

THE I_p QUANTUM INVARIANT OF PERIODIC 3-MANIFOLDS

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ABSTRACT. We use the periodicity of the Jones polynomial introduced first by Murasugi to give a congruence that relates the invariant of a periodic 3-manifold and the corresponding invariant of its orbit 3-manifold. This would provide a criterion for the periodicity of 3-manifolds.

1. Introduction

The discovery of the Jones polynomial led to a revolution in the field of knot theory and 3-dimensional manifolds. Since then, many invariants of knots and 3-manifolds have been discovered and many unanswered questions showed up in this field. We like to explore the relation between the I_p -invariants for a periodic 3-manifold and its orbit manifold.

Let r be an odd prime, and let G be the finite cyclic group \mathbb{Z}_r . In [8], Gilmer was interested in studying the relation between the $SU(2)$ -invariants of r -periodic 3-manifolds and their quotient manifolds. He obtained a congruence relating these invariants. His result was obtained by using the trace formula of topological quantum field theory and studying Gaussian sums. Chbili used the results about the Jones polynomial and the Kauffman multi-bracket of r -periodic links to obtain a similar result for rational homology 3-spheres for the $SO(3)$ -invariants in [5]. Also in [4], he gave similar results for the $SU(3)$ and the MOO-invariants. Chen and Le generalized the above results for rational homology spheres using any complex simple Lie algebra in [7].

We generalized all of the above results for a general modular category with an integrally closed ground ring in [10]. In this paper, we give a sharper result in the case of the modular categories derived from the Kauffman bracket by generating the given ideal with only two elements. The surgery descriptions of a periodic 3-manifold and its orbit manifold, obtained in [14], is the first motivation of our result. The second motivation is the regularity of the Kauffman bracket first introduced in [6, 11].

In this paper, we consider the invariant $I_p(M)$ which is a normalization of the invariant that belongs to the TQFT derived from the Kauffman bracket introduced in [1]. It is also

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a normalization of the θ_p invariant that is derived from the Kauffman bracket introduced in [2, 3]. This invariant belongs to the ring of integers of the ring k_p defined by

$$k_p = \begin{cases} \mathbb{Z}[\zeta_{2p}, \frac{1}{p}], & \text{if } p \equiv -1 \pmod{4}; \\ \mathbb{Z}[\zeta_{4p}, \frac{1}{p}], & \text{if } p \equiv 1 \text{ or } 2 \pmod{4}, \end{cases}$$

here and elsewhere ζ_{2p} and ζ_{4p} are $2p^{th}$ and $4p^{th}$ roots of unity respectively for an odd prime or twice an odd prime p . This result was proved first in [13, 12].

We recall the definition of the invariant first introduced in [2, 3]. Let M be a closed oriented 3-manifold. Lickorish and Wallace proved that M can be obtained by surgery on S^3 along a framed link L . We let $b_+(L)$, $b_-(L)$, $b_0(L)$ denote the number of positive, negative, zero eigenvalues of the framing matrix of L .

The Kauffman bracket skein module of the solid torus is known to be generated as an algebra by z over the ring $\mathbb{Z}[A, A^{-1}]$, where z correspond to a standard loop around the hole in the solid torus. This module has a standard basis $\{z^i : i \geq 0\}$. Another basis is given by $\{e_i : i \geq 0\}$, where e_i satisfies the following recursive relation

$$e_{i+1} = ze_i - e_{i-1} \quad \text{for } i \geq 1,$$

and $e_0 = 1, e_1 = z$.

From now on we fix a $2p^{th}$ root of unity A_p . In [2, 3] a special element Ω_p of the Kauffman bracket skein module of the torus was defined, and it is given by $\Omega_p = \sum_{i=0}^{\lfloor \frac{p-3}{2} \rfloor} \langle e_i \rangle e_i$, where $\langle e_i \rangle$ is the Kauffman bracket evaluation of e_i in the plane at the root of unity A_p . Now, the invariant $I_p(M)$ is defined by the following formula

$$I_p(M) = \eta^{b_0(L)} \frac{\langle \Omega_p, \Omega_p, \dots, \Omega_p \rangle_L}{\langle t(\Omega_p) \rangle^{b_+(L)} \langle t^{-1}(\Omega_p) \rangle^{b_-(L)}},$$

