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EXISTENCE OF BOUNDED SOLUTIONS FOR A NONLINEAR PARABOLIC SYSTEM WITH NONLINEAR GRADIENT TERM

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Abstract. In this note we show the existence of bounded solutions of the nonlinear parabolic system

$$(u_1)_t + \mathcal{A}_1 u_1 = a_1(x)\,|\nabla u_1|^{p_1} + f_1(x, u_1, u_2)$$
$$(u_2)_t + \mathcal{A}_2 u_2 = a_2(x)\,|\nabla u_2|^{p_2} + f_2(x, u_1, u_2)$$

where $\mathcal{A}_i$ is the pseudo-Laplacian operator and a_i, f_i are given functions, $i = 1, 2$. We prove the existence of weak bounded solutions using a priori $L^\infty-$ estimate and the theory of nonlinear parabolic approximation.

1. Introduction

Let Ω be an open and bounded subset in $\mathbb{R}^N$ with smooth boundary Γ and let T be a positive real number. In the cylinder $Q_T = \Omega\times]0, T[$, with lateral boundary $S_T = \Gamma\times]0, T[$, we consider the nonlinear system $(\boldsymbol{S})$

$$\frac{\partial u_1}{\partial t} - \mathcal{A}_1 u_1 = a_1(x)\,|\nabla u_1|^{p_1} + f_1(x, u_1, u_2) \quad (x, t) \in Q_T, \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} - \mathcal{A}_2 u_2 = a_2(x)\,|\nabla u_2|^{p_2} + f_2(x, u_1, u_2) \quad (x, t) \in Q_T, \tag{1.2}$$

$$u_1(x, t) = u_2(x, t) = 0 \quad (x, t) \in S_T, \tag{1.3}$$

$$(u_1(x, 0), u_2(x, 0)) = (u_{10}(x), u_{20}(x)) \quad x \in \Omega, \tag{1.4}$$

where

$$\mathcal{A}_i u = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\Big(|\frac{\partial u}{\partial x_j}|^{p_i-2}\frac{\partial u}{\partial x_j}\Big), \quad p_i \geq 2,\ i = 1, 2$$

is the pseudo-Laplacian operator. Precise conditions on a_i, f_i and u_{i0} will be given later.

The prototype systems $(\boldsymbol{S})$ is only weakly coupled in the reaction terme f_i' s and turns up in many mathematical settings as non-Newtonian fluids, nonlinear filtration, population evolution, reaction diffusion problems, porus media and so forth. Therefore, it is important to obtain information about the existence of solutions for this problem.

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When $a_i \equiv 0$, much attention has been given to the existence and the regularity of solutions of systems ($\boldsymbol{S}$), by using different approaches (see, for exemple,[13, 14, 15] and references therein.

The case of a single equation of the type ($\boldsymbol{S}$) is studied in [10, 11]. The purpose of this paper is the natural extension to system ($\boldsymbol{S}$) of the result by [5], which concerns the single equation $\frac{\partial u}{\partial t} - \Delta_p u = d\,|\nabla u|^p + f(x,t)$.

We organize this paper in the following framework. In Section 2, we define a concept of weak solution and present our main result. The proof of the weak bounded solutions existence is postponed to Section 3 and is obtained as the limit of approximated solutions corresponding to regularized problems.

Notation. We represent the Sobolev space of order m in Ω by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \forall |\alpha| \le m\},$$

with the norm

$$\|u\|_{m,p} = \Big(\sum_{|\alpha| \le m} |D^\alpha u|^p_{L^p(\Omega)} \Big)^{1/p}, u \in W^{m,p}(\Omega), 1 \le p < \infty.$$

Let $\mathcal{D}(\Omega)$ be the space of test functions in Ω and by $W_0^{m,p}(\Omega)$ we represent the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. The dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$ with p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. We use the symbols $(\cdot,\cdot)$ and $|\cdot|$, to indicate the inner product and the norm in $L^2(\Omega)$. We use $\langle \cdot,\cdot \rangle_{W^{-1,p}(\Omega), W_0^{1,p}(\Omega)}$ to indicate the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $\|\cdot\|_0$ to indicate the norm $W_0^{1,p}(\Omega)$.

