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LIPSCHITZ ESTIMATES FOR COMMUTATORS OF SINGULAR INTEGRAL OPERATORS ON WEIGHTED HERZ SPACES

YAN LIN, ZONGGUANG LIU, MINGMING SONG

Abstract. In this paper, we establish the boundedness of commutators generated by weighted Lipschitz functions and Calderón-Zygmund singular integral operators on weighted Herz spaces.

1. Introduction

The standard singular integral operator is defined by

$$Tf(x) = \text{p.v.} \int_{\boldsymbol{R}^n} K(x-y)f(y)dy. \tag{1.1}$$

A well-known result of Stein ([9]) states that if T is bounded on $L^q(\boldsymbol{R}^n)$, $1 < q < \infty$, and

$$|K(x)| \le \frac{C}{|x|^n}, \quad \forall x \neq 0, \tag{1.2}$$

then T is also bounded on the weighted spaces $L^q_{|x|^\beta}(\boldsymbol{R}^n)$, $-n < \beta < n(q-1)$, where the range of β is the best.

In 1994, the above Stein's result was developed by Soria and Weiss ([8]) in the following way. The singular integral operator satisfying (1.2) will be replaced by any sublinear operator T satisfying the following size condition: For any $f \in L^1(\boldsymbol{R}^n)$ with compact support and for $x \notin \text{supp} f$,

$$|Tf(x)| \le C \int_{\boldsymbol{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

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Throughout this paper we focus on the Calderón-Zygmund singular integral operator,

$$(1.3) \qquad Tf(x) = \text{p.v.} \int_{\boldsymbol{R}^n} K(x-y)f(y)dy,$$

with the kernel K satisfying

(1)$|K(x)| \le C|x|^{-n}$, $x \ne 0$,

(2)$|K(x-y) - K(x)| \le C\dfrac{|y|}{|x|^{n+1}}$, $2|y| \le |x|$.

Let b be a locally integrable function on $\boldsymbol{R}^n$ and let T be a Calderón-Zygmund singular integral operator. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x) = bT(f)(x) - T(bf)(x).$$

Janson ([4]) proved that $[b,T]$ is bounded from $L^p(\boldsymbol{R}^n)$ to $L^q(\boldsymbol{R}^n)$ for $1 < p < q < \infty$, $0 < \beta < 1$ and $1/q = 1/p - \beta/n$ if and only if $b \in Lip_\beta$. Lu and Yang ([7]) proved the boundedness of $[b,T]$ on Herz spaces. On the other hand, the necessary and sufficient conditions for boundedness of some commutators of singular integrals with weighted Lipschitz functions on Lebesgue spaces recently are obtained by Hu and Gu ([3]). Since Herz spaces are generalizations of Lebesgue spaces, a natural question is whether this kind of commutators also have boundedness on Herz spaces. The answer is affirmative. The main purpose of this paper is to generalize the above results to the case of weighted Herz spaces.

Let $B_k = \{x \in \boldsymbol{R}^n : |x| \le 2^k\}$, $E_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$, where by χ_E we denote the characteristic function of a set E. Let $f_B = \frac{1}{|B|}\int_B f(x)dx$.

Definition 1.1. *([1]) Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbf{R}^n)$ is defined by*

$$\dot{K}_q^{\alpha,p}(\mathbf{R}^n) = \{f \in L_{loc}^q(\mathbf{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|_q^p\right)^{1/p}$$

with usual modifications made when $p = \infty$.

A non-negative function μ defined on $\boldsymbol{R}^n$ is called a weight if it is locally integrable. A weight μ is said to belong to the Muckenhoupt class $A_p(\boldsymbol{R}^n)$ for $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B \mu(x)dx\right)\left(\frac{1}{|B|}\int_B (\mu(x))^{-\frac{1}{p-1}}dx\right)^{p-1} dx \le C,$$

for every ball $B \subset \boldsymbol{R}^n$. The class $A_1(\boldsymbol{R}^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|}\int_B \mu(x)dx \le C\mu(x), \qquad a.e. x \in \boldsymbol{R}^n.$$

A function $\mu \in A_\infty$ if it satisfies the condition of A_p for some $p > 1$. It is well-known that if $1 < p < q < \infty$, then we have $A_1 \subset A_p \subset A_q$.

