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ON 2-λ-NUCLEARITY MAPS

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Abstract. In this paper we give an answer for the following problem: Is 2-quasi-$\lambda(P)$-nuclear maps between normed spaces equivalent to quasi-$\lambda(P)$-nuclear maps for a non nuclear G_∞-space $\lambda(P)$? [3]. Also, we define the k-Köthe space $\lambda^{(k)}$, and we prove that if $\lambda^{(k)}$ is generated from power sets of infinite type $P_1, P_2, \ldots, P_k$ and $\lambda^{(k)}$ is nuclear, then 2-quasi-$\lambda^{(k)}$-nuclear maps between normed spaces is equivalent to quasi-$\lambda^{(k)}$-nuclear maps.

1. Introduction

The area of the sequence space is one of the important areas in mathematics. In some sense, a Banach space with fixed base is equivalent to a sequence space for this reason many mathematicians are interested in this area. An example of sequence space is the Köthe space. The first pioneer worker in Köthe space is G. Köthe infact he is the first mathematician who introduced the notion of Köthe space. After that many authors worked in Köthe spaces such as T. Terzioğlu who introduced the notions of G_1 and G_∞ spaces and he established their nuclearity criterion [4]. While E. Dubinsky and M. S. Ramanujan in [1] studied λ-nuclearity and quasi-λ nuclear maps between Normed spaces. In [3], W. Shatanawi introduced the notion of 2-quasi-λ-nuclear maps between normed spaces and he studies the properties of 2-quasi-λ-nuclear maps.

2. Basic Concepts

A set P of sequences of non-negative real numbers is called a **Köthe set**, if it satisfies the following conditions:

(1) For each pair of elements $a, b \in P$ there is $c \in P$ with $a_n = O(c_n)$ and $b_n = O(c_n)$, where $a_n = O(b_n)$ means that there is a constant $\rho > 0$ such that $a_n \leq \rho b_n$ for all $n \in \mathbf{N}$.

(2) For every integer $r \in N$ there exists $a \in P$ with $a_r > 0$.

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The space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n| \, a_n < +\infty$$

for all $a \in P$, is called the **Köthe space**, $\lambda(P)$, generated by P.

Theorem 2.1. (Grothendieck-Pietsch criterion for nuclearity) *A Köthe space $\lambda(A)$ is nuclear if and only if for every $a \in A$, there is $b \in A$ such that*

$$\left(\frac{a_n}{b_n}\right) \in \ell_1,$$

where $\ell_1 = \{(x_n) \colon \sum_n |x_n| < \infty\}$.

A Köthe set P will be called a **power set of infinite type** if it satisfies the following conditions:

(1) For each $a \in P$, $1 \leq a_n \leq a_{n+1}$ for all n.
(2) For each $a \in P$, there exists $b \in P$ such that $a_n^2 = O(b_n)$.

A Köthe space of the form $\lambda(P)$ where P is a power set of infinite type is called a G_∞**-space** or a **smooth sequence space of infinite type** [4].

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then $P_\infty = \{(k^{\alpha_n}) \colon k \in \mathbf{N}\}$ is countable Köthe set. The corresponding Köthe space $\Lambda_\infty(\alpha) = \lambda(P_\infty)$ is called the **power series of infinite type**.

By the sequence e_n, we mean the sequence whose nth term is 1 and all other terms are 0.

Definition 2.2. *A linear map T of a normed space E into a normed space F is called a* **quasi-λ-nuclear map** *if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E'; the dual space of E, such that*

$$||Tx|| \leq \sum\nolimits_n |\alpha_n| \, |\langle x, a_n \rangle|,$$

for all x in E [1].

Definition 2.3. *A linear map T of a normed space E into a normed space F is called a* **2-quasi-λ-nuclear map** *if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E'; the dual space of E, such that*

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n| \, |\langle x, a_n \rangle|^2\right)^{1/2},$$

for all x in E [3].