The pseudo-Laplacian operator $\mathcal{A}_i$ is such that for $i = 1, 2$

$$\begin{array}{rccc} \mathcal{A}_i : & W_0^{1,p}(\Omega) & \to & W^{-1,p_i'}(\Omega) \\ & u & \mapsto & \mathcal{A}_i u \end{array}$$

and it satisfies the following properties:

- $\mathcal{A}_i$ is monotonic, that is, $\langle \mathcal{A}_i u - \mathcal{A}_i v, u - v \rangle \ge 0, \forall u, v \in W_0^{1,p}(\Omega)$;
- $\mathcal{A}_i$ is hemicontinuous, that is, for each $u, v, w \in W_0^{1,p}(\Omega)$ the function $\lambda \mapsto \langle \mathcal{A}_i(u + \lambda v), w \rangle$ is continuous in $\mathbb{R}$;
- $\langle \mathcal{A}_i u(t), u(t) \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \|u\|_0^p$;
- $\langle \mathcal{A}_i u(t), u'(t) \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \frac{1}{p}\frac{d}{dt}\|u\|_0^p$;
- $\|\mathcal{A}_i u(t)\|_{W^{-1,p'}(\Omega)} \le C\|u\|_0^{p-1}$, where C is a constant.

2. **Assumptions and main results**

First we specify our notion of weak solution.

Definition 2.1. *A pair $(u_1,\ u_2)$ is said to be a weak solution of $(\boldsymbol{S})$ if for $i=1,2$*

$$u_i \in C(0,T;L^2(\Omega));$$
$$\frac{\partial u_i}{\partial t} \in L^{p_i'}(0,T;W^{-1,p_i'}(\Omega)) + L^1(Q_T);$$
$$\int_\Omega u_i(x,\tau)w_i(x,\tau)dx - \int_\Omega u_i(x,0)w_i(x,0)dx - \int_0^\tau \langle v_{i1}, u_i\rangle dt - \int_0^\tau \int_\Omega v_{i2}u_i dxdt +$$
$$\int_0^\tau \int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \nabla w_i dxdt = \int_0^\tau \int_\Omega a_i(x) |\nabla u_i|^{p_i} w_i dxdt + \int_0^\tau \int_\Omega f_i(x,u_1,u_2)w_i dxdt.$$
$$\forall \tau \in [0,T], \forall w_i \in L^\infty(\Omega\times(0,\tau)) \cap L^{p_i}(0,\tau;W_0^{1,p_i}(\Omega))$$
$$\text{and} \quad \frac{\partial w_i}{\partial t} = v_{i1} + v_{i2} \in L^{p'}(0,\tau;W^{-1,p'}(\Omega)) + L^1(\Omega\times(0,\tau)).$$

We consider the following assumptions on the data:

(H1) $p_i \in [2,N[\ ,(i=1,2)$.

(H2)$u_{i0} \in L^{+\infty}(\Omega),(i=1,2)$.

(H3)$a_i \in L^\infty(\Omega),(i=1,2)$.

(H4) $f_i \in C^1(\Omega\times\mathbb{R}\times\mathbb{R}),(i=1,2)$.

The next lemma plays a central role in the proof of the existence theorem. Its proof can be found in $[D-G-S]$.

Lemma 2.2. *For every $\beta, f \in L^\infty(\Omega)$, $0 \le \beta(s) \le M, \forall s \in \mathbb{R}$ and $\alpha \in [2,N]$, the problem*
$$u_t - div(|\nabla u|^{\alpha-2}\nabla u) = \beta(u)|\nabla u|^\alpha + f, \quad u=0 \text{ on } \partial\Omega,$$
possesses a solution u such that $u \in L^\infty(Q_T) \cap L^\infty(0,T;L^2(\Omega)) \cap L^\alpha(0,T;W_0^{1,\alpha}(\Omega))$.

