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ON FRACTIONAL DIFFERENTIABLE s–CONVEX FUNCTIONS

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Abstract. In this paper some properties of s–convex functions are considered. A combination between local fractional α–derivative and s–convexity are introduced and investigated.

1. Introduction

In [4], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense. This class is defined in the following way: a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{1.1}$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2. It is convenient to mention that, Hudzik and Maligranda (see [4]), proved that the functions in K_s^2 are nonnegative. Also, it can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [3], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense.

Theorem 1.1. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s–convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[0, 1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2). The above inequalities are sharp.

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In literature, for a continuous function f on (a,b) and for all $x \in [a,b]$, $\alpha \in \mathbb{R}_+$, the left (respectively right) *Riemann–Liouville integral* at the point x is defined by

$$I_{a,-}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt,$$

$$I_{b,+}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt.$$

The left (respectively right) Riemann–Liouville derivative at x is given by

$$D_{a,-}^{\alpha}(f)(x) = \frac{d}{dx} I_{a,-}^{1-\alpha}(f)(x),$$

$$D_{b,+}^{\alpha}(f)(x) = \frac{d}{dx} I_{b,+}^{1-\alpha}(f)(x).$$

Therefore, the function f admits a fractional derivative of order α, $0 < \alpha < 1$ by below (above) if $D_{-}^{\alpha}(f)(x)$ exists (if $D_{+}^{\alpha}(f)(x)$ exists).

In [1], Adda and Cresson, have introduced a local fractional derivative as follows:

Definition 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, f is said to have right (resp. left) local fractional derivative of order $\alpha \in (0,1)$ at $y \in [a,b]$ if*

$$d_{\sigma}^{\alpha} f(x) = \lim_{x \to y^{\sigma}} D_{y,-\sigma}^{\alpha} \left[\sigma\left(f(x) - f(y)\right)\right],$$

for $\sigma = \pm$, respectively.

One can deduce the following properties for f.

(1) If f is differentiable at x, we have

$$\lim_{\alpha \to 1} d_{\sigma}^{\alpha} f(x) = f'(x), \ \sigma = \pm.$$

(2) We have $d_{\sigma}^{\alpha}(C) = 0$, for all $C \in \mathbb{R}$ and $\sigma = \pm$.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the right (resp. left) local fractional derivative $d_{\sigma}^{\alpha} f(x)$, $0 < \alpha < 1$ at $y \in [a,b]$ is given by*

$$d_{\sigma}^{\alpha} f(x) = \Gamma(1+\alpha) \lim_{y \to x^{\sigma}} \frac{\sigma\left(f(y) - f(x)\right)}{|y-x|^{\alpha}}.$$

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $d_{\sigma}^{\alpha} f(x)$ exists for $\alpha > 0$, $\sigma = \pm$, then*

$$f(x) = f(y) + \sigma \frac{d_{\sigma}^{\alpha} f(x)}{\Gamma(1+\alpha)} \left[\sigma(y-x)\right]^{\alpha} + R_{\sigma}(x,y),$$

with

$$R_{\sigma}(x,y) = \sigma \frac{1}{\Gamma(1+\alpha)} \int_0^{x-y} \frac{d}{dt} F_{\sigma}(y, \sigma t, \alpha) \left(\sigma(x-y-t)\right)^{\alpha} dt,$$

and

$$\lim_{x\to y^\sigma}\frac{R_\sigma(x,y)}{(\sigma(x-y))^\alpha}=0,$$

where,

$$F_\sigma(y,\sigma(x-y),\alpha)=D^\alpha_{y,-\sigma}\left[\sigma(f-f(y))\right](x).$$

Also, in [1], the notion of a local α–derivative is introduced as follows:

Definition 1.5. *Let I be an open interval of $\mathbb{R}$, $\alpha\in(0,1]$ and let f be a function on I. Then, f is said to have a right (resp. left) locally α–derivative at $t_0\in I$ iff the function $t\to\frac{f(t)-f(t_0)}{\sigma(\sigma(t-t_0)^\alpha)}$, $\sigma=+$ (resp. $\sigma=-$), admits, a limit in $\mathbb{R}$ when $t\to t_0^\sigma$.*