where $\langle \Omega_p, \Omega_p, \dots, \Omega_p \rangle_L$ is the Kauffman bracket evaluation of the link L whose components are colored Ω_p in the plane at the root of unity A_p , t is the twist map on the Kauffman bracket skein module of the solid torus, and η is defined by $\langle \Omega_p \rangle^{-1/2}$ as an element of k_p . After substituting $\langle t^\epsilon(\Omega_p) \rangle = \eta^{-1} \kappa^{3\epsilon}$ and $\kappa^6 = A_p^{-6-p(p+1)/2}$ from [1, 2] into the previous formula, we obtain

$$I_p(M) = (\kappa^3)^{b_-(L)-b_+(L)} \langle \eta \Omega_p, \eta \Omega_p, \dots, \eta \Omega_p \rangle_L.$$

We introduce the following definitions that will be used in the rest of this paper.

Definition 1.1. Let $J_r = (r, (-A_p^2 - A_p^{-2})^r - (-A_p^2 - A_p^{-2}))$ be the ideal generated by r and $(-A_p^2 - A_p^{-2})^r - (-A_p^2 - A_p^{-2})$ in the ring k_p , where r is relatively prime to p .

Definition 1.2. Let $[i] = \frac{A_p^{2i} - A_p^{-2i}}{A_p^2 - A_p^{-2}}$. This is called the i^{th} quantum integer.

2. Periodic links and periodic 3-manifolds

As before, we have M a closed oriented 3-manifold that is a result of surgery on S^3 along the framed link L .

Definition 2.1. A framed link L in S^3 is said to be r -periodic if there exists a \mathbb{Z}_r -action on S^3 , with a fixed point set equal to a circle, that maps L to itself under this action and L is assumed to be disjoint from the circle.

Definition 2.2. M is said to be r -periodic if there is an orientation preserving \mathbb{Z}_r -action with fixed point set equal to a circle, and the action is free outside this circle.

Now we list the following two results from [14] that will be used later.

Theorem 2.1. *There is a \mathbb{Z}_r -action on M with a fixed point set equal to a circle iff M can be obtained as a result of surgery on a r -periodic link L in S^3 and \mathbb{Z}_r acts freely on the set of the components of L .*

By the positive solution of the Smith conjecture, we can represent any framed r -periodic link as a closure of some graph such that the rotation of this graph about the z -axis in \mathbb{R}^3 (or the circle in S^3) by $2\pi/r$ leaves it invariant, i.e $L_* = \overline{\Omega}$ (where the bar means the closure of the graph) see figure 1.

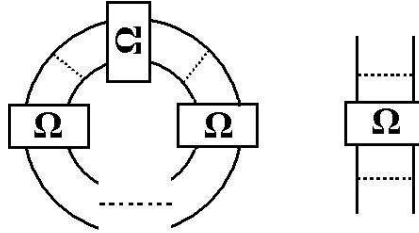


FIGURE 1. Periodic Link and its Quotient

Let $M_* = M/\mathbb{Z}_r$ denote the orbit space, then M_* is obtained by surgery on S^3 along the link $L_* = L/\mathbb{Z}_r$.

Lemma 2.1. *Let L a r -periodic link in S^3 . The following are equivalent*

- (1) \mathbb{Z}_r acts freely on the set of components of L ;
- (2) The linking number of each component of the L_* the axis of the action is congruent to zero modulo r ;
- (3) The number of components of L is equal to r times the number of components of L_* .

3. Main Result

We state the main result of this paper and its consequences and then we give the proofs.

Theorem 3.1. *Suppose that M is an r -periodic 3-manifold with quotient manifold M_* . Then the following congruence holds*

$$I_p(M) \equiv \kappa^{3m} I_p(M_*)^r \pmod{J_r},$$

for some integer m .

Corollary 3.1. *Suppose that M is an r -periodic 3-manifold with quotient manifold M_* , where $r = 2p + 1$. Then the following congruence holds*

$$I_p(M) \equiv \kappa^{3m} I_p(M_*)^r \pmod{r},$$

for the same integer m as above.

The above theorem generalizes the following main result of Chbili in [5] for the case of the $SO(3)$ and the $SU(2)$ -invariants.

Corollary 3.2. [5] *Suppose M is a 3-dimensional rational homology sphere, and that M is r -periodic then*

$$I_p(M) \equiv (A^{-6-p(p+1)/2})^l (I_p(M_*))^r \pmod{J_r},$$

for some integer l .

Our proof relies basically on the main result of [11].

