T_2$ *then* Y *is* T_2*.*

Theorem 3.11. *Let* $f : (X, \tau) \to (Y, \rho)$ *be a quasi* $b-$*open bijection. Then the following hold*

a) If Y *is compact, then* X *is* $b-$*compact.*
b) If Y *is Lindelöf, then* X *is* $b-$*Lindelöf.*

Theorem 3.12. *If* $f : (X, \tau) \to (Y, \rho)$ *is a quasi* $b-$*open surjection and* Y *is connected then* X *is* $b-$*connected.*

4. Strongly $b-$Open and Strongly $b-$Closed Functions

Definition 4.1. *A function $f:(X,\tau)\rightarrow(Y,\rho)$ is said to be strongly $b-open$ if $f(U)\in BO(Y)$ for every $U\in BO(X)$.*

Clearly, every *strongly $b-open$* function is *$b-open$*.

Theorem 4.2. *Let $f:(X,\tau)\rightarrow(Y,\rho)$ and $g:(Y,\rho)\rightarrow(Z,\sigma)$ be two strongly $b-open$ functions. Then the composition function $gof:(X,\tau)\rightarrow(Z,\sigma)$ is strongly $b-open$.*

Proof. Let $U\in BO(X)$. Then $f(U)\in BO(Y)$ since f is *strongly $b-open$*. But g is *strongly $b-open$* so $g(f(U))\in BO(Z)$. Hence gof is *strongly $b-open$*. □

Theorem 4.3. *A function $f:(X,\tau)\rightarrow(Y,\rho)$ is strongly $b-open$ if and only if for each $x\in X$ and for any $U\in BO(X)$ with $x\in U$, there exists $V\in BO(Y)$ such that $f(x)\in V$ and $V\subseteq f(U)$.*

Proof. It is obvious. □

Theorem 4.4. *A function $f:(X,\tau)\rightarrow(Y,\rho)$ is strongly $b-open$ if and only if for each $x\in X$ and for any $b-neighborhood$ U of x in X, there exists a $b-neighborhood$ V of $f(x)$ in Y such that $V\subseteq f(U)$.*

Proof. $\Rightarrow$). Let $x\in X$ and let U be a *$b-neighborhood$* of x. Then there exists $W\in BO(X)$ such that $x\in W\subseteq U$. Then $f(x)\in f(W)\subseteq f(U)$. But , $f(W)\in BO(Y)$ since f is *strongly $b-open$*. Hence $V=f(W)$ is a *$b-neighborhood$* of $f(x)$ and $V\subseteq f(U)$.

$\Leftarrow$). Let $U\in BO(X)$ and $x\in U$. Then U is a *$b-neighborhood$* of x. So by assumption, there exists a *$b-neighborhood$* $V_{f(x)}$ of $f(x)$ such that, $f(x)\in V_{f(x)}\subseteq f(U)$. It follows that $f(U)$ is a *$b-neighborhood$* of each of its points. Therefore, $f(U)\in BO(Y)$. Hence f is *strongly $b-open$*. □

Theorem 4.5. *A function $f:(X,\tau)\rightarrow(Y,\rho)$ is strongly $b-open$ if and only if $f[bInt(A)]\subseteq bInt[f(A)]$, for every $A\subseteq X$.*

Proof. $\Rightarrow$). Let $A\subseteq X$ and $x\in bInt(A)$. Then there exists $U_x\in BO(X)$ such that $x\in U_x\subseteq A$. So $f(x)\in f(U_x)\subseteq f(A)$ and by assumption, $f(U_x)\in BO(Y)$. Hence, $f(x)\in bInt[f(A)]$. Thus $f[bInt(A)]\subseteq bInt[f(A)]$.