In general, the α–right or α–left local fractional derivative may not exist. However, the following quantities are always defined:

$$\begin{aligned}\overline{\lim_{x\to x_0^+}}\frac{f(x)-f(x_0)}{(x-x_0)^\alpha}&=\Lambda^\alpha_+(x_0),\\ \underline{\lim_{x\to x_0^+}}\frac{f(x)-f(x_0)}{(x-x_0)^\alpha}&=\lambda^\alpha_+(x_0),\\ \overline{\lim_{x\to x_0^-}}\frac{f(x)-f(x_0)}{-(-(x-x_0))^\alpha}&=\Lambda^\alpha_-(x_0),\\ \underline{\lim_{x\to x_0^-}}\frac{f(x)-f(x_0)}{-(-(x-x_0))^\alpha}&=\lambda^\alpha_-(x_0).\end{aligned}$$

If $\Lambda^\alpha_+(x_0)$ and $\lambda^\alpha_+(x_0)$ are finite and equal, then they are equal to the α–right local derivative at x_0. Similarly, if $\Lambda^\alpha_-(x_0)$ and $\lambda^\alpha_-(x_0)$ are finite and equal, then they are equal to the α–left local derivative at x_0.

Let us assume that $\lim\limits_{t\to x^\sigma}u_x(t)=d^\alpha_\sigma f(x)$, then $\lim\limits_{t\to x^\sigma}\{u_x(t)-d^\alpha_\sigma f(x)\}=0$. Set $\lim\limits_{t\to x^\sigma}u_x(t)=0$. Then,

$$u_x(t)=d^\alpha_\sigma f(x)-\frac{f(t)-f(x)}{\sigma(\sigma(t-x))^\alpha},\qquad t\neq x.$$

and we write

$$\frac{f(t)-f(x)}{\sigma(\sigma(t-x))^\alpha}=d^\alpha_\sigma f(x)-u_x(t)$$

which is equivalent to

$$f(t)=f(x)+\sigma(\sigma(t-x))^\alpha\left[d^\alpha_\sigma f(x)-u_x(t)\right].$$

Simply, we show that f is right (resp. left) α–differentiable at x, if $\lim\limits_{t\to x^\sigma}u_x(t)$; exists.

In this paper we study some properties of s–convex functions, and we give a combination between the local fractional α–derivative and s–convexity for some function f defined on real interval.

2. Some Properties of s–Convex Functions

We begin with the following theorem, see also [2]:

Theorem 2.1. *Let f be an s–convex function on (a,b) and let $x_i \in (a,b)$, $i = 1,2,3,....$ If $\alpha_i > 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, then*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i^s f\left(x_i\right). \tag{2.1}$$

Proof. Let $x_i \in (a,b)$, $\alpha_i > 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, for all $i = 1,2,3,....$ The proof will be done by induction. For $n = 2$, the result holds by the assumptions, since f is s–convex. Suppose that (2.1) holds for $n = k$, that is

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum_{i=1}^{k} \alpha_i^s f\left(x_i\right). \tag{2.2}$$

We want to show that (2.2) is true for $n = k+1$. Therefore, by induction we have,

$$\begin{aligned} f\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) &= f\left(\sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}\right) \\ &\leq \sum_{i=1}^{k} \alpha_i^s f\left(x_i\right) + \alpha_{k+1}^s f\left(x_{k+1}\right) \\ &= \sum_{i=1}^{k+1} \alpha_i^s f\left(x_i\right), \end{aligned}$$

which is required. □

Theorem 2.2. *Fix $s \in (0,1]$. Let f be an s–convex function on the open (a,b) and let $x(t) : [c,d] \to \mathbb{R}^+$ be integrable with $a < x(t) < b$. If $\alpha(t) : [c,d] \to \mathbb{R}^+$ is positive, $\int_c^d \alpha\left(t\right) dt = 1$, and $\alpha x(t)$ is integrable on $[c,d]$, then*

$$f\left(\int_c^d \alpha\left(t\right) x\left(t\right) dt\right) \leq \int_c^d \alpha^s\left(t\right) f\left(x\left(t\right)\right) dt. \tag{2.3}$$