3. **Existence of weak bounded solutions**

Our main result is the following:

Theorem 3.1. *Let (H1) to (H4) be satisfied. Then there exists at least one weak bounded solution (u_1,u_2) of problem (S) such that for $i=1,2$, we have $u_i \in L^{p_i}(0,T;W_0^{1,p_i}(\Omega)) \cap C(0,T;L^{q_i}(\Omega)) \cap L^\infty(Q_T)$, for all $q_i \in [1,+\infty)$.*

We will prove Theorem 3.1 by means of nonlinear parabolic regularization.

Starting from a suitable initial iteration $(u_1^0,u_2^0) = (u_{10},u_{20})$, we construct a sequence $\{(u_1^n,u_2^n)\}_{n=1}^\infty$ from the iteration process

$$\frac{\partial u_i^n}{\partial t} - \mathcal{A}_i u_i^n = a_i(x) \min\{|\nabla u_i^n|^{p_i}, n\} + f_i(x, u_1^{n-1}, u_2^{n-1}) \quad (x,t) \in Q_T, \tag{3.1}$$

$$u_i^n(x,t) = 0 \quad (x,t) \in S_T, \tag{3.2}$$

$$u_i^n(x,0) = u_{i0}(x) \quad x \in \Omega \tag{3.3}$$

where $i = 1, 2$. It is clear that for each $n = 1, 2, ...,$ the above systems consists of two uncoupled initial boundary-value problems. By classical results, the existence of weak solution $u_i^n \in C(0,T; L^2(\Omega)) \cap L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$ follows from [17] By lemma2.2 we assert that

$$u_i^n \in L^\infty(Q_T), i = 1, 2. \tag{3.4}$$

To find a limit function $(u_1(x,t), u_2(x,t))$ of $(u_1^n(x,t), u_2^n(x,t))$ we will divide our proof in the following four lemmas.

Lemma 3.2. *There exist a constant c independent of n such that for $\tau \in [0,T]$*

$$\sup_{0 \le \tau \le T} \int_\Omega |u_i^n(x,\tau)|^2 \, dx + \int_0^T \int_\Omega |\nabla u_i^n|^{p_i} \, dxdt \le c. \tag{3.5}$$

Proof. Since $u_i^n \in L^\infty(Q_T) \cap L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$, $\sinh(\lambda u_i^n) \in L^\infty(Q_T) \cap L^{p_i}(0,T; W_0^{1,p_i}(\Omega))$ $(\lambda = \max(\|a_1\|_\infty, \|a_2\|_\infty))$ is a testing function for (3.1). For each $\tau \in [0,T]$, we derive

$$\int_\Omega \int_0^{u_i^n(x,\tau)} \sinh(\lambda s) ds dx + \lambda \int_0^\tau \int_\Omega \cosh(\lambda u_i^n) |\nabla u_i^n|^{p_i} \, dxdt \le$$

$$\lambda \int_0^\tau \int_\Omega |\sinh(\lambda u_i^n)| \, |\nabla u_i^n|^{p_i} \, dxdt + \int_0^\tau \int_\Omega |\sinh(\lambda u_i^n)| \left| f_i(x, u_1^{n-1}, u_2^{n-1}) \right| dxdt$$

$$+ \int_\Omega \int_0^{u_{i0}(x)} \sinh(\lambda s) ds dx. \tag{3.6}$$

It is not difficult to cheek that

$$\int_0^w \sinh(\lambda s) ds = \frac{1}{2\lambda} [\exp(\lambda s) + \exp(-\lambda s)]_0^w = \frac{1}{\lambda} [\cosh(\lambda w) - 1] \ge \frac{\lambda}{2} (w)^2, \tag{3.7}$$

$$\cosh(s) \ge |\sinh(s)|, \cosh(s) \ge 1. \tag{3.8}$$

By (3.4), (H2), $(H4)$, (3.7), (3.8), we obtain

$$\int_\Omega |u_i^n|^2 (x,\tau) dx + \int_0^T \int_\Omega |\nabla u_i^n|^{p_i} \, dxdt \le c_i(T). \tag{3.9}$$

Taking the supremum for $\tau \in [0,T]$, we have, for $\forall n \in \mathbb{N}$

$$\sup_{0 \le \tau \le T} \int_\Omega |u_i^n|^2 (x,\tau) dx + \int_0^T \int_\Omega |\nabla u_i^n|^{p_i} \, dxdt \le c_i(T). \tag{3.10}$$

