Jordan Journal of Mathematics and Statistics (JJMS) 2(2), 2009, pp. 55- 66

THE GENERALIZED STIELTJES AND FOURIER TRANSFORMS OF CERTAIN SPACES OF GENERALIZED FUNCTIONS

S.K.Q.AL-OMARI

Abstract. Inspired of Roumieu and Beurling definitions, we defined spaces of ultradifferentiable functions of rapid descents in L^p spaces which shown to be closed with respect to the classical Stieltjes transform. The generalized Stieltjes transform is thereby defined on the duals through the generalization of the Parseval's equation. Further, we obtained spaces and derived certain related results justifying multiplication. Adopting the concept of Boehmian spaces, the celebrated Fourier transforms were extended to a so-called space of ultradifferentiable Boehmians .

1. Introduction

The Schwartz' Theory of distributions and its applications are well known in the literature. Spaces of generalization of the theory were obtained and developed by many authors in the recent past. As a space of generalized functions the theory of ultradistributions, being more general than distributions, considered by Roumieu [17], [18] and Beurling [6] is formulated so that it generalizes the Schwartz' space D' of distributions. Various integral transforms for various spaces of ultradistributions have been obtained and the corresponding properties are developed in Refs [1], [2] and [3] and, many others. Tempered ultradistributions or ultradistributions of slow growth established in this note and the ultradistributions which are employed in [3] and [4] generalize the Schwartz space S' of tempered distributions [5]. However, the ultradistributions described in [2] are, indeed, expand the space $E'$of distributions of bounded support [22]. For detailed treatment and relevant properties (see [6] , [17] and references therein).

For a conventional function f, the Stieltjes transform, as a function of y, is given by [21]

$$(\tau f)(y) = \int_0^\infty f(x)(y+x)^{-1}\,dx, (y>0) \tag{1.1}$$

2000 *Mathematics Subject Classification.* : 40H05, 46A45.

Key words and phrases. : Generalized Function, Ultradistribution, Tempered Distribution, Stieltjes Transform, Roumieu Ultradistribution .

Received: May 1, 2009. Accepted : Dec. 6 ,2009.

provided the integral is convergent. Its generalized transform for an arbitrary complex number p except the zero and negative integers and, all z in the z-plane cut from the origin along the negative real axis is defined in [7, 18] by

$$(\tau_p f)(z) = \int_0^\infty f(x)(z+x)^{-p}\, dx \tag{1.2}$$

Authors such as Erdelyi [9], Pandey [13], Pathak [15] and others extended (1.2) to certain spaces of generalized functions. The generalized Stieltjes transform , as a well-known fact, can be formulated as an iterated Laplace transform and therefore its inverse is , explicitly, expressed as an iterated inverse Laplace transform [19]. Authors, such as Love and Byrne [11] and Pollard [16] obtain formulae for the inverse of the transform for real values . The complex inversion formulae are established in [8] and [20] as well .

A complex valued measurable function f is said to belong to the space $L^p(R)$ if

$$\|f\|_p = \left\{ \int_R |f(x)|^p\, dx \right\}^{\frac{1}{p}}$$

is finite .Two functions are identified whenever they are equal almost everywhere in the Lebesgue sense.

2. **Definitions and Notations**

A base of this paper will be certain spaces of test functions of ultradifferentiable functions and tempered ultradistributions which correspond to spaces D_{L^s} and D'_{L^s} [14] and spaces of type S [10] . Sequences (a_i) and (b_i) wherever they appear together with conditions employed are to be treated as in [4] .

Definition 2.1. *Let α be a fixed real number. Then, the following have a meaning in the sense of results*

(i) By $S^{L^r}_{\alpha,(a_i),a}(0,\infty)$ (respectively, $S^{L^r}_{\alpha,\{a_i\},a}(0,\infty)$), $1 \le r \le \infty$, we denote the set of all complex valued infinitely differentiable functions $\varphi(x)$ such that there is a constant $m > 0$ for which

$$\left\| (1+x)^\alpha x^k D^k \varphi(x) \right\|_{L^r} \le m a^\alpha a_\alpha \tag{2.1}$$

for all $a > 0$ (respectively, for some $a > 0$).

(ii) $\varphi(x) \in S^{L^r,(b_j),b}_{\alpha}(0,\infty)$ (respectively, $S^{L^r,\{b_j\},b}_{\alpha}(0,\infty)$), $1 \le r \le \infty$ if and only if it is infinitely smooth and for some constant $n > 0$,

$$\left\| (1+x)^\alpha x^k D^k \varphi(x) \right\|_{L^r} \le m b^k b_k \tag{2.2}$$

for all $b > 0$ (respectively, for some $b > 0$).