Proof. The proof follows from the discrete version (2.1) by considering Riemann sums. The details are left to the interested reader. □

Theorem 2.3. *Let $f:[a,b]\rightarrow\mathbb{R}_{+}$, be an s–convex function in $[a,b]$, then, for all distinct $x_1,x_2,x_3\in[a,b]$, such that $x_1<x_2<x_3$, the following inequality*

$$f\left(x_2\right)\left(x_3-x_1\right)^s\leq\left(x_3-x_2\right)^s f\left(x_1\right)+\left(x_2-x_1\right)^s f\left(x_3\right), \tag{2.4}$$

holds, for all $s\in(0,1]$.

Proof. Let x_1,x_2,x_3 be a distinct points in $[a,b]$. Setting $\lambda=\frac{x_3-x_2}{x_3-x_1}$, $x_2=\lambda x_1+\left(1-\lambda\right)x_3$, we have,

$$\begin{aligned}f\left(x_2\right) &= f\left(\lambda x_1+\left(1-\lambda\right)x_3\right)\\ &\leq \lambda^s f\left(x_1\right)+\left(1-\lambda\right)^s f\left(x_3\right)\\ &= \left(\frac{x_3-x_2}{x_3-x_1}\right)^s f\left(x_1\right)+\left(\frac{x_2-x_1}{x_3-x_1}\right)^s f\left(x_3\right),\end{aligned}$$

which gives the required result. □

Theorem 2.4. *If $f:[a,b]\rightarrow\mathbb{R}_{+}$ is s–convex, and $a<t<\frac{u+t}{2}\leq r<u<b$, then,*

$$\frac{f\left(r\right)-f\left(t\right)}{\left(r-t\right)^s}\leq\frac{f\left(u\right)-f\left(t\right)}{\left(u-t\right)^s}, \tag{2.5}$$

for all $s\in(0,1]$.

Proof. Suppose that f is s–convex. Let $a<t<\frac{u+t}{2}\leq r<u<b$, set $\lambda=\frac{r-t}{u-t}$ and $r=\lambda u+\left(1-\lambda\right)t$, then we have

$$\begin{aligned}f\left(r\right)=f\left(\lambda u+\left(1-\lambda\right)t\right) &\leq \lambda^s f\left(u\right)+\left(1-\lambda\right)^s f\left(t\right)\\ &\leq \left(\frac{r-t}{u-t}\right)^s f\left(u\right)+\left[1-\left(\frac{r-t}{u-t}\right)\right]^s f\left(t\right)\\ &= \left(\frac{r-t}{u-t}\right)^s f\left(u\right)+\left(\frac{u-r}{u-t}\right)^s f\left(t\right)\end{aligned}$$

However, $\frac{u+t}{2}\leq r$, which is equivalent to write, $r-t\geq u-r$, and this implies that $\frac{r-t}{u-t}\geq\frac{u-r}{u-t}$, for all $t<\frac{u+t}{2}\leq r<u$, therefore the inequalities

$$\begin{aligned}\left(\frac{r-t}{u-t}\right)^s f\left(u\right)+\left(\frac{u-r}{u-t}\right)^s f\left(t\right) &\leq \left(\frac{r-t}{u-t}\right)^s\left(f\left(u\right)-f\left(t\right)\right)\\ &\leq \left(\frac{r-t}{u-t}\right)^s\left(f\left(u\right)-f\left(t\right)\right)+f\left(t\right),\end{aligned}$$

hold, since f is nonnegative. Thus,

$$f\left(r\right)\leq\left(\frac{r-t}{u-t}\right)^s\left(f\left(u\right)-f\left(t\right)\right)+f\left(t\right),$$

and we write,

$$f(r) - f(t) \leq \left(\frac{r-t}{u-t}\right)^s (f(u) - f(t)),$$

hence,

$$\frac{f(r) - f(t)}{(r-t)^s} \leq \frac{f(u) - f(t)}{(u-t)^s},$$

which holds if f is s–convex, and the proof is completed. □

3. Fractional Derivatives and s–Convexity

In Definition 1.5, we assumed that f has an α–derivative of order α if $d_\sigma^\alpha f$ exists and $d_-^\alpha f = d_+^\alpha f$. We denote the α–derivative of f by $d^\alpha f$.