This proves (3.5). By lemma 3.3, there exit a subsequence $\{u_i^n\}, i = 1, 2$ (denoted again by $\{u_i^n\}$) and a function $u_i(x,t) \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$ such that as $n \to +\infty$,

$$u_i^n \rightharpoonup u_i \text{ weakly in } L^{p_i}(0,T; W_0^{1,p_i}(\Omega)); \tag{3.11}$$

$$u_i^n \rightharpoonup u_i \text{ weakly * in } L^\infty(Q_T), \tag{3.12}$$

and u_i satifies (3.4) and (3.5) by the weak lower semicontinuity.

Lemma 3.3.

$$u_i^n \rightharpoonup u_i \textit{ strongly in } L^{p_i}(Q_T); \tag{3.13}$$

$$u_i^n \rightharpoonup u_i \textit{ a.e. in } Q_T. \tag{3.14}$$

Proof.
We have $\frac{\partial u_i^n}{\partial t} = \mathcal{A}_i u_i^n + \left[a_i(x) \min\{|\nabla u_i^n|^{p_i}, n\} + f_i(x, u_1^{n-1}, u_2^{n-1})\right].$
By (3.4) and (3.10) we derive

$$\|\mathcal{A}_i u_i^n\|_{L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega))} \leq c; \tag{3.15}$$

$$\left\|a_i(x) \min\{|\nabla u_i^n|^{p_i}, n\} + f_i(x, u_1^{n-1}, u_2^{n-1})\right\|_{L^1(Q_T)} \leq c. \tag{3.16}$$

By virtue of lemma 4.2 in , [5] we obtain

$$u_i^n \rightharpoonup u_i \text{ strongly in } L^{p_i}(Q_T). \tag{3.17}$$

Taking a subsequence of $\{u_i^n\}, i = 1, 2$ (denoted again by $\{u_i^n\}$) further, we have

$$u_i^n \rightharpoonup u_i \text{ a.e. in } Q_T. \tag{3.18}$$

By Vitali's theorem, we have, for any $r \in (1, +\infty)$,

$$f_i(., u_1^n, u_2^n) \rightharpoonup f_i(., u_1, u_2) \text{ strongly in } L^r(Q_T). \tag{3.19}$$

Lemma 3.4. $\nabla u_i^n \rightharpoonup \nabla u_i$ *a.e. in* Q_T.

Proof.
Fix $\mu, \varepsilon > 0$. Due to (3.18), Egoroff's theorem implies that there exists a measurable set $A_\varepsilon \subset Q_T$ such that $L^{N+1}(Q_T \backslash A_\varepsilon) \leq \varepsilon$ and $u_i^n \rightharpoonup u_i$ uniformly on A_ε, which follows that

$$|u_i^n \rightharpoonup u_i^m| < \mu \text{ on } A_\varepsilon \tag{3.20}$$

if $n, m > M$. Let ξ be a cutoff function such that $\xi \equiv 1$ on A_ε, $spt\xi \subset Q_T$.
By subtracting $(3.1)_n$ and $(3.1)_m$ we have

$$\frac{\partial u_i^n}{\partial t} - \frac{\partial u_i^m}{\partial t} - (\mathcal{A}_i u_i^n - \mathcal{A}_i u_i^m) =$$

$$a_i(x) \left(\min\{|\nabla u_i^n|^{p_i}, n\} - \min\{|\nabla u_i^m|^{p_i}, m\}\right)$$

$$+f_i(x,u_1^{n-1},u_2^{n-1})-f_i(x,u_1^{m-1},u_2^{m-1}). \tag{3.21}$$

Choosing a testing function $\xi T_\varepsilon(u_i^n-u_i^m)=\xi\max\{-\varepsilon,\min\{(u_i^n-u_i^m),\varepsilon\}\}$ for (3.21) and noting that T_ε is an odd function satisfying $|T_\varepsilon|\le\varepsilon$, we conclude that