3. **Main results**

In [3], W. Shatanawi proved that: For a nuclear G_∞-space $\lambda(P)$, a bounded linear map between normed spaces is a quasi-$\lambda(P)$-nuclear map if and only if it is a 2-quasi-$\lambda(P)$-nuclear map. He asked the following question:
Is the above result still valid for any G_∞-space $\lambda(P)$ which is not nuclear?

In this section, we introduced an example to show that the nuclearity of G_∞-space is essential in the above result.

Let $P_0 = \{(1,1,1,\ldots),(2,2,2,\ldots),\ldots\}$. Then P_0 is a power set of infinity type. Infact one can easily show that $\lambda(P_0)$ is a G_∞-space. By using Grothendieck-Pietsch criterion for nuclearity one can see that $\lambda(P_0)$ is not nuclear.

Example **3.1.** *Define a map* $T : c_0 \to \ell_2$ *by*

$$Tx = \left(\frac{x_n}{\sqrt[3]{n^2}}\right).$$

Then T *is a 2-quasi-*$\lambda(P_0)$*-nuclear map which is not quasi-*$\lambda(P_0)$*-nuclear.*

Proof. To Show that T is a 2-quasi-$\lambda(P_0)$-nuclear map,

$$\begin{aligned} ||Tx||_2^2 &= \left|\left|\left(\frac{x_n}{\sqrt[3]{n^2}}\right)\right|\right|_2^2 \\ &= \sum_{n=1}^{\infty} \frac{|x_n|^2}{n^{\frac{4}{3}}} \\ &= \sum_{n=1}^{\infty} \frac{|\langle x, e_n\rangle|^2}{n^{\frac{4}{3}}}. \end{aligned}$$

Let $\alpha_n = \frac{1}{n^{\frac{4}{3}}}$. Then

$$||Tx||_2^2 = \sum_n |\alpha_n||\langle x, e_n\rangle|^2.$$

Since (e_n) is a bounded sequence in $c_0' = \ell_1$ and the sequence $(\alpha_n) \in \lambda(P_0)$, we conclude that T is a 2-quasi-$\lambda(P_0)$-nuclear map. To prove that T is not a quasi-$\lambda(P_0)$-nuclear map, we assume the contrary; that is, T is a quasi-$\lambda(P_0)$-nuclear map. Then there is a bounded sequence (a_n) in $c_0' = \ell_1$ and a sequence (α_n) in $\lambda(P_0)$ such that for $x \in c_0$, we have

$$||Tx||_2 \le \sum_{n=1}^{\infty} |\alpha_n||\langle x, a_n\rangle|.$$

Since $a_n \in \ell_1$, then a_n has the form

$$a_n = \left(a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, \ldots\right).$$

Given $k \in \mathbf{N}$. Since $e_k \in c_0$, we have

$$||Te_k||_2 \leq \sum_{n=1}^{\infty} |\alpha_n||\langle e_k, a_n\rangle|.$$

Therefore

$$\frac{1}{\sqrt[3]{k^2}} \leq \sum_{n=1}^{\infty} |\alpha_n||a_n^{(k)}|.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} \leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n||a_n^{(k)}|.$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} \leq \sum_{n=1}^{\infty} \left(|\alpha_n| \sum_{k=1}^{\infty} |a_n^{(k)}| \right).$$

Since

$$||a_n||_1 = \sum_{k=1}^{\infty} \left|a_n^{(k)}\right|$$

and (a_n) is a bounded sequence in ℓ_1, we have

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} \leq \sup\{||a_n|| \ : \ n \in \mathbf{N}\} \sum_{n=1}^{\infty} |\alpha_n|.$$

The last inequality implies that $(\alpha_n) \notin \lambda(P_0)$, which is a contradiction. So T is not quasi-$\lambda(P_0)$-nuclear.