Lemma 3.1. *If $f : I \to \mathbb{R}^+$ is an s–convex function, then f is s–Hölder $(0 < s < 1)$ on any compact interval $[a, b] \subseteq I^\circ$.*

Proof. By Theorem 2.4, we have

$$d_+^s f(a) \leq d_+^s f(x) \leq \frac{f(y) - f(x)}{(y-x)^s} \leq d_-^s f(x) \leq d_-^s f(b),$$

for all $x, y \in [a, b]$ with $x < y$, hence f verifies the Hölder conditions with $H = \frac{1}{\Gamma(1+s)} \max\left\{\left|d_+^s f(a)\right|, \left|d_-^s f(b)\right|\right\}$. □

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}^+$, be an s–convex function, then f is s–Hölder on $I^\circ := (a, b)$ and $d_-^s(f)(x)$ and $d_+^s(f)(x)$ exist and are finite at each point in I°.*

Proof. According to Theorem 2.4 and Lemma 3.1, we have

$$\frac{f(x) - f(a)}{(x-a)^s} \leq \frac{f(y) - f(a)}{(y-a)^s} \leq \frac{f(z) - f(a)}{(z-a)^s}$$

for all $x \leq y < a < z \in I$. It follows that

$$d_-^s f(a) \leq \frac{f(z) - f(a)}{(z-a)^s}.$$

A symmetric argument will then yield the existence of $d_+^s f(a)$ and the availability of the relation $d_-^s f(a) \leq d_+^s f(a)$. On the other hand, starting with $x < u \leq v < y \in I^\circ$, Theorem 2.4 and Lemma 3.1 yield

$$\frac{f(u) - f(x)}{(u-x)^s} \leq \frac{f(v) - f(x)}{(v-x)^s} \leq \frac{f(y) - f(v)}{(y-v)^s}.$$

Since f admits finite s–derivatives at each interior point, it will be s-Hölder continuous at each interior point. □

Theorem 3.3. *A function $f : (a,b) \to \mathbb{R}^+$ is s–convex iff there is an increasing function $g : (a,b) \to \mathbb{R}^+$ and a point $c \in (a,b)$ such that for all $x \in (a,b)$,*

$$f(x) - f(c) = \int_c^x g(t)\,dt. \tag{3.1}$$

Proof. ($\Rightarrow$) Suppose that f is s–convex. Choose $g = d_+^s f$, which exists and is increasing (follows by Theorems 2.4, 3.2) and let $c \in (a,b)$, then f is absolutely continuous on $[c,x]$. By elementary calculus

$$f(x) - f(c) = \int_c^x d_+^s f(t)\,dt = \int_c^x g(t)\,dt.$$

($\Leftarrow$) Conversely, suppose that (3.1) holds with g increasing. Let α, β be positive with $\alpha + \beta = 1$. Then for $x < y$ in (a,b),

$$\begin{aligned}
& \alpha^s f(x) + \beta^s f(y) - f(\alpha x + \beta y) \\
= \;& \alpha^s f(x) + \beta^s f(y) - (\alpha + \beta) f(\alpha x + \beta y) \\
= \;& \alpha^s f(x) - \alpha f(\alpha x + \beta y) + \beta^s f(y) - \beta f(\alpha x + \beta y) \\
\geq \;& \alpha f(x) - \alpha f(\alpha x + \beta y) + \beta f(y) - \beta f(\alpha x + \beta y) \\
= \;& \beta \int_{\alpha x + \beta y}^{y} g(t)\,dt - \alpha \int_x^{\alpha x + \beta y} g(t)\,dt.
\end{aligned}$$