(iii) Similarly ,the C^{∞}*-function* $\varphi(x) \in S^{L^r,(b_j),b}_{\alpha,(a_i),a}(0,\infty)$ *(respectively,* $S^{L^r,\{b_j\},b}_{\alpha,\{a_i\},a}(0,\infty)$ *if and only if for some constant L and for all* $a > 0, b > 0$ *(respectively, for some* $a > 0, b > 0$*) the following*

$$\left\|(1+x)^{\alpha} x^k D^k \varphi(x)\right\|_{L^r} \leq L a^{\alpha} b^k a_{\alpha} b_k \tag{2.3}$$

holds good .

Obviously, due to definitions, subsets

$$S^{L^r}_{\alpha,(a_i),a} \subset S^{L^r}_{\alpha,\{a_i\},a}, S^{L^r,(b_j),b}_{\alpha} \subset S^{L^r,\{b_j\},b}_{\alpha} and\ S^{L^r,(b_j),b}_{\alpha,(a_i),a} \subset S^{L^r,\{b_j\},b}_{\alpha,\{a_i\},a} \tag{2.4}$$

are true and possess analysis which is similar in application and thus the concern will be on $S^{L^r}_{\alpha,(a_i),a}, S^{L^r,(b_j),b}_{\alpha}$ and $S^{L^r,(b_j),b}_{\alpha,(a_i),a}$ for further investigations.

Natural topologies on $S^{L^r}_{\alpha,(a_i),a}, S^{L^r,(b_j),b}_{\alpha}$ and $S^{L^r,(b_j),b}_{\alpha,(a_i),a}$ can be defined as

$$\|\varphi\|_{r,a} = \sup \|\alpha x \in (0,\infty) \frac{\left\|(1+x)^{\alpha} x^k D^k \varphi(x)\right\|_{L^r}}{a^{\alpha} a_{\alpha}}, \quad b > 0 \tag{2.5}$$

$$\|\varphi\|_{r,b} = \sup \|k x \in (0,\infty) \frac{\left\|(1+x)^{\alpha} x^k D^k \varphi(x)\right\|_{L^r}}{b^k b_k}, \quad a > 0 \tag{2.6}$$

and

$$\|\varphi\|_{r,a} = \sup \|\alpha, k x \in (0,\infty) \frac{\left\|(1+x)^{\alpha} x^k D^k \varphi(x)\right\|_{L^r}}{a^{\alpha} b^k a_{\alpha} b_k}, \quad a, b > 0 \tag{2.7}$$

A sequence $(\varphi_n) \in S^{L^r}_{\alpha,(a_i),a}, S^{L^r,(b_j),b}_{\alpha}$ and $S^{L^r,(b_j),b}_{\alpha,(a_i),a}$
converges to $S^{L^r}_{\alpha,(a_i),a}, S^{L^r,(b_j),b}_{\alpha}$ and $S^{L^r,(b_j),b}_{\alpha,(a_i),a}$ if

$$\lim_{n\to\infty} \left\|(1+x)^{\alpha} x^k D^k (\varphi_n(x) - \varphi(x))\right\|_{L^r} = 0$$

and there is a constant $m > 0$ independent of n such that

$$\lim_{n\to\infty} \left\|(1+x)^{\alpha} x^k D^k (\varphi_n(x) - \varphi(x))\right\|_{L^r} \leq m a^{\alpha} a_{\alpha}$$

for all $a > 0$.

Denoting by $S'^{L^r}_{\alpha,(a_i),a}, S'^{L^r,(b_j),b}_{\alpha}$ and $S'^{L^r,(b_j),b}_{\alpha,(a_i),a} \left(S'^{L^r}_{\alpha,\{a_i\},a}, S'^{L^r,\{b_j\},b}_{\alpha} and\ S'^{L^r,\{b_j\},b}_{\alpha,\{a_i\},a}\right)$.
The set of continuous linear forms on $S^{L^r}_{\alpha,(a_i),a}, S^{L^r,(b_j),b}_{\alpha}$ and
$S^{L^r,(b_j),b}_{\alpha,(a_i),a} \left(S^{L^r}_{\alpha,\{a_i\},a}, S^{L^r,\{b_j\},b}_{\alpha} and\ S^{L^r,\{b_j\},b}_{\alpha,\{a_i\},a}\right)$.
The resulting spaces are the tempered (temperate) ultradistribution spaces of Beurling - type (of Roumieu-type respectively).

3. **Stieltjes Transform for Slow Growth Ultradistributions**

For our results the following is the main theorem in this section which enables to define the generalized Stieltjes transform in certain L^p spaces

Theorem 3.1. *Let z be a complex number, neither zero nor negative real, then*

$$(z+x)^{-p} \in S^{L^r}_{\alpha,(a_i),a}(0,\infty)$$

where ,
$\alpha r + kr + 1 < 0, k = 0, 1, 2, ...$*and* $\alpha \in \operatorname{Re} p$.