□

Let $P_1, P_2, \ldots, P_k$ be Köthe sets. We define the k-Köthe set $P^{(k)}$ by

$$P^{(k)} = \left\{ \begin{bmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & \cdots \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ a_1^{(k)} & a_2^{(k)} & a_3^{(k)} & \cdots \end{bmatrix} : \ (a_n^{(i)})_{n=1}^{\infty} \in P_i \ \text{ for } \ i = 1, 2, \ldots, k \right\},$$

and the corresponding k-Köthe space $\lambda^{(k)}$ by

$$\lambda^{(k)} = \left\{ \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \cdots \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ x_1^{(k)} & x_2^{(k)} & x_3^{(k)} & \cdots \end{bmatrix} : (x_n^{(i)})_{n=1}^{\infty} \in \lambda(P_i) \, \text{for} \, i = 1, 2, \ldots, k \right\}.$$

One can see the following lemma from Grothendieck-Pietsch criterion for nuclearity.

Lemma 3.2. *Let $P_1, P_2, \ldots, P_k$ be Köthe sets.Then the k-Köthe space $\lambda^{(k)}$ is nuclear if and only if $\lambda(P_i)$ is nuclear for all $i = 1, 2, \ldots, k$.*

Our next result indicates the relationship between 2-quasi-λ-nuclear maps and quasi-λ-nuclear maps in case λ stands to a fixed k-Köthe space $\lambda^{(k)}$.

By $x = (x_n) \in \lambda^{(k)}$, we mean $x = (x_n)$ is an element in $\lambda^{(k)}$ of the form

$$x = (x_n) = \begin{pmatrix} x_1 & x_{1+k} & x_{1+2k} & \cdots \\ x_2 & x_{2+k} & x_{2+2k} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ x_k & x_{2k} & x_{3k} & \cdots \end{pmatrix}.$$

Theorem 3.3. *Suppose that $P_1, P_2, \ldots, P_k$ be power sets of infinite type. If the corresponding k-Köthe space $\lambda^{(k)}$ is nuclear, then a bounded linear map between normed spaces E and F is a quasi-$\lambda^{(k)}$-nuclear map if and only if it is a 2-quasi-$\lambda^{(k)}$-nuclear map.*

Proof. ($\Longrightarrow$) Follows from Theorem 2.1 [3].

($\Longleftarrow$) Assume that T is a 2-quasi-$\lambda^{(k)}$-nuclear map from a normed space E into a normed space F. Then there exist sequence (α_n) in $\lambda^{(k)}$ and sequence (a_n) in E' such that

$$||Tx||^2 \leq \sum_n |\alpha_n||\langle x, a_n\rangle|^2,$$

and hence

$$||Tx|| \leq \sum_n \sqrt{|\alpha_n|}|\langle x, a_n\rangle|.$$

So it is enough to show that $(\sqrt{|\alpha_n|})$ is a sequence in $\lambda^{(k)}$. Since $\alpha_n \in \lambda^{(k)}$, then α_n has the form

$$\alpha_n = \begin{pmatrix} \alpha_1 & \alpha_{1+k} & \alpha_{1+2k} & \cdots \\ \alpha_2 & \alpha_{2+k} & \alpha_{2+2k} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \alpha_k & \alpha_{2k} & \alpha_{3k} & \cdots \end{pmatrix}.$$

Our mission is to prove that

$$\sqrt{|\alpha_n|} = \begin{pmatrix} \sqrt{|\alpha_1|} & \sqrt{|\alpha_{1+k}|} & \sqrt{|\alpha_{1+2k}|} & \cdots \\ \sqrt{|\alpha_2|} & \sqrt{|\alpha_{2+k}|} & \sqrt{|\alpha_{2+2k}|} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \sqrt{|\alpha_k|} & \sqrt{|\alpha_{2k}|} & \sqrt{|\alpha_{3k}|} & \cdots \end{pmatrix}$$

is an element in $\lambda^{(k)}$. To do this it is enough to prove that

$$(\sqrt{|\alpha_i|}, \sqrt{|\alpha_{i+k}|}, \sqrt{|\alpha_{i+2k}|}, \ldots) \in \lambda(P_i)$$

for all $i = 1, 2, 3, \ldots, k$. Given $a \in P_i$. Since $\lambda^{(k)}$ is nuclear, we have $\lambda(P_i)$ is nuclear. Hence by Grothendieck-Pietsch criterion for nuclearity, there is $b \in P_i$ such that