To bound the last expression, since g is increasing, we simply replace both integrands by the constant $g(\alpha x + \beta y)$, this being the smallest value of the first integrand and the largest of the second. Thus,

$$\begin{aligned}
& \alpha^s f(x) + \beta^s f(y) - f(\alpha x + \beta y) \\
\geq \;& \beta \int_{\alpha x + \beta y}^{y} g(t)\,dt - \alpha \int_x^{\alpha x + \beta y} g(t)\,dt \\
\geq \;& \beta g(\alpha x + \beta y)\left[y - (\alpha x + \beta y)\right] - \alpha g(\alpha x + \beta y)(\alpha x + \beta y - x) \geq 0.
\end{aligned}$$

which is equivalent to the inequality that defines s-convexity. □

Theorem 3.4. *Let $\alpha_n = \frac{1}{2n+1}$, $n \in \mathbb{N}$. If $f : [a,b] \to \mathbb{R}^+$, is locally α_n–differentiable on (a,b), and has a local maximum (minimum) at x_0 then $d_\sigma^{\alpha_n}(f)(x_0) = 0$, $\sigma = \pm$.*

Proof. Without loss of generality, assume that f has a local maximum at x_0. Then, there exists a $\delta > 0$ such that $f(x_0) \geq f(x)$, $\forall x \in (x_0 - \delta, x_0 + \delta)$. If $x \in (x_0, x_0 + \delta)$, then $x - x_0 > 0 \Rightarrow (x - x_0)^{\alpha_n} > 0$ and $f(x) - f(x_0) < 0$, which means that

$$d_+^{\alpha_n} f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{\alpha_n}} \leq 0, \qquad x \in (x_0, x_0 + \delta). \tag{3.2}$$

If $x \in (x_0 - \delta, x_0)$, then $x - x_0 < 0 \Rightarrow (x - x_0)^{\alpha_n} < 0$ and $f(x) - f(x_0) > 0$, which means that

$$d_{+}^{\alpha_n} f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{\alpha_n}} \geq 0, \qquad x \in (x_0 - \delta, x_0). \tag{3.3}$$

Therefore, by (3.2) and (3.3) we have $d_{+}^{\alpha_n}(f)(x_0) = 0$, for all $n = 1, 2, 3, \ldots$. The proof where f has a local minimum at x_0 goes likewise. □

Theorem 3.5. *Let $\alpha_n = \frac{1}{2n+1}$, $n \in \mathbb{N}$. If $f : [a,b] \to \mathbb{R}^+$ is a continuous function, then there exists a point $c \in (a,b)$, such that*

$$\underline{d_{\sigma}^{\alpha_n}} f(c) \leq \frac{f(b) - f(a)}{\sigma(\sigma(b-a)^{\alpha_n})} \leq \overline{d_{\sigma}^{\alpha_n}} f(c). \tag{3.4}$$

Here,

$$\underline{d_{\sigma}^{\alpha_n}} f(x) = \liminf_{x \to c} \frac{f(x) - f(c)}{\sigma(\sigma(x-c)^{\alpha_n})}, \quad \text{and} \quad \overline{d_{\sigma}^{\alpha_n}} f(x) = \limsup_{x \to c} \frac{f(x) - f(c)}{\sigma(\sigma(x-c)^{\alpha_n})},$$

are respectively the lower derivative and the upper derivative of f at c, $\sigma = \pm$.

Proof. As in the smooth case, and since f is continuous, we consider the function

$$F(x) = f(x) - \frac{f(b) - f(a)}{(b-a)^{\alpha_n}} (x-a)^{\alpha_n}, \quad x \in [a,b].$$

Clearly, F is continuous and $F(a) = F(b)$. If F attains its supremum at $c \in (a,b)$, then $\underline{d_{\sigma}^{\alpha_n}} f(c) \leq 0 \leq \overline{d_{\sigma}^{\alpha_n}} f(c)$, and the conclusion is immediate. The same is true when F attains its infimum at an interior point of $[a,b]$. If both extremes are attained at the endpoints, then F is constant and the conclusion works for all $c \in (a,b)$. □

Theorem 3.6. *Let $0 < s < 1$. Suppose that f is s–differentiable on (a,b). Then, f is s–convex iff $d^s f$ is increasing.*

Proof. ($\Rightarrow$) done by Theorem 3.3.