Proof. Assume $z = \sigma + iw$. The fact we shall need is that

$$|z|^{\operatorname{Re} p} e^{-\pi|\operatorname{Im} p|} \leq |z^p| \leq |z|^{\operatorname{Re} p} e^{\pi|\operatorname{Im} p|} \tag{3.1}$$

Employing (3.1) we have

$$\begin{aligned} \left|(1+x)^{\alpha} x^k D^k (z+x)^{-p}\right| &= \frac{(1+x)^{\alpha} x^k \left|(p)_k\right|}{\left|(z+x)^{p+k}\right|} \\ &\leq \frac{\left|(p)_k\right| (1+x)^{\alpha} x^k e^{\pi|\operatorname{Im} p|}}{\left[(\sigma+x)^2 + w^2\right]^{\frac{1}{2(\operatorname{Re}(p+k))}}} \qquad (3.2) \end{aligned}$$

where $(p)_k = p(p+1)(p+2)\ldots(p+k-1)$.
We consider the following cases
Case I. Let $\sigma > 0$. Properties of integration then yields

$$\begin{aligned} \int_0^{\infty} \left|(1+x)^{\alpha} x^k D^k (z+x)^{-p}\right|^r dx &= \int_0^{\sigma} \left|(1+x)^{\alpha} x^k D^k (z+x)^{-p}\right|^r dx \\ &\quad + \int_{\sigma}^{\infty} \left|(1+x)^{\alpha} x^k D^k (z+x)^{-p}\right|^r dx \qquad (3.3) \\ &\equiv I_1(\alpha,k,r) + I_2(\alpha,k,r) \end{aligned}$$

We evaluate the integral $I_2(\alpha,k,r)$ as follows :

$$\begin{aligned} \int_{\sigma}^{\infty} \left|(1+x)^{\alpha} x^k D^k (z+x)^{-p}\right|^r dx &< \left|(p)_k\right| \int_{\sigma}^{\infty} \frac{(1+x)^{\alpha r} x^{kr} e^{r\pi|\operatorname{Im} p|}}{(2\sigma)^{r(\operatorname{Re}(p+k))}} dx \\ &< \frac{e^{r\pi|\operatorname{Im} p|}\left|(p)_k\right|}{(2\sigma)^{r(\operatorname{Re}(p+k))}} \int_{\sigma}^{\infty} (1+x)^{(\alpha+k)^r} dx \\ &< \frac{\left|(p)_k\right| e^{r\pi|\operatorname{Im} p|} (1+\sigma)^{ar+kr+1}}{(ar+kr+1)(2\sigma)^{r(\operatorname{Re}(p+k))}} \qquad (3.4) \end{aligned}$$

The values of x are finite in $(0,\sigma]$ and hence, the integral $I_1(\alpha,k,r)$ will be reduced to the form

$$\int_0^{\sigma}\left|(1+x)^{\alpha}x^kD^k(z+x)^{-p}\right|^r dx \le |(p)_k|\int_0^{\sigma}\frac{(1+\sigma)^{r\alpha}}{\sigma^{r\,\mathrm{Re}\,p}}e^{r\pi|\mathrm{Im}\,p|}dx$$

Hence,

$$\int_0^{\sigma}\left|(1+x)^{\alpha}x^kD^k(z+x)^{-p}\right|^r dx \le |(p)_k|\frac{\sigma(1+\sigma)^{r\alpha}}{\sigma^{r\,\mathrm{Re}\,p}}e^{\pi|\mathrm{Im}\,p|}. \tag{3.5}$$

Invoking (3.4) and (3.5) into (3.3) and multiplying by $\dfrac{1}{a^{\alpha}a_{\alpha}}$, for any $a>0$, and considering supremum over $\alpha\le\mathrm{Re}\,p$ implies that

$$\|\varphi\|_{r,a}<M$$

where

$$M=\left[\sigma^{1-\mathrm{Re}\,p}(1+\sigma)^{\alpha}-\frac{(1+\sigma)^{a+k+\frac{1}{r}}}{2\sigma^{(\mathrm{Re}(p+k))}(\alpha r+kr+1)^{\frac{1}{r}}}\right]|(p)_k|\,e^{\pi|\mathrm{Im}\,p|}$$

Case II. if $\sigma=0$ then $z=iw$. Set $v=|w|$.Therefore

$$\begin{aligned}\int_0^{\infty}\left|(1+x)^{\alpha}x^kD^k(z+x)^{-p}\right|^r dx &= \int_0^{\nu}\left|(1+x)^{\alpha}x^kD^k(z+x)^{-p}\right|^r dx\\ &\quad+\int_{\nu}^{\infty}\left|(1+x)^{\alpha}x^kD^k(z+x)^{-p}\right|^r dx \qquad (3.6)\\ &\equiv I_1^*(\alpha,k,r)+I_2^*(\alpha,k,r)\end{aligned}$$