Definition 1.2. *([6]) Let $0 < \alpha < \infty$ and $0 < p < \infty$, $1 < q < \infty$, and let μ_1 and μ_2 be non-negative weighted functions.*

The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\mu_1,\mu_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mu_1,\mu_2) = \{f \in L^q_{loc}(\mathbf{R}^n \setminus \{0\},\, \mu_2) : \|f\|_{\dot{K}_q^{\alpha,p}(\mu_1,\mu_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mu_1,\mu_2)} = \left(\sum_{k=-\infty}^{\infty} \mu_1(B_k)^{\alpha p/n}\|f\chi_k\|^p_{L^q(\mu_2)}\right)^{\frac{1}{p}}$$

with usual modifications made when $p = \infty$.

It is easy to see that $\dot{K}_q^{0,q}(\boldsymbol{R}^n) = L^q(\boldsymbol{R}^n)$ and $\dot{K}_q^{\alpha/q,q}(\boldsymbol{R}^n) = L^q_{|x|^\alpha}(\boldsymbol{R}^n)$ for $0 < q \le \infty$ and $\alpha \in \mathbb{R}$. Actually, for $1 < q < \infty$, $L^q_{|x|^\alpha}(\boldsymbol{R}^n)$ is a Lebesgue space with power weight if and only if $-n < \alpha < n(q-1)$. Thus, Herz spaces are generalizations of Lebesgue spaces. Moreover, the homogeneous Herz spaces include the Lebesgue spaces with power weights as special cases. Obviously, when $\mu_1 = \mu_2 = 1$, $\dot{K}_q^{\alpha,p}(\mu_1,\mu_2) = \dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ for all $0 < \alpha < \infty$, $0 < p \le \infty$, $1 < q < \infty$. On the other hand, when $\alpha = 0$, $\dot{K}_q^{\alpha,q}(\mu_1,\mu_2) = L^q(\mu_2)$ for $1 < q \le \infty$.

2. Main Results

In order to obtain our main results, first we need introduce some necessary notations and requisite lemmas.

Definition 2.1. *([2]) We say that a locally integrable function f belongs to the weighted Lipschitz space $Lip^p_{\beta,\mu}$ for $1 \le p \le \infty$, $0 < \beta < 1$ and $\mu \in A_\infty$, if*

$$\sup \frac{1}{\mu(B)^{\beta/n}}\left[\frac{1}{\mu(B)}\int_B |f(x) - f_B|^p \mu(x)^{1-p}dx\right]^{1/p} \le C < \infty,$$

where the supremum is taken over all balls $B \subset \mathbf{R}^n$.

Modulo constants, the Banach space of such functions are denoted by $Lip^p_{\beta,\mu}$. The smallest bound C satisfying conditions above is then taken to be the norm of f in this space, and is denoted by $\|f\|_{Lip^p_{\beta,\mu}}$. Put $Lip_{\beta,\mu} = Lip^1_{\beta,\mu}$. Obviously, for the case $\mu = 1$, the space $Lip_{\beta,\mu}$ is the classical Lipschitz space Lip_β. Thus, weighted Lipschitz spaces are generalizations of classical Lipschitz spaces.

If $\mu \in A_1(\boldsymbol{R}^n)$, García-Cuerva in [2] proved that the space $Lip^p_{\beta,\mu}$ coincide, and the norm of $\|.\|_{Lip^p_{\beta,\mu}}$ are equivalent with respect to different values of p provided that $1 \le p \le \infty$. That is $\|f\|_{Lip^p_{\beta,\mu}} \sim \|f\|_{Lip_{\beta,\mu}}$, where $1 \le p \le \infty$.

Lemma 2.2. ([5]) *Let $\mu \in A_1$, then there are constants C_1, C_2 and $0 < \delta < 1$ depending only on A_1-constant of μ, such that for any measurable subset E of a ball B,*

$$C_1 \frac{|E|}{|B|} \le \frac{\mu(E)}{\mu(B)} \le C_2 \left(\frac{|E|}{|B|}\right)^{\delta}.$$

Lemma 2.3. *Let $\mu \in A_1$ and $b \in Lip_{\beta,\mu}$, then there is a constant C such that for $j > k$,*

$$|b_{B_j} - b_{B_k}| \le C(j-k)\|b\|_{Lip_{\beta,\mu}} \mu(B_j)^{\frac{\beta}{n}} \frac{\mu(B_k)}{|B_k|}.$$