$$\sum_{m=0}^{\infty} \frac{a_{m+1}}{b_{m+1}} < \infty.$$

Since P_i is a power set of infinite type there is an element c in P_i and a positive constant γ such that $b_{m+1}^2 \leq \gamma c_{m+1}$ for all $m \in \{0, 1, 2, 3, \ldots\}$. So

$$|\alpha_{i+mk}| b_{m+1}^2 \leq \gamma |\alpha_{i+mk}| c_{m+1}$$

for all $m \in \{0, 1, 2, 3, \ldots\}$. Hence

$$\sqrt{|\alpha_{i+mk}|} b_{m+1} \leq \sqrt{\gamma |\alpha_{i+mk}| c_{m+1}}$$

for all $m \in \{0, 1, 2, 3, \ldots\}$. Since $(\alpha_{i+mk})_{m=0}^{\infty} \in \lambda(P_i)$ and $(c_{m+1})_{m=0}^{\infty} \in P_i$, we have

$$\sum_{m=0}^{\infty} |\alpha_{i+mk}| c_{m+1} < \infty.$$

Therefore there is a positive constant η such that $|\alpha_{i+mk}| c_{m+1} \leq \eta$ for all $m \in \{0, 1, 2, 3, \ldots\}$. Hence

$$\sqrt{|\alpha_{i+mk}|} b_{m+1} \leq \sqrt{\gamma \eta}$$

for all $m \in \{0, 1, 2, 3, \ldots\}$. Therefore

$$\begin{aligned} \sum_{m=0}^{\infty} \sqrt{|\alpha_{i+mk}|} a_{m+1} &= \sum_{m=0}^{\infty} \sqrt{|\alpha_{i+mk}|} \frac{a_{m+1}}{b_{m+1}} b_{m+1} \\ &\leq \sqrt{\gamma \eta} \sum_{m=0}^{\infty} \frac{a_{m+1}}{b_{m+1}}. \end{aligned}$$

Since

$$\sum_{m=0}^{\infty} \frac{a_{m+1}}{b_{m+1}} < \infty,$$

we conclude that $(\sqrt{\alpha_{i+mk}})_{m=0}^{\infty} \in \lambda(P_i)$. Hence $(\sqrt{\alpha_n})$ is a sequence in $\lambda^{(k)}$. Therefore T is quasi-$\lambda^{(k)}$-nuclear. □

Next we present an example to show that the nuclearity of the k-Köthe space is essential in Theorem 3.1. Let $P_1 = \{(m^n) : m \in \mathbf{N}\}$, $P_2 = \{(m^{n^2}) : m \in \mathbf{N}\}$, ..., $P_{k-1} = \{(m^{n^{k-1}}) : m \in \mathbf{N}\}$, $P_k = \{(1, 1, 1, \ldots), (2, 2, 2, \ldots), \ldots\}$. Then P_i, $i = 1, 2, 3, \ldots, k$ are power sets of infinite type. Let $\lambda^{(k)}$ be the correspondence k-Köthe space. Since $\lambda(P_k)$ is not nuclear, by Lemma 3.1 we conclude that $\lambda^{(k)}$ is not nuclear.

Example **3.4.** *Define a map*

$$T : c_0 \to \ell_2$$

by

$$Tx = (z_1, z_2, z_3, \ldots),$$

where

$$z_1 = \left(\frac{x_1}{2^2}, \frac{x_2}{3^{3^2}}, \ldots, \frac{x_{k-1}}{k^{k^{k-1}}}, \frac{x_k}{k+1} \right),$$

$$z_2 = \left(\frac{x_{k+1}}{(k+2)^{k+2}}, \frac{x_{k+2}}{(k+3)^{(k+3)^2}}, \ldots, \frac{x_{2k-1}}{(2k)^{(2k)^{(k-1)}}}, \frac{x_{2k}}{2k+1} \right),$$

and

$$z_3 = \left(\frac{x_{2k+1}}{(2k+2)^{2k+2}}, \frac{x_{k+2}}{(2k+3)^{(2k+3)^2}}, \ldots, \frac{x_{3k-1}}{(3k)^{(3k)^{(k-1)}}}, \frac{x_{3k}}{3k+1} \right), \ldots.$$

Then the map T is 2-quasi-$\lambda^{(k)}$-nuclear which is not quasi-$\lambda^{(k)}$-nuclear.