$\Leftarrow$). Let $U\in BO(X)$. Then by assumption, $f[bInt(U)]\subseteq bInt[f(U)]$. Since $bInt(U)=U$ and $bInt[f(U)]\subseteq f(U)$. Hence, $f(U)=bInt[f(U)]$. Thus, $f(U)\in BO(Y)$. □

Theorem 4.6. *A function $f:(X,\tau)\rightarrow(Y,\rho)$ is strongly $b-open$ if and only if $bInt[f^{-1}(B)]\subseteq f^{-1}[bInt(B)]$, for every $B\subseteq Y$.*

Proof. $\Rightarrow$). Let $B \subseteq Y$. Since $bInt[f^{-1}(B)] \in BO(X)$ and f is *strongly* $b-open$, $f[bInt(f^{-1}(B))] \in BO(Y)$. Also we have $f[bInt(f^{-1}(B))] \subseteq f[f^{-1}(B)] \subseteq B$. Hence, $f[bInt(f^{-1}(B))] \subseteq bInt(B)$. Therefore, $bInt[f^{-1}(B)] \subseteq f^{-1}[bInt(B)]$.

$\Leftarrow$). Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by assumption, we obtain,

$$bInt(A) \subseteq bInt[f^{-1}(f(A)] \subseteq f^{-1}[bInt(f(A))].$$

This implies that,

$$f[bInt(A)] \subseteq f[f^{-1}(bInt(f(A))] \subseteq bInt[f(A)].$$

Thus, $f[bInt(A)] \subseteq bInt[f(A)]$, for all $A \subseteq X$. Hence, by Theorem 4.5, f is *strongly* $b-open$. □

Theorem 4.7. *A function* $f : (X, \tau) \to (Y, \rho)$ *is strongly* $b-open$ *if and only if* $f^{-1}[bCl(B)] \subseteq bCl[f^{-1}(B)]$, *for every* $B \subseteq Y$.

Proof. $\Rightarrow$). Let $B \subseteq Y$ and $x \in f^{-1}[bCl(B)]$. Then $f(x) \in bCl(B)$. Let $U \in BO(X)$ such that $x \in U$. By assumption, $f(U) \in BO(Y)$ and $f(x) \in f(U)$. Thus $f(U) \cap B \neq \phi$. Hence $U \cap f^{-1}(B) \neq \phi$. Therefore, $x \in bCl[f^{-1}(B)]$. So we obtain $f^{-1}[bCl(B)] \subseteq bCl[f^{-1}(B)]$.

$\Leftarrow$). Let $B \subseteq Y$. Then $Y - B \subseteq Y$. By assumption,

$$f^{-1}[bCl(Y - B)] \subseteq bCl[f^{-1}(Y - B)].$$

This implies that,

$$X - bCl[f^{-1}(Y - B)] \subseteq X - f^{-1}[bCl(Y - B)].$$

Hence,

$$X - bCl[X - f^{-1}(B)] \subseteq f^{-1}[Y - bCl(Y - B)].$$

Then, $bInt[f^{-1}(B)] \subseteq f^{-1}[bInt(B)]$. Now by Theorem 4.6, it follows that f is *strongly* $b-open$. □

Definition 4.8. [3]**.** *A function* $f : (X, \tau) \to (Y, \rho)$ *is said to be* $b-irresolute$ *if* $f^{-1}(V) \in BO(X)$ *for every* $V \in BO(Y)$.

Theorem 4.9. *Let* $f : (X, \tau) \to (Y, \rho)$ *be a function and* $g : (Y, \rho) \to (Z, \sigma)$ *be a strongly* $b-open$ *injection. If* $gof : (X, \tau) \to (Z, \sigma)$ *is* $b-irresolute$, *then* f *is* $b-irresolute$.

Proof. Let $U \in BO(Y)$. Then $g(U) \in BO(Z)$ since g is *strongly* $b-open$. Also gof is $b-irresolute$, so we have $(gof)^{-1}[g(U)] \in BO(X)$. Since g is an *injection*, we have $(gof)^{-1}[g(U)] = (f^{-1}og^{-1})[g(U)] = f^{-1}[g^{-1}(g(U))] = f^{-1}(U)$. Then, $f^{-1}(U) \in BO(X)$. So f is $b-irresolute$. □

Theorem 4.10. *Let $f : (X, \tau) \to (Y, \rho)$ be strongly $b-open$ surjection and $g : (Y, \rho) \to (Z, \sigma)$ be any function. If $gof : (X, \tau) \to (Z, \sigma)$ is $b - irresolute$, then g is $b - irresolute$.*