The integral $I_1^*(\alpha,k,r)<|(p)_k|\int_0^{\nu}\dfrac{(1+\nu)^{ar}}{\nu^{r(\mathrm{Re}\,p)}}e^{\pi|\mathrm{Im}\,p|r}dx$. That is,

$$I_1^*(\alpha,k,r)<|(p)_k|\int_0^{\nu}\frac{(1+\nu)^{ar}}{\nu^{\left(r(\mathrm{Re}\,p)+1\right)}}e^{\pi|\mathrm{Im}\,p|r}dx \tag{3.7}$$

Estimation of the improper integral $I_2^*(\alpha,k,r)$ can be confirmed as follows :

$$\begin{aligned}I_2^*(\alpha,k,r) &< \int_{\sigma}^{\infty}|(p)_k|\frac{(1+x)^{\alpha}x^ke^{r\pi|\mathrm{Im}\,p|}}{[v^2+w^2]^{\frac{1}{2(\mathrm{Re}\,p+k)}}}dx\\ &< \int_{\sigma}^{\infty}|(p)_k|\frac{(1+x)^{\alpha+k}e^{r\pi|\mathrm{Im}\,p|}}{(\sqrt{2}v)^{\mathrm{Re}(p+k)}}dx \qquad (3.8)\end{aligned}$$

and , therefore, (3.8) assumes the expression

$$I_2^*\left(\alpha,k,r\right)<\frac{\left|(p)_k\right|-e^{\pi|\operatorname{Im}p|r}\left(1+x\right)^{(a+k)r+1}}{\left((a+k)\,r+1\right)\left(\sqrt{2}\nu\right)^{\operatorname{Re}(p+k)}}\tag{3.9}$$

provided $r\alpha+rk+1<0$.

The right hand side part of (3.9) is a positive value for any non-negative integer k, $\alpha\leq\operatorname{Re}p$. Thus, combining (3.7), (3.9), Multiplying by $\frac{1}{a_\alpha a^\alpha}$, (3.6) can be formed as

$$\left\|(1+x)^\alpha x^kD^k\left(z+x\right)^{-p}\right\|_{L^r}<Ca_\alpha a^\alpha\tag{3.10}$$

where

$$C=\left[\frac{(1+\nu)^{\alpha r}}{\nu^{[r(\operatorname{Re}p)+1]}}-\frac{(1+\nu)^{\alpha r+kr+1}}{\left(\alpha r+kr+1\right)\left(\sqrt{2}\nu\right)^{\operatorname{Re}(p+k)}}\right]\left|(p)_k\right|e^{\pi|\operatorname{Im}p|r}$$

Letting α traverse the set of real numbers such that $\alpha\leq\operatorname{Re}p$, $x\in(0,\infty)$ we have

$$\|\varphi\|_{r,a}<\infty$$

for any $a>0$. To complete the proof of the theorem it remains to be shown that the theorem holds true for $\sigma<0$. For this end assume $\sigma=-\beta,\beta>0$

Let $\gamma>\beta>0$. We have

$$\int_0^\infty\left|(1+x)^\alpha x^kD^k\left(z+x\right)^{-p}\right|^r dx=\left\{\int_0^\gamma+\int_\gamma^\infty\right\}\left|(1+x)^\alpha x^kD^k\left(z+x\right)^{-p}\right|^r dx$$

Similar proof to the proof considered in the Cases I and II yields

$$\begin{aligned}&\int_0^\infty\left|(1+x)^\alpha x^kD^k\left(z+x\right)^{-p}\right|^r dx\\&<\left[\frac{(1+\gamma)^{\alpha r}\gamma^{kr+1}}{|w|^{r(\operatorname{Re}p)}}-\frac{(1+\gamma)^{(\alpha+k)r+1}}{|w|^{r(\operatorname{Re}p+k)}\left(\alpha r+kr+1\right)}\right]\left|(p)_k\right|e^{\pi|\operatorname{Im}p|}\end{aligned}\tag{3.11}$$

provided $\alpha r+kr+1<0$.

Allowing α traverse the set of real numbers which are less or equal to $\operatorname{Re}p$ and $x\in(0,\infty)$ implies

$$\|\varphi\|_{r,a}<\infty$$

for any $a>0$.

Thus, the theorem is completely proved. □

Due to analysis employed above we state

Theorem 3.2. *For any complex number p , neither zero nor negative real, we have*

$$\left(z+x\right)^{-p}\in S_\alpha^{L^r,(b_j),b}\left(S_{\alpha,(a_i),a}^{L^r,(b_j),b}\right)$$

where $\alpha r+kr+1<0,\alpha\leq\operatorname{Re}p,k=0,1,2,...,$ and $x\in(0,\infty)$.