Proof. Note that

$$||Tx||^2 = \sum_{m=1}^{k-1} \sum_{j=0}^{\infty} \frac{|x_{m+kj}|^2}{(m+1+kj)^{2(m+1+kj)^m}} + \sum_{j=0}^{\infty} \frac{|x_{k(j+1)}|^2}{(1+k(j+1))^2}.$$

Let

$$(\beta_n) = \begin{pmatrix} \frac{1}{2^4} & \frac{1}{(k+2)^{2(k+2)}} & \frac{1}{(2k+2)^{2(2k+2)}} & \cdots \\ \frac{1}{3^{2(3^2)}} & \frac{1}{(k+3)^{2(k+3)^2}} & \frac{1}{(2k+3)^{2(2k+3)^2}} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \frac{1}{k^{2k^{(k-1)}}} & \frac{1}{(2k)^{2(2k)^{k-1}}} & \frac{1}{(3k)^{2(3k)^{k-1}}} & \cdots \\ \frac{1}{(k+1)^2} & \frac{1}{(2k+1)^2} & \frac{1}{(3k+1)^2} & \cdots \end{pmatrix}.$$

Hence $||Tx||^2$ can be written in the form

$$||Tx||^2 = \sum_n |\beta_n| |\langle x, e_n \rangle|^2.$$

Since (e_n) is a bounded sequence in $c_0' = \ell_1$, and $(\beta_n) \in \lambda^{(k)}$, we conclude that T is 2-quasi-$\lambda^{(k)}$-nuclear. To show that T is quasi-$\lambda^{(k)}$-nuclear, we assume the contrary; that is, T is quasi-$\lambda^{(k)}$-nuclear. So there is $(\alpha_n) \in \lambda^{(k)}$ and a bounded sequence (a_n) in c_0' such that

$$||Tx|| \leq \sum_n |\alpha_n| |\langle x, a_n \rangle|^2.$$

Since $a_n \in \ell_1$, a_n has the form

$$a_n := (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots).$$

Take $x = e_j$, $j \in \{k, 2k, 3k, \ldots\}$, then

$$||Te_j|| \leq \sum_{n=1}^{\infty} |\alpha_n||\langle e_j, a_n\rangle.$$

So

$$\frac{1}{1+j} \leq \sum_{n=1}^{\infty} |\alpha_n||a_j^{(n)}|.$$

Therefore

$$\sum_{j\in\{k,2k,3k,\ldots\}} \frac{1}{1+j} \leq \sum_{j\in\{k,2k,3k,\ldots\}} \sum_{n=1}^{\infty} |\alpha_n||a_j^{(n)}|.$$

Which implies that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{1+km} &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n||a_{km}^{(n)}| \\
&= \sum_{n=1}^{\infty} |\alpha_n| \sum_{m=1}^{\infty} |a_{km}^{(n)}| \\
&\leq \sum_{n=1}^{\infty} |\alpha_n|||a_n||_1.
\end{aligned}$$

Since (a_n) is a bounded sequence in ℓ_1, we have

$$\sum_{m=1}^{\infty} \frac{1}{1+km} \leq (\sup_{n\in\mathbf{N}} ||a_n||) \sum_{n=1}^{\infty} |\alpha_n|.$$

Since $\sum_{m=1}^{\infty} \frac{1}{1+km}$ diverges, we conclude that $(\alpha_n) \notin \lambda^{(k)}$. Which is a contradiction. So T is not quasi-$\lambda^{(k)}$-nuclear. □

Problem. Note that $P_1, P_2, \ldots, P_k$ in Theorem 3.1 are assumed to be power set of infinite type.
Does Theorem 3.1 still valid for any nuclear k-Köthe space $\lambda^{(k)}$ generated by the Köthe sets $P_1, P_2, \ldots, P_k$ some of them are not power set of infinite type?

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