Proof Write

$$\begin{aligned} |b_{2B} - b_B| &= \left| \frac{1}{|B|} \int_B b(x)dx - b_{2B} \right| \\ &\le \frac{1}{|B|} \int_B |b(x) - b_{2B}| dx \\ &\le \frac{1}{|B|} \int_{2B} |b(x) - b_{2B}| dx \\ &\le C \frac{1}{|B|} \|b\|_{Lip_{\beta,\mu}} \mu(2B)^{\frac{\beta}{n}+1}. \end{aligned}$$

Thus, by Lemma 2.2, we obtain

$$\begin{aligned} |b_{B_j} - b_{B_k}| &\le |b_{B_j} - b_{B_{j-1}}| + |b_{B_{j-1}} - b_{B_{j-2}}| + \cdots + |b_{B_{k+1}} - b_{B_k}| \\ &\le \frac{1}{|B_{j-1}|} \int_{B_j} |b(x) - b_{B_j}| dx + \frac{1}{|B_{j-2}|} \int_{B_{j-1}} |b(x) - b_{B_{j-1}}| dx \\ &\qquad + \cdots + \frac{1}{|B_k|} \int_{B_{k+1}} |b(x) - b_{B_{k+1}}| dx \\ &\le C|b\|_{Lip_{\beta,\mu}} \left(\frac{\mu(B_j)^{\frac{\beta}{n}+1}}{|B_{j-1}|} + \frac{\mu(B_{j-1})^{\frac{\beta}{n}+1}}{|B_{j-2}|} + \cdots + \frac{\mu(B_{k+1})^{\frac{\beta}{n}+1}}{|B_k|} \right) \\ &\le C|b\|_{Lip_{\beta,\mu}} \mu(B_j)^{\frac{\beta}{n}} \left(\frac{\mu(B_j)}{|B_{j-1}|} + \frac{\mu(B_{j-1})}{|B_{j-2}|} + \cdots + \frac{\mu(B_{k+1})}{|B_k|} \right) \\ &\le C(j-k)|b\|_{Lip_{\beta,\mu}} \mu(B_j)^{\frac{\beta}{n}} \frac{\mu(B_k)}{|B_k|}. \end{aligned}$$

Lemma 2.4. *Let $\mu \in A_1$, then for $1 < p < \infty$,*

$$\int_B \mu(x)^{1-p'} dx \le C|B|^{p'} \mu(B)^{1-p'},$$

where $1/p + 1/p' = 1$.

Proof Since $A_1 \subset A_p$ $(1 < p)$, μ satisfies the condition of the weight A_p,

$$\left(\frac{1}{|B|}\int_B \mu(x)dx\right)\left(\frac{1}{|B|}\int_B (\mu(x))^{-\frac{1}{p-1}} dx\right)^{p-1} dx \le C,$$

for every ball $B \subset \boldsymbol{R}^n$.

We obtain

$$\int_B \mu(x)^{1-p'} dx \le C\left(\frac{|B|}{\mu(B)}\right)^{\frac{1}{p-1}} |B| = C|B|^{p'} \mu(B)^{1-p'}.$$

Recently, the authors in [3] discussed the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Lebesgue spaces.

Theorem A. ([3]) *Let T be a Calderón-Zygmund singular integral operator. Let $\mu \in A_1$, $1/q = 1/p - \beta/n$ for $0 < \beta < 1$ and $1 < p < q < \infty$. Let $b \in Lip_{\beta,\mu}$. Then the commutator $[b, T]$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$.*

The purpose of this paper is to state the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Herz spaces. Our theorem is as follows:

Theorem 2.5. *Let T be a Calderón-Zygmund singular integral operator. Let $b \in Lip_{\beta,\mu}$, $\mu \in A_1$, $0 < \beta < 1$, $0 < p \le \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - \beta/n$ and $-n/q_2 < \alpha < n(\delta - 1/q_1)$, where δ is defined as in Lemma 2.2, then the commutator $[b, T]$ is bounded from $\dot{K}^{\alpha,p}_{q_1}(\mu,\mu)$ to $\dot{K}^{\alpha,p}_{q_2}(\mu,\mu^{1-q_2})$.*

Proof We only consider the case $0 < p < \infty$ and omit the details of the case $p = \infty$ since their similarity.