Proposition 3.1. *If L is an r -periodic link, then*

$$\langle L \rangle \equiv \langle L_* \rangle^r \pmod{J_r},$$

where $\langle L \rangle, \langle L_* \rangle$ are the Kauffman bracket of the links L, L_* respectively.

Murasugi's result is for the Jones polynomial, but this is an equivalent form of it because of the relationship between the Jones polynomial and the Kauffman bracket. We need this form to prove the next lemma that uses the following definition.

Definition 3.1. Let L be an r -periodic link, and λ be a coloring of L . If the colored graph $L(\lambda)$ is invariant under the rotation by $2\pi/r$, then λ is called an r -periodic coloring.

Lemma 3.1. *If λ is an r -periodic coloring of an r -periodic link L , then*

$$\langle L(\lambda) \rangle_L \equiv \langle L_*(\lambda_*) \rangle_{L_*}^r \pmod{J_r},$$

where $\lambda_* = \lambda/\mathbb{Z}_r$.

Proof. It follows from the fact that each colored component in the colored link can be replaced by a linear combination of links and the above result of Murasugi. □

Now we state the following lemma that was first introduced by Chbili in [5], and then we give its proof using the last lemma.

Lemma 3.2. *If L be an r -periodic link, then*

$$\langle \Omega_p, \Omega_p, \dots, \Omega_p \rangle_L \equiv \langle \Omega_p, \Omega_p, \dots, \Omega_p \rangle_{L_*}^r \pmod{J_r},$$

and

$$\langle \eta \Omega_p, \eta \Omega_p, \dots, \eta \Omega_p \rangle_L \equiv \langle \eta \Omega_p, \eta \Omega_p, \dots, \eta \Omega_p \rangle_{L_*}^r \pmod{J_r}.$$

Proof. Let us start with any coloring of L say λ , either λ is r -periodic or not. Let us assume that λ is not r -periodic, i.e $L(\lambda)$ is not invariant under the rotation by $2\pi/r$ about the z -axis. Hence the i -th rotation of $L(\lambda)$ (the rotation by $2i\pi/r$) represents a colored graph with the same Kauffman bracket and different coloring denoted by λ_i . So the term with a non-periodic coloring occurs r times. Hence we reduce the summation on the left-hand side to the periodic colorings. Now the result follows from lemma (3.1) and the fact that the periodic colorings of L are in one-to-one correspondence with the colorings of L_* (by restriction). \square

Now we are ready to give the proof of our main statement.

$$\begin{aligned} I_p(M) &= (\kappa^3)^{b_-(L)-b_+(L)} \langle \eta \Omega_p, \eta \Omega_p, \dots, \eta \Omega_p \rangle_L \\ &\equiv \kappa^{3m} (\kappa^3)^{rb_-(L_*)-rb_+(L_*)} \langle \eta \Omega_p, \eta \Omega_p, \dots, \eta \Omega_p \rangle_{L_*}^r \pmod{J_r} \\ &\equiv \kappa^{3m} (I_p(M_*))^r \pmod{J_r}, \end{aligned}$$

where $m = b_-(L) - b_+(L) - rb_-(L_*) + rb_+(L_*)$. For the proof of corollary 3.2, we use the fact $b_+(L) + b_-(L) = rb_+(L_*) + rb_-(L_*)$, and the formula $\kappa^6 = A_p^{-6-p(p+1)/2}$.

Remark 1. The main result of this paper confirms the main result of [15] and it shows that the ideal generated only by two elements as shown below for the case of the modular categories derived from the Kauffman bracket. Also, this result is obtained without using TQFTs and by using only the regularity of the Kauffman bracket.

Lemma 3.3. *We have $[i]^r \equiv [i] \pmod{J_r}$, for $1 \leq i \leq [\frac{p-3}{2}]$. Therefore, the ideal generated by r and $[i]^r - [i]$, for $0 \leq i \leq [\frac{p-3}{2}]$ is equal to the ideal J_r .*

Proof. The result follows from the fact that

$$[i]_p = \begin{cases} \sum_{k=0}^{[\frac{i-1}{2}]} c_k (A^2 + A^{-2})^{2k+1}, & \text{if } i \text{ is odd;} \\ \sum_{k=0}^{[\frac{i}{2}]} c_k (A^2 + A^{-2})^{2k}, & \text{if } i \text{ is even.} \end{cases}$$

for $c_k \in \mathbb{Z}$. \square

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