Proof. Let $V \in BO(Z)$. Then $(gof)^{-1}(V) \in BO(X)$ since gof is $b-irresolute$. Also f is *strongly b−open*, so $f[(gof)^{-1}(V)] \in BO(Y)$. Since f is $surjective$, we note that $f[(gof)^{-1}(V)] = [fo(gof)^{-1}](V) = [fo(f^{-1}og^{-1})](V) = [(fof^{-1})og^{-1}](V) = g^{-1}(V)$. Hence g is $b - irresolute$. □

Definition 4.11. *A function $f : (X, \tau) \to (Y, \rho)$ is said to be strongly $b - closed$ if $f(C) \in BC(Y)$ for every $C \in BC(X)$.*

The straight forward proof of the following theorem is omitted.

Theorem 4.12. *If $f : (X, \tau) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, \sigma)$ are two strongly $b - closed$ functions, then $gof : (X, \tau) \to (Z, \sigma)$ is a strongly $b - closed$ function.*

Theorem 4.13. *Let $f : (X, \tau) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, \sigma)$ be two functions such that $gof : (X, \tau) \to (Z, \sigma)$ is a strongly $b - closed$ function. Then*

1) If f is $b - irresolute$ and $surjection$ then g is strongly $b - closed$.
2) If g is $b - irresolute$ and $injection$, then f is strongly $b - closed$.

Proof. (1). Let $F \in BC(Y)$. Since f is $b-irresolute$, $f^{-1}(F) \in BC(X)$. Now gof is *strongly $b - closed$* and f is $surjection$, then $(gof)(f^{-1}(F)) = g(F) \in BC(Z)$. This implies that g is *strongly $b - closed$*.

(2). Let $C \in BC(X)$. Since gof is *strongly $b - closed$*, $(gof)(C) \in BC(Z)$.Now g is $b - irresolute$ and $injection$, so $g^{-1}[(gof)(C)] = f(C) \in BC(Y)$. This shows that f is *strongly $b - closed$*. □

Theorem 4.14. *A function $f : (X, \tau) \to (Y, \rho)$ is strongly $b - closed$ if and only if $bCl[f(A)] \subseteq f[bCl(A)]$, for every $A \subseteq X$.*

Proof. ⇒). Let f be *strongly $b - closed$* and $A \subseteq X$. Then $f[bCl(A)] \in BC(Y)$. Since $f(A) \subseteq f[bCl(A)]$, we obtain $bCl[f(A)] \subseteq f[bCl(A)]$.

⇐). Let $C \in BC(X)$. By assumption, we obtain,

$$f(C) \subseteq bCl[f(C)] \subseteq f[bCl(C)] = f(C).$$

Hence $f(C) = bCl[f(C)]$. Thus, $f(C) \in BC(Y)$. It follows that f is *strongly $b - closed$*. □

Theorem 4.15. *Let $f : (X, \tau) \to (Y, \rho)$ be a function such that $Int[Cl(f(A))] \subseteq f[bCl(A)]$ for every $A \subseteq X$. Then f is strongly $b - closed$.*

Proof. Let $C \in BC(X)$. Then by assumption we have,

$$Int[Cl(f(C)] \subseteq f[bCl(C)] = f(C).$$

Put $F = Cl[f(C)]$. Then F is closed in Y. Also it implies that $Int(F) \subseteq f(C) \subseteq F$. Hence, $f(C)$ is *semi closed* in Y. Since $SO(Y) \subseteq BO(Y)$, $f(C) \in BC(Y)$. This implies that f is *strongly* $b-$*closed*. □

Theorem 4.16. *Let $f : (X, \tau) \to (Y, \rho)$ be a strongly $b-$closed function and $B \subseteq Y$. If $U \in BO(X)$ with $f^{-1}(B) \subseteq U$, then there exists $V \in BO(Y)$ with $B \subseteq V$ such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.*

Proof. Let $V = Y - f(X-U)$. Then $Y - V = f(X-U)$. Since f is *strongly* $b-$*closed*, $V \in BO(Y)$. Since $f^{-1}(B) \subseteq U$, we have $Y-V = f(X-U) \subseteq f[f^{-1}(Y-B)] \subseteq Y-B$. Hence, $B \subseteq V$. Also $X - U \subseteq f^{-1}[f(X - U)] = f^{-1}(Y - V) = X - f^{-1}(V)$. So $f^{-1}(V) \subseteq U$. □