Set $f=\sum_{j=-\infty}^{\infty} f\chi_j := \sum_{j=-\infty}^{\infty} f_j$. Then, by the Minkowski inequality, we write

$$\begin{aligned}\|[b,T]f\|^p_{\dot K^{\alpha,p}_{q_2}(\mu,\mu^{1-q_2})} &= \sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big\|\Big(\sum_{j=-\infty}^{\infty}[b,T]f\chi_j\Big)\chi_k\Big\|^p_{L^{q_2}(\mu^{1-q_2})}\\ &\le C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-2}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Big)^p\\ &\quad+C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=k-1}^{k+1}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Big)^p\\ &\quad+C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=k+2}^{\infty}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Big)^p\\ &:= I_1+I_2+I_3.\end{aligned}$$

For I_2, by Theorem A and Lemma 2.2, we find

$$\begin{aligned}I_2 &\le C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=k-1}^{k+1}\|[b,T]f\chi_j\|_{L^{q_2}(\mu^{1-q_2})}\Big)^p\\ &\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=k-1}^{k+1}\|f\chi_j\|_{L^{q_1}(\mu)}\Big)^p\\ &\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\|f\chi_k\|^p_{L^{q_1}(\mu)}\\ &= C\|b\|^p_{Lip_{\beta,\mu}}\|f\|^p_{\dot K^{\alpha,p}_{q_1}(\mu,\mu)}.\end{aligned}$$

To obtain the estimates for I_1 and I_3, we first observe that for $x\in E_k$ and $|j-k|\ge 2$,

$$([b,T]f\chi_j)\chi_k(x) = (b(x)-b_{B_j})T(f\chi_j)(x) - T((b-b_{B_j})f\chi_j)(x).$$

Thus,

$$\begin{aligned}&\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\\ &\le C\Big(\int_{E_k}|(b(x)-b_{B_j})T(f\chi_j)(x)-T((b-b_{B_j})f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\Big)^{\frac{1}{q_2}}\\ &\le C\Big(\int_{E_k}|b(x)-b_{B_j}|^{q_2}|T(f\chi_j)(x)|^{q_2}|\mu(x)^{1-q_2}dx\Big)^{\frac{1}{q_2}}\\ &\quad+C\Big(\int_{E_k}|T((b-b_{B_j})f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\Big)^{\frac{1}{q_2}}\end{aligned}$$

$$
\begin{aligned}
&\leq C\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\bigg|\int_{E_j}K(x-y)f(y)dy\bigg|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\quad +C\bigg(\int_{E_k}\bigg|\int_{E_j}K(x-y)(b(y)-b_{B_j})f\chi_j(y)dy\bigg|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&:= D_1(j,k)+D_2(j,k).
\end{aligned}
$$

Now, let us estimate I_1. If $j \leq k-2$, using Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

$$
\begin{aligned}
&\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\leq\bigg(\int_{B_k}|b(x)-b_{B_k}|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}+\bigg(\int_{B_k}|b_{B_k}-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}}\mu(B_k)^{\frac{1}{q_2}}+|b_{B_k}-b_{B_j}|\bigg(\int_{B_k}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}}\mu(B_k)^{\frac{1}{q_2}}+C(k-j)\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}}\frac{\mu(B_j)}{|B_j|}\bigg(\int_{B_k}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}}\mu(B_k)^{\frac{1}{q_2}}+C(k-j)\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}}|B_k|\mu(B_k)^{\frac{1-q_2}{q_2}}\frac{\mu(B_j)}{|B_j|}\\
&\leq C(k-j)\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{\beta}{n}+\frac{1}{q_2}}\frac{|B_k|}{|B_j|}\frac{\mu(B_j)}{\mu(B_k)}\\
&\leq C(k-j)\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{1}{q_1}}\frac{|B_k|}{|B_j|}\bigg(\frac{|B_j|}{|B_k|}\bigg)^{\delta}\\
&= C(k-j)\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{1}{q_1}}2^{(k-j)n(1-\delta)}.
\end{aligned}
$$