Proof. The proof is simillar to that in theorem 3.1 □

Definition 3.3. *Let* $f \in S'^{L^r}_{\alpha,(a_i),a}, S'^{L^r,(b_j),b}_{\alpha}$ *and* $S'^{L^r,(b_j),b}_{\alpha,(a_i),a}$, $\alpha \leq \operatorname{Re} p$. *By virtue of Theorem* 3.1 *and* 3.2 *we define the ultradistributional Stieltjes transform of slow growth of Beurling-type to be the map of* z *such that*

$$F(z) = \left\langle f(x), (z+x)^{-p} \right\rangle \tag{3.12}$$

where, $\alpha r + kr + 1 < 0, k = 0, 1, 2, ...$

Proofs of Theorem 3.1 and Theorem 3.2 for ultradifferentiable functions of Roumieu-type are similar and thus, the generalized Stieltjes transform of Roumieu-type $f \in S'^{L^r}_{\alpha,(a_i),a}, S'^{L^r,(b_j),b}_{\alpha}$ and $S'^{L^r,(b_j),b}_{\alpha,(a_i),a}$, is justified and can be defined as

$$S(z) = \left\langle f(x), (z+x)^{-p} \right\rangle$$

for some constants $a, b > 0$.

4. **Multiplicity of Ultradistributions**

Definition 4.1. *Denote by* $M^{L^r}_{(a_i),a}$ *the set of complex-valued infinitely smooth* $\theta(x)$, *on* $I(0,\infty)$, *such that* $\left\| (1+x)^{\ell} D^i \theta(x) \right\|_{L^r} \leq C a^i a_i$ *for some positive constant* C *and arbitrary* ℓ *and* a .

Theorem 4.2. *Let* $\theta \in M^{L^u}_{(a_i),a}$ *and* $\phi \in S^{L^\nu}_{\alpha,(a_i),a}(0,\infty)$. *Let* $u \geq r$ *and* $\frac{1}{u} + \frac{1}{v} = \frac{1}{r}$. *Then*

$$\phi \to \theta\phi$$

maps $S^{L^v}_{\alpha-\ell,(a_i),a}(0,\infty)$ *into* $S^{L^r}_{\alpha,(a_i),a}(0,\infty)$, *continuously* .

Proof. With the aid of Leibnitz' rule and the triangle inequality we have

$$\begin{aligned} \left|(1+x)^{\alpha} x^k D^k(\theta\phi)(x)\right| &\leq \left|(1+x)^{\alpha} x^k \sum_{j=0}^{k} D^j\theta(x) D^{k-j}\phi(x)\right| \\ &\leq \sum_{j=0}^{k} (1+x)^{\ell} \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\phi(x)\right| \end{aligned} \tag{4.1}$$

Therefore ,

$$\left|(1+x)^{\alpha} x^k D^k(\theta\phi)(x)\right|^r \leq \left|\sum_{j=0}^{k} (1+x)^{\ell} \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\theta(x)\right|\right|^r .$$

Hence,

$$\int_0^\infty \left|(1+x)^\alpha x^k D^j (\theta\phi)(x)\right|^r dx \tag{4.2}$$

$$\leq \int_0^\infty \left|\sum_{j=0}^k (1+x)^\ell \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\theta(x)\right|\right|^r dx$$

i.e

$$\left(\int_0^\infty \left|(1+x)^\alpha x^k D^k (\theta\phi)(x)\right|^r dx\right)^{\frac{1}{r}} \tag{4.3}$$

$$\leq \left(\int_0^\infty \left|\sum (1+x)^\ell D^j\theta(x) (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\phi(x)\right|\right|^r dx\right)^{\frac{1}{r}}$$

Employing Minkowski for (4.2) yields

$$\left\|(1+x)^\alpha x^k D^k (\theta\phi)(x)\right\|_{L^r} \tag{4.4}$$

$$\leq \sum_{j=0}^k \left(\int_0^\infty \left|(1+x)^\ell \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\phi(x)\right|\right|^r dx\right)^{\frac{1}{r}}$$

$$\leq \sum_0^k \left\|(1+x)^\ell \left|D^j\theta(x) (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\phi(x)\right|\right|\right\|_{L^r}$$

By using the Hölder's inequality we find

$$\int_0^\infty \left|\left\{(1+x)^\ell \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k \left|D^{k-j}\phi(x)\right|\right\}^r\right| dx \tag{4.5}$$

$$\leq \left(\int_0^\infty \left|(1+x)^\ell D^j\theta(x)\right|^{rr'} dx\right)^{\frac{1}{r'}} \left(\int_0^\infty \left|(1+x)^{\alpha-\ell} x^k D^{k-j}\phi(x)\right|^{r r''} dx\right)^{\frac{1}{r''}}$$

where $\frac{1}{r'} + \frac{1}{r''} = 1, r' \geq 1.$

Therefore, from(4.5) we have

$$\left\|(1+x)^\ell \left|D^j\theta(x)\right| (1+x)^{\alpha-\ell} x^k D^{k-j}\phi(x)\right\|_{L^r} \tag{4.6}$$

$$\leq \left\|(1+x)^\ell D^j\theta(x)\right\|_{L^{rr'}} \left\|(1+x)^{\alpha-\ell} x^k D^{k-j}\phi(x)\right\|_{L^{rr''}}$$

Setting $u = rr'$ and $v = rr''$ implies $\frac{1}{u} + \frac{1}{v} = \frac{1}{r}$ and $r' \geq 1$ implies $u \geq r$.