Theorem 4.17. *Let $f : (X, \tau) \to (Y, \rho)$ be a surjective strongly $b-$closed function and $B, C \subseteq Y$. If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $b-$neighborhoods, then so have B and C.*

Proof. Let E and F be the disjoint $b-$*neighborhood* of $f^{-1}(B)$ and $f^{-1}(C)$ respectively. Then by the last theorem There exist two sets $U, V \in BO(Y)$ with $B \subseteq U$ and $C \subseteq V$ such that $f^{-1}(B) \subseteq f^{-1}(U) \subseteq bInt(E)$ and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq bInt(F)$. Since E and F are disjoint, so are $bInt(E)$ and $bInt(F)$, and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too since f is a *surjective* function. □

Theorem 4.18. *A surjective function $f : (X, \tau) \to (Y, \rho)$ is strongly $b-$closed if and only if for each subset B of Y and each set $U \in BO(X)$ containing $f^{-1}(B)$, there exists a set $V \in BO(Y)$ containing B, such that $f^{-1}(V) \subseteq U$.*

Proof. $\Rightarrow$). This follows from Theorem 4.16.

$\Leftarrow$). Let $C \in BC(X)$ and $y \in Y - f(C)$. Then $f^{-1}(y) \subseteq X - f^{-1}(f(C)) \subseteq X - C$ and $X-C \in BO(X)$. Hence by assumption, there exists a set $V_y \in BO(Y)$ containing y such that $f^{-1}(V_y) \subseteq X - C$. This implies that $y \in V_y \subseteq Y - f(C)$. Thus, $Y - f(C) = \cup\{V_y : y \in Y - f(C)\}$. Hence, $Y - f(C) \in BO(Y)$. Therefore, $f(C) \in BC(Y)$. □

Theorem 4.19. *Let $f : (X, \tau) \to (Y, \rho)$ be a bijection. Then the following are equivalent:*

1) f is strongly $b-$closed.
2) f is strongly $b-$open.
3) f^{-1} is $b-$irresolute.

Proof. (1) $\rightarrow$ (2). Let $U \in BO(X)$. Then $X - U \in BC(X)$. By (1), $f(X-U) \in BC(Y)$. But $f(X-U) = f(X)-f(U) = Y-f(U)$. Thus $f(U) \in BO(Y)$.

(2) $\rightarrow$ (3). Let $A \subseteq X$. Since f is *strongly* $b-open$, so by Theorem 4.7, $f^{-1}[bCl(f(A))] \subseteq bCl[f^{-1}(f(A))]$. It implies that $bCl[f(A)] \subseteq f[bCl(A)]$. Thus $bCl[(f^{-1})^{-1}(A)] \subseteq (f^{-1})^{-1}[bCl(A)]$, for all $A \subseteq X$. Then, it follows that f^{-1} is $b-irresolute$.

(3) $\rightarrow$ (1). Let $C \in BC(X)$. Then $X - C \in BO(X)$. Since f^{-1} is $b-irresolute$, $(f^{-1})^{-1}(X-C) \in BO(Y)$. But $(f^{-1})^{-1}(X-C) = f(X-C) = Y - f(C)$. Thus $f(C) \in BC(Y)$. □

In a similar way used in proving Theorem 2.15, Theorem 2.17 and Theorem 2.19 we can prove the following three theorems

Theorem 4.20. *Let $f : (X,\tau) \rightarrow (Y,\rho)$ be a strongly $b-open$ bijection. Then the following hold*

a) If X is $b-T_1$ then Y is $b-T_1$.
b) If X is $b-T_2$ then Y is $b-T_2$.

Theorem 4.21. *Let $f : (X,\tau) \rightarrow (Y,\rho)$ be a strongly $b-open$ bijection. Then the following hold*

a) If Y is $b-compact$, then X is $b-compact$.
b) If Y is $b-Lindelöf$, then X is $b-Lindelöf$.

Theorem 4.22. *If $f : (X,\tau) \rightarrow (Y,\rho)$ is a strongly $b-open$ surjection and Y is $b-connected$ then X is $b-connected$.*

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