By Hölder's inequality, denoting that $1/q_1+1/q_1'=1$, we obtain

$$
\begin{aligned}
\int_{E_j}|f\chi_j(y)|dy &\leq \bigg(\int_{B_j}|f\chi_j(y)|^{q_1}\mu(y)^{\frac{1}{q_1}q_1}dy\bigg)^{\frac{1}{q_1}}\bigg(\int_{B_j}\mu(y)^{-\frac{1}{q_1}q_1'}dy\bigg)^{\frac{1}{q_1'}}\\
&= \bigg(\int_{B_j}|f\chi_j(y)|^{q_1}\mu(y)dy\bigg)^{\frac{1}{q_1}}\bigg(\int_{B_j}\mu(y)^{1-q_1'}dy\bigg)^{\frac{1}{q_1'}}\\
&\leq C\|f\chi_j\|_{L^{q_1}(\mu)}|B_j|\mu(B_j)^{-\frac{1}{q_1}}.
\end{aligned}
$$

By the above two estimates and Lemma 2.2, we have

$$\begin{aligned}
D_1(j,k) &\leq C\left[\int_{E_k}|b(x)-b_{B_j}|^{q_2}\left(\int_{E_j}\frac{1}{|x-y|^n}|f\chi_j(y)|dy\right)^{q_2}\mu(x)^{1-q_2}dx\right]^{\frac{1}{q_2}}\\
&\leq C2^{-kn}\left(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\left(\int_{E_j}|f\chi_j(y)|dy\right)\\
&\leq C(k-j)2^{-kn}\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{1}{q_1}}2^{(k-j)n(1-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}2^{jn}\mu(B_j)^{-\frac{1}{q_1}}\\
&= C(k-j)2^{-(k-j)n\delta}\|b\|_{Lip_{\beta,\mu}}\left[\frac{\mu(B_k)}{\mu(B_j)}\right]^{\frac{1}{q_1}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\leq C(k-j)2^{-(k-j)n\delta}\|b\|_{Lip_{\beta,\mu}}\left[\frac{|B_k|}{|B_j|}\right]^{\frac{1}{q_1}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|b\|_{Lip_{\beta,\mu}}\|f\chi_j\|_{L^{q_1}(\mu)},
\end{aligned}$$

and

$$\begin{aligned}
D_2(j,k) &= C\left(\int_{E_k}\left|\int_{E_j}K(x-y)(b(y)-b_{B_j})f\chi_j(y)dy\right|^{q_2}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\
&\leq C2^{-kn}\left(\int_{B_k}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\left(\int_{B_j}|b(y)-b_{B_j})||f\chi_j(y)|dy\right)\\
&\leq C2^{-kn}\left[|B_k|^{q_2}\mu(B_k)^{1-q_2}\right]^{\frac{1}{q_2}}\left(\int_{B_j}|b(y)-b_{B_j}|^{q_1'}\mu(y)^{\frac{1}{-q_1}q_1'}dy\right)^{\frac{1}{q_1'}}\\
&\qquad\times\left(\int_{B_j}|f\chi_j(y)|^{q_1}\mu(y)dy\right)^{\frac{1}{q_1}}\\
&\leq C2^{-kn}2^{kn}\mu(B_k)^{\frac{1}{q_2}-1}\|b\|_{Lip_{\beta,\mu}}\mu(B_j)^{\frac{1}{q_1'}}\mu(B_j)^{\frac{\beta}{n}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C\|b\|_{Lip_{\beta,\mu}}\left[\frac{\mu(B_j)}{\mu(B_k)}\right]^{1-\frac{1}{q_2}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\left[\frac{|B_j|}{|B_k|}\right]^{\delta(1-\frac{1}{q_2})}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C\|b\|_{Lip_{\beta,\mu}}2^{(j-k)n\delta(1-\frac{1}{q_2})}\|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}$$

Combining the estimates for $D_1(j,k)$ and $D_2(j,k)$, we obtain that if $j\leq k-2$, then

$$\begin{aligned}
&\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\\
&\quad\le C(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|b\|_{Lip_{\beta,\mu}}\|f\chi_j\|_{L^{q_1}(\mu)}+C\|b\|_{Lip_{\beta,\mu}}2^{(k-j)n\delta(\frac{1}{q_2}-1)}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\quad\le C(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|b\|_{Lip_{\beta,\mu}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\quad= C\|b\|_{Lip_{\beta,\mu}}2^{(j-k)\alpha}(k-j)2^{(k-j)(\alpha+\frac{n}{q_1}-n\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}$$