Putting (4.6) into (4.4), then Definition 3.3 and multiplying (4.4) by $1 \setminus a^{\alpha} a_{\alpha}$ leads to the relation

$$\frac{\left\| (1+x)^{\alpha} x^k D^k (\theta\phi)(x) \right\|_{L^r}}{a^{\alpha} a_{\alpha}} \leq \sum_{j=0}^{k} C a^j a_j \frac{\left\| (1+x)^{\alpha-\ell} x^k D^{k-j} \phi(x) \right\|_{L^v}}{a^{\alpha} a_{\alpha}} = \sum_{j=0}^{k} C a^j a_j a^{-\ell} \frac{\left\| (1+x)^{\alpha-\ell} x^k D^{k-j} \phi(x) \right\|_{L^v}}{a^{\alpha-\ell} a_{\alpha}} \tag{4.7}$$

.From $[4, (1.2)]$we have

$$a_{\alpha-\ell} \leq a_0 a_\ell / a_\alpha.$$

Using this in (4.7) leads to

$$\frac{\left\| (1+x)^{\alpha-\ell} x^k D^k (\theta\phi)(x) \right\|_{L^r}}{a^{\alpha} a_{\alpha}} \leq C \sum_{j=0}^{k} \frac{C a^j a_j}{a^{\ell}} \frac{a_0}{a_\ell} \frac{\left\| (1+x)^{\alpha-\ell} x^k D^{k-j} \phi(x) \right\|_{L^v}}{a^{\alpha-\ell} a_{\alpha-\ell}}$$

Considering the supremum over all $x \in (0, \infty)$ and $\alpha \leq \operatorname{Re} p$ yields

$$\|\theta\phi\|_{r,a} \leq C \sum_{j=0}^{k} \frac{a^j a_j a_0}{a^{\ell} a_{\alpha}} \|\phi\|_{v,a}. \tag{4.8}$$

Thus, from (4.8) we observe that $(\theta\phi_n) \to 0$, as $n \to \infty$ in the topology of $S^{L^r}_{\alpha-\ell,(a_i),a}$ when $\phi_n \to 0$ in the topology of $S^{L^v}_{\alpha,(a_i),a}$. This completes the proof of the theorem . □

The theorem, above, suggests a space of operators for multiplication for a space $M^{L^r}_{(a_i),a}$ of Beurling-type ultradifferentiable functions of rapid descents in $S^{L^r}_{\alpha-\ell,(a_i),a}$ for the Stieltjes transforms . However the theorem includes no evident that $M^{L^r}_{(a_i),a}$ admits all multipliers of such spaces .

Definition 4.3. *To define multiplication for* $S^{L^v,(b_j),b}_{\alpha}$ $\left(S^{L^v,(b_j),b}_{\alpha,(a_i),a}\right)$ *denote, respectively, by* $M^{L^u,(b_j),b}$ $\left(M^{L^u,(b_j),b}_{(a_i),a}\right)$ *the spaces of all infinitely smooth* θ *such that*

$$(I) \left\| (1+x)^{\ell} D^j \theta(x) \right\|_{L^r} \leq C b^j b_j$$

and

$$(II) \left\| (1+x)^{\ell} D^j \theta(x) \right\|_{L^r} \leq C a^j b^j a_j b_j$$

with arbitrary constants a *and* b*. Then ,*

Theorem 4.4. *Let* $\theta \in M^{L^u,(b_j),b}\left(M^{L^u,(b_j),b}_{(a_i),a}\right)$ *and* $\phi \in S_\alpha^{L^v,(b_j),b}\ \left(S^{L^v,(b_j),b}_{\alpha,(a_i),a}\right)$*then if* $u \geq r$ *and* $\frac{1}{u}+\frac{1}{v}=\frac{1}{r}$ *the mapping*

$$\phi \rightarrow \theta\phi$$

resp., maps $\phi \in S_{\alpha-\ell}^{L^v,(b_j),b}\ \left(S^{L^v,(b_j),b}_{\alpha-\ell,(a_i),a}\right)$ *into* $\phi \in S_\alpha^{L^r,(b_j),b}\ \left(S^{L^r,(b_j),b}_{\alpha,(a_i),a}\right)$ *continuously .*

Proof. The proof is analogous to that of theorem 4.2 and thus avoided. □

Having permitted the constants a and b to be suitably selected in Definition 4.1 and 4.3 defines spaces $M^{L^r}_{\{a_i\},a}, M^{L^u,\{b_j\},b}$ and $M^{L^u,\{b_j\},b}_{\{a_i\},a}$ in the sense of Roumieu which leads to the following results

Theorem 4.5. *Let* $\theta \in M^{L^u}_{\{a_i\},a}$ *and* $\phi \in S^{L^v}_{\alpha,\{a_i\},a}$*. The map* $\phi \rightarrow \theta\phi$ *is continuous from* $S^{L^v}_{\alpha-\ell,\{a_i\},a}$ *into* $S^{L^r}_{\alpha,\{a_i\},a}$ *where* $u \geq r$ *and* $\frac{1}{u}+\frac{1}{v}=\frac{1}{r}$.