($\Leftarrow$) Suppose that $d^s f(x)$ is increasing, then, the fundamental theorem of calculus assures that

$$f(x) - f(c) = \int_c^x d^s f(t)\, dt,$$

for any $c \in (a,b)$. It follows that f is s–convex. □

Definition 3.7. *Let $0 < s < 1$. We say that a function f defined on $[a,b]$ has a fractional support of order s at x_0 if there exists a function S of the form $S_{\sigma}(x) = f(x_0) + \sigma m_s (\sigma(x - x_0))^s$, such that $S_{\sigma}(x) \leq f(x)$ for every $x \in [a,b]$, where $m_s = d^s f$.*

Theorem 3.8. *A function $f : (a,b) \to \mathbb{R}^+$, is s–convex iff there is at least one fractional support of order s at each $x_0 \in (a,b)$.*

Proof. If f is s–convex then by Theorem 3.2, $d_{-}^{s}f(x), d_{+}^{s}f(x)$ exist.
Let $p = \min\left\{d_{-}^{s}f(x), d_{+}^{s}f(x)\right\}$, and $P = \max\left\{d_{-}^{s}f(x), d_{+}^{s}f(x)\right\}$.
For $x_0 \in (a,b)$, choose $m_s \in [p,P]$. Then

$$\frac{f(x) - f(x_0)}{\sigma\left(\sigma\left(x - x_0\right)\right)^s} \geq m_s \ \ (\ \leq m_s\ )$$

as $x > x_0$ (or $x < x_0$). In either case, $f(x) - f(x_0) \geq \sigma m_s \left(\sigma\left(x - x_0\right)\right)^s$, that is, $f(x) \geq f(x_0) + \sigma m_s \left(\sigma\left(x - x_0\right)\right)^s$. Conversely, suppose that f has fractional support of order s at each point of (a,b). Let $x, y \in (a,b)$. For $x_0 = \lambda x + (1-\lambda)y$, $\lambda \in [0,1]$, let $S_\sigma(x) = f(x_0) + \sigma m_s \left(\sigma(x - x_0)\right)^s$ be the fractional support for f at x_0. Then

$$f(x_0) = S_\sigma(x_0) = \lambda^s S_\sigma(x) + (1-\lambda)^s S_\sigma(y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

as desired. □

Corollary 3.9. *If $f : (a,b) \to \mathbb{R}^+$, is an s–convex function, then for all $x, y \in [a,b]$, we have*

$$f(x) - f(y) \geq m_s \left(\sigma(x-y)\right)^s. \tag{3.5}$$

Proof. Follows directly from Definition 3.7.

Theorem 3.10. *Let $f : (a,b) \to \mathbb{R}^+$, be an s–convex function. Then f is s–differentiable at x_0 iff the s–fractional support for f at x_0 is unique. Moreover, $S(x) = f(x_0) + \sigma m \left(\sigma(x - x_0)\right)^s$ provides this unique fractional support.*

Proof. It is clear from the proof of Theorem 3.8 that corresponding to each $m_s \in [p,P]$, there is a fractional support of order s for f at x_0. Uniqueness of the fractional support means $d_{-}^{s}f(x), d_{+}^{s}f(x)$; that is $d^s f(x)$ exists. Any fractional of support $S_\sigma(x) = f(x_0) + \sigma m_s \left(\sigma\left(x - x_0\right)\right)^s$, gives that $f(x) - f(x_0) \geq \sigma m_s \left(\sigma\left(x - x_0\right)\right)^s$. For $x_1 < x_0 < x_2$, we have

$$\frac{f(x_1) - f(x_0)}{\sigma\left(\sigma\left(x_1 - x_0\right)\right)^s} \leq m_s \leq \frac{f(x_2) - f(x_0)}{\sigma\left(\sigma\left(x_2 - x_0\right)\right)^s}.$$

Taking the limit as $x_1 \to x_0^-$ and $x_2 \to x_0^+$, gives $d_{-}^{s}f(x) \leq m_s \leq d_{+}^{s}f(x)$, so s–differentiability of f at x_0 implies uniqueness of m_s, hence the support $S = S_\sigma$ at x_0. □

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