Let $W(j,k)=(k-j)2^{(k-j)(\alpha+\frac{n}{q_1}-n\delta)}$. We have

$$\begin{aligned}
I_1 &\le C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)\alpha}W(j,k)\|b\|_{Lip_{\beta,\mu}}\|f\chi_j\|_{L^{q_1}(\mu)}\Big)^p\\
&\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)\alpha}W(j,k)\|f\chi_j\|_{L^{q_1}(\mu)}\Big)^p.
\end{aligned}$$

If $0<p\le 1$, then by Lemma 2.2 and $\alpha<n(\delta-\frac{1}{q_1})$,

$$\begin{aligned}
I_1 &\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}\mu(B_j)^{\alpha p/n}W(j,k)^p\|f\chi_j\|^p_{L^{q_1}(\mu)}\\
&\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{j=-\infty}^{\infty}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}\sum_{k=j+2}^{\infty}W(j,k)^p\\
&= C\|b\|^p_{Lip_{\beta,\mu}}\|f\|^p_{\dot K^{\alpha,p}_{q_1}(\mu,\mu)}.
\end{aligned}$$

When $1<p<\infty$, by Hölder's inequality, Lemma 2.2 and $\alpha<n(\delta-\frac{1}{q_1})$,

$$\begin{aligned}
I_1 &\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{\alpha p}{n}}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)\alpha}W(j,k)\|f\chi_j\|_{L^{q_1}(\mu)}\Big)^p\\
&\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-2}W(j,k)\Big)^{\frac{p}{p'}}\Big[\sum_{j=-\infty}^{k-2}\Big(2^{(j-k)\alpha}W(j,k)^{\frac{1}{p}}\|f\chi_j\|_{L^{q_1}(\mu)}\Big)^p\Big]^{\frac{1}{p}\cdot p}\\
&\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)\alpha p}W(j,k)\|f\chi_j\|^p_{L^{q_1}(\mu)}\Big)\\
&\le C\|b\|^p_{Lip_{\beta,\mu}}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}\mu(B_j)^{\alpha p/n}W(j,k)\|f\chi_j\|^p_{L^{q_1}(\mu)}\Big)\\
&= C\|b\|^p_{Lip_{\beta,\mu}}\sum_{j=-\infty}^{\infty}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}\Big(\sum_{k=j+2}^{\infty}W(j,k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= C\|b\|^p_{Lip_{\beta,\mu}} \sum_{j=-\infty}^{\infty} \mu(B_j)^{\alpha p/n} \|f\chi_j\|^p_{L^{q_1}(\mu)} \\
&= C\|b\|^p_{Lip_{\beta,\mu}} \|f\|^p_{\dot{K}^{\alpha,p}_{q_1}(\mu,\mu)}.
\end{aligned}
$$

For I_3, by analogy to the estimates of I_1, we find, for $j \geq k+2$,

$$
D_1(j,k) \leq C(j-k)2^{(k-j)n\delta\frac{1}{q_2}} \|b\|_{Lip_{\beta,\mu}} \|f\chi_j\|_{L^{q_1}(\mu)},
$$

$$
D_2(j,k) \leq C2^{\frac{(k-j)n}{q_2}} \|b\|_{Lip_{\beta,\mu}} \|f\chi_j\|_{L^{q_1}(\mu)},
$$

and

$$
\begin{aligned}
\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})} &\leq C\|b\|_{Lip_{\beta,\mu}} (j-k)2^{\frac{(k-j)n\delta}{q_2}} \|f\chi_j\|_{L^{q_1}(\mu)} \\
&= C\|b\|_{Lip_{\beta,\mu}} (j-k)2^{(j-k)\alpha\delta} 2^{(k-j)(\alpha+n/q_2)\delta} \|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}
$$

Let $V(j,k) = (j-k)2^{(k-j)(\alpha+\frac{n}{q_2})\delta}$. Thus,

$$
\begin{aligned}
I_3 &\leq C \sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p/n} \left(\sum_{j=k+2}^{\infty} \|b\|_{Lip_{\beta,\mu}} 2^{(j-k)\alpha\delta} V(j,k) \|f\chi_j\|_{L^{q_1}(\mu)} \right)^p \\
&= C\|b\|^p_{Lip_{\beta,\mu}} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} \mu(B_k)^{\alpha/n} 2^{(j-k)\alpha\delta} V(j,k) \|f\chi_j\|_{L^{q_1}(\mu)} \right)^p.
\end{aligned}
$$