Theorem 4.6. *Let* $\theta \in M^{L^u,(b_j),b}\left(M^{L^u,(b_j),b}_{(a_i),a}\right)$ *and* $\phi \in S_\alpha^{L^v,(b_j),b}\ \left(S^{L^v,\{b_j\},b}_{\alpha,(a_i),a}\right)$ *then if* $u \geq r$ *and* $\frac{1}{u}+\frac{1}{v}=\frac{1}{r}$ *the mapping*

$$\phi \rightarrow \theta\phi$$

maps $\phi \in S_{\alpha-\ell}^{L^v,(b_j),b}\ \left(S^{L^v,\{b_j\},b}_{\alpha-\ell,\{a_i\},a}\right)$ *into* $\phi \in S_\alpha^{L^r,\{b_j\},b}\ \left(S^{L^r,\{b_j\},b}_{\alpha,\{a_j\},a}\right)$ *continuously .*

The poof of Theorems 4.5 and 4.6 is similar to that considered for Theorem 4.2.

5. Fourier Transform of Ultradifferentiable Boehmians

The concept of Boehmians is motivated by regular operators introduced by Boehme [7]. Boehmians have an algebraic character of Mikusinski operators and at the same time do not have restriction on the support. Applying the general construction of Boehmians to various function spaces yields various spaces of Boehmians. General Boehmians contain the Schwartz space of distributions , Roumieu ultradistributions, regular operators and tempered distributions as well.

The construction of Boehmians is similar to the construction of field of quotients. The space of Boehmians we consider in this note contains the space of the tempered Boehmians [1] which in turn contains the space of tempered distributions and, the obtained definition of the Fourier transform coincides with the definition of the Fourier transform of the locally integrable Boehmian which appears in [12] .

Definition 5.1. *An infinitely smooth function* ϕ *is said to be in* $S_{a_i,a}(\mathbb{R})$ *if for some positive constants* a *and* A *its derivatives are estimated by*

$$\left|x^i D^j \phi(x)\right| \leq A(a+\alpha)^i a^i$$

for all $i, j \in \mathbb{N}_0$*, and* $\alpha > 0$ *being arbitrary constant. Elements in* $S_{a_i,a}(\mathbb{R})$ *are ,indeed ,ultradifferentiable functions of rapid descent. By* D *denote the space of all infinitely* *differentiable complex-valued functions on* $\mathbb{R}$ *with bounded support. A delta sequence* Δ *is a sequence of real-valued functions* $\delta_1, \delta_2, ..., \delta_n \in D$ *such that*

$i-\int \delta_n(x)\,dx = 1$ *for all* $n \in \mathbb{N}$ *.*

$ii-\int |\delta_n(x)|\,dx \le M$ *for some* $M > 0, n \in \mathbb{N}$ *.*

iii- for every $\varepsilon > 0$ *there is* $n_0 \in \mathbb{N}$ *such that* $\delta_n(x) = 0$ *for* $|x| \ge \varepsilon$ *and* $n > n_0$*.*

The convolution product of two functions f *and* g *is defined by*

$$(f * g)(x) = \int_{\mathbb{R}} f(u)\, g(x-u)\, du$$

whenever the integral exists. A pair of sequences (f_n, φ_n) *, or* f_n/φ_n *,is said to be a quotient of sequences if* $f_n \in S_{a_i,a}, (\varphi_n) \in \Delta$ *, and* $f_n * \varphi_m = f_m * \varphi_n$ *for all* $m, n \in \mathbb{N}$*.*

Two quotients of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n * \psi_m = g_m * \varphi_n$ for all $m, n \in \mathbb{N}$. The equivalence class of f_n/φ_n is denoted by $[f_n/\varphi_n]$.The space of all equivalence classes of quotients is denoted by $B_{s_{a_i},a}$. Its elements are called ultradifferentiable Boehmians of rapid descents. The space $B_{s_{a_i},a}$, indeed, contains the space of tempered Boehmians defined in [1]. Properties of addition and multiplication and, convergence and the same as in [1, 12]. Retaining the set of delta sequences, concept of quotients and equivalence classes we construct, similarly, a space of C^∞ - Boehmians B_{C^∞}.