When $0 < p \leq 1$, by Lemma 2.2 and $\alpha > \frac{n}{q_2}$,

$$
\begin{aligned}
I_3 &\leq C\|b\|^p_{Lip_{\beta,\mu}} \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} \mu(B_k)^{\alpha p/n} 2^{(j-k)\alpha p\delta} V(j,k)^p \|f\chi_j\|^p_{L^{q_1}(\mu)} \\
&\leq C\|b\|^p_{Lip_{\beta,\mu}} \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} \mu(B_j)^{\alpha p/n} V(j,k)^p \|f\chi_j\|^p_{L^{q_1}(\mu)} \\
&= C\|b\|^p_{Lip_{\beta,\mu}} \sum_{j=-\infty}^{\infty} \mu(B_j)^{\alpha p/n} \|f\chi_j\|^p_{L^{q_1}(\mu)} \sum_{k=-\infty}^{j-2} V(j,k)^p \\
&= C\|b\|^p_{Lip_{\beta,\mu}} \sum_{j=-\infty}^{\infty} \mu(B_j)^{\alpha p/n} \|f\chi_j\|^p_{L^{q_1}(\mu)} \\
&= C\|b\|^p_{Lip_{\beta,\mu}} \|f\|^p_{\dot{K}^{\alpha,p}_{q_1}(\mu,\mu)}.
\end{aligned}
$$

When $1 < p < \infty$, by Hölder's inequality, we obtain

$$\begin{aligned} I_3 &\le C\|b\|^p_{Lip_{\beta,\mu}} \sum_{k=-\infty}^{\infty} \mu(B_k)^{\frac{\alpha p}{n}} \left(\sum_{j=k+2}^{\infty} V(j,k) \right)^{\frac{p}{p'}} \left[\sum_{j=k+2}^{\infty} \left(2^{(j-k)\alpha\delta} V(j,k)^{\frac{1}{p}} \|f\chi_j\|_{L^{q_1}(\mu)} \right)^p \right]^{\frac{1}{p}\cdot p} \\ &\le C\|b\|^p_{Lip_{\beta,\mu}} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} V(j,k)\mu(B_j)^{\alpha p/n} \|f\chi_j\|^p_{L^{q_1}(\mu)} \right) \\ &= C\|b\|^p_{Lip_{\beta,\mu}} \sum_{j=-\infty}^{\infty} \mu(B_j)^{\alpha p/n} \|f\chi_j\|^p_{L^{q_1}(\mu)} \\ &= C\|b\|^p_{Lip_{\beta,\mu}} \|f\|^p_{\dot{K}^{\alpha,p}_{q_1}(\mu,\mu)}. \end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, we complete the proof of Theorem 2.5.

As the special case of Theorem 2.5, we can deduce the following conclusion immediately. When $\mu = 1$, $Lip_{\beta,\mu}(\boldsymbol{R}^n) = Lip_\beta(\boldsymbol{R}^n)$, $\delta = 1, \dot{K}^{\alpha,p}_{q_1}(\mu,\mu) = \dot{K}^{\alpha,p}_{q_1}(\boldsymbol{R}^n)$, $\dot{K}^{\alpha,p}_{q_2}(\mu,\mu^{1-q_2}) = \dot{K}^{\alpha,p}_{q_2}(\boldsymbol{R}^n)$. Thus, we have the boundedness of commutators generated by singular integrals and classical Lipschitz functions on Herz spaces, which was established by Lu and Yang in [7].

Corollary 2.6. *Suppose that $b \in Lip_\beta(\mathbf{R}^n)$, $0 < \beta < 1$. Let $0 < p \le \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - \beta/n$ and $-n/q_2 < \alpha < n(1 - 1/q_1)$. Then $[b,T]$ maps $\dot{K}^{\alpha,p}_{q_1}(\mathbf{R}^n)$ continuously into $\dot{K}^{\alpha,p}_{q_2}(\mathbf{R}^n)$.*

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(Lin) School of Sciences, China University of Mining and Technology, Beijing 100083, P. R. China
E-mail address: linyan@cumtb.edu.cn (Corresponding author)

(Liu) School of Sciences, China University of Mining and Technology, Beijing 100083, P. R. China
E-mail address: liuzg@cumtb.edu.cn

(Song) School of Sciences, China University of Mining and Technology, Beijing 100083, P. R. China
E-mail address: dodosmm@126.com