Lemma 5.2. *Let* f_n/φ_n *be a quotient in* $B_{s_{a_i},a}$ *.Then* $\hat{f}_n/\hat{\varphi}_n$ *is a quotient in* B_{C^∞} *.*

Proof. f_n/φ_n quotient in $\hat{B}_{s_{a_i},a}$ implies $f_i * \varphi_j = f_j * \varphi_i$ for all $i, j \in \mathbb{N}$. Employing the Fourier transform to both sides implies $\hat{f}_i \hat{\varphi}_j = \hat{f}_j \hat{\varphi}_i$ for all $i, j \in \mathbb{N}$. This completes the proof of the theorem. □

Let the Boehmian $B = [f_n/\varphi_n] \in B_{s_{a_i},a}$ then $\hat{B} = \hat{f}_n/\hat{\varphi}_n$. Since $\varphi_n \in \Delta, \hat{\varphi}_n \to 1$ as $n \to \infty$.Hence, the defined Fourier transform of the ultradifferentiable Boehmian B can simply be simplified to

$\hat{B} = \lim \hat{f}_n$.

References

(1) . K. Q. Al-Omari, D. Loonker, P.K. Banerji and S. L. Kalla, Fourier sine(cosine) transform for ultradistributions and their extensions to tempered and ultra-Boehmian spaces, Integ. Trans. Spl. Funct. 19(6) (2008), 453 - 462.

(2) S. K. Q. Al-Omari, Certain Class of Kernels for Roumieu-Type Convolution Transform of Ultradistribution of Compact Growth, J. Concr. Appli. Math. 7(4) (2009), 310-316.

(3) S. K. Q. Al-Omari , Cauchy and Poison Integrals of Tempered Ultradistributions of Roumieu and Beurling Types , J. Concr. Appli. Math. 7(1)(2009), 36-47.

(4) P. K. Banerji and S. K. Q. Al-Omari, Multipliers and Operators on the Tempered Ultradistribution Spaces of Roumieu Type for the Distributional Hankel-type transformation spaces, Internat. J. Math. Math. Sci.(2006), Article ID 31682 , 1-7.
(5) P. K. Banerji, S. K. Alomari and L. Debnath,Tempered Distributional Fourier Sine(Cosine)Transform, Integ. Trans. Spl. Funct. 17(11)(2006), 759-768.
(6) A. Beurling, Quasi-analiticity and generalized distributions, Lectures 4 and 5, A. M. S. Summer Institute, Stanford, 1961.
(7) T. K. Boehme, The Support of Mikusinski Operators, Tran. Amer. Math. Soc. 176(1973), 319-334.
(8) A. Byrne and E. R. Love, Complex Inversion Theorems for Generalized Stieltjes Transforms , J. Austral. Math. Soc. 18(1974), 328-358.
(9) A. Erdélyi, The Stieltjes Transforms of Generalized Functions, Proc. Roy. Soc. Edinburgh, 76A(1977), 231-249.
(10) I. M. Gel'fand, and G. E. Shilov., Generalized Functions, Vol. 2. Academic Press , New York, 1968.
(11) E. R. Love and A. Byrne , Real inversion Theorems for Stieltjes Transforms, J. London Math. Soc. 2(22)(1980), 285-306.
(12) P. Mikusinski, Fourier Transform for Integrable Boehmians, Rocky Mountain J. Math. 17(3)(1987), 577-582.
(13) J. N. Pandy, On the Stieltjes Transform of Generalized Functions , Proc. Cambridge Phil. Soc. 71(1972), 85-96.
(14) R. S. Pathak, Integral transforms of generalized functions and their applications,Gordon and Breach Science Publishers, Australia, Canada, India, Japan, 1997.
(15) R. S. Pathak, Distributional Stieltjes Transformation ,Proc. Edinburgh Math. Soc. Ser., 20 (1)(1976), 15-22.
(16) H. Pollard , Studies on the Stieltjes Transform ,Bull. Amer. Math. Soc. 48(1942), 214.
(17) C. Roumieu, Ultradistributions defines sur (R^n) etsurcertaines classea de variétése differetiables, J. d'Analyse Math. 10(1962-63), 153-192.
(18) C. Roumieu, Sur quelques extensions de la notin de distribution, Ann. Ecole Norm. Sup. 77(1960), 41-121.
(19) J. H. Schwartz, The generalized Stieltjes Transform and its Inverse, J. Math. Phys. 46(2005), 013501
(20) D. B. Summer, An inversion Formula for the Generalized Stieltjes Transform, Bull. Amer. Math. Soc. 55(1949), 174-183.
(21) D. V. Widder, The Laplace Transform , Princeton University Press, 1944.
(22) A. H. Zemanian, Distribution Theory and Transform analysis, Dover Publications Inc. New York, 1987.

Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa' Applied University, Jordan
E-mail address: shrideh2001@yahoo.co.in