$$\begin{aligned}&\int_{Q_T}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)\xi T_\varepsilon'(u_i^n-u_i^m)\ dz\\&\le\int_{Q_T}(u_i^n-u_i^m)T_\varepsilon(u_i^n-u_i^m)\xi_t dz\\&+\int_{Q_T}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).\nabla\left(\xi T_\varepsilon(u_i^n-u_i^m)\right)\ dz\\&+\int_{Q_T}a_i(x)\left(\min\{|u_i^n|^{p_i},n\}-\min\{|u_i^m|^{p_i},m\}\right)\xi T_\varepsilon(u_i^n-u_i^m)\ dz\\&+\int_{Q_T}\left(f_i(x,u_1^{n-1},u_2^{n-1})-f_i(x,u_1^{m-1},u_2^{m-1})\right)\xi T_\varepsilon(u_i^n-u_i^m)\ dz\\&\le c_i(\mu)\varepsilon.\end{aligned}$$

By virtue of (3.4) and (3.5). It follows that from (3.19) that

$$\lim_{n,m\to+\infty}\sup\int_{A_\varepsilon}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)dz\le C_i(\mu)\varepsilon.$$

We deduce that

$$\lim_{n,m\to+\infty}\sup\int_{A_\varepsilon}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)dz=0. \tag{3.22}$$

Since $p_i>2$, we obtain

$$\int_{A_\varepsilon}|\nabla u_i^n-\nabla u_i^m|^{p_i}\,dz\le c\int_{A_\varepsilon}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)dz.$$

Therefore it follows from (3.22) that

$$\lim_{n,m\to+\infty}\sup\int_{A_\varepsilon}|\nabla u_i^n-\nabla u_i^m|^{p_i}\,dz=0. \tag{3.23}$$

We deduce that

$$\nabla u_i^n\rightharpoonup\nabla u_i\ \ a.e.\text{ in }\ A_\varepsilon.$$

This is true for each $\varepsilon>0$ and so

$$\nabla u_i^n\rightharpoonup\nabla u_i\ \ a.e.\text{ in }\ Q_T. \tag{3.24}$$

By (3.5) and Vitali's theorem, we have, for any $r_i\in(1,p_i)\,,i=1,2$

$$\nabla u_i^n \rightharpoonup \nabla u_i \ \textit{strongly in} \ \ L^{r_i}(Q_T), i=1,2. \tag{3.25}$$

Lemma 3.5. $\nabla u_i^n \rightharpoonup \nabla u_i$ *strongly in* $L^{p_i}(Q_T), i=1,2.$

Taking a testing function $sinh(\lambda(u_i^n-u_i^m)) \in L^\infty(Q_T)\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ for (3.21) $(\lambda = \max(\|a_1\|_\infty, \|a_2\|_\infty)+1)$, we deduce that

$$\begin{aligned}
&\lambda \int_{Q_T} \cosh(\lambda(u_i^n-u_i^m))\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)dz \\
&\le A\int_{Q_T} \sinh(\lambda(u_i^n-u_i^m))\left(|\nabla u_i^n|^{p_i}+|\nabla u_i^m|^{p_i}\right)dz \\
&+\int_{Q_T} |\sinh(\lambda(u_i^n-u_i^m))|\left|\left(f_i(x,u_1^{n-1},u_2^{n-1})-f_i(x,u_1^{m-1},u_2^{m-1})\right)\right|dz,
\end{aligned} \tag{3.26}$$

where $A=\max(\|a_1\|_\infty, \|a_2\|_\infty)$.

Since $\sinh(\lambda s)$ is an odd function. The above inequality becomes

$$\begin{aligned}
&\int_{Q_T} \lambda\cosh(\lambda(u_i^n-u_i^m)) - A\left|\sinh(\lambda(u_i^n-u_i^m))\right|\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n-\nabla u_i^m)dz \\
&\le A\int_{Q_T} |\sinh(\lambda(u_i^n-u_i^m))|\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n.\nabla u_i^m + |\nabla u_i^m|^{p_i-2}\nabla u_i^m.\nabla u_i^n\right)dz \\
&+\int_{Q_T} |\sinh(\lambda(u_i^n-u_i^m))|\left|\left(f_i(x,u_1^{n-1},u_2^{n-1})-f_i(x,u_1^{m-1},u_2^{m-1})\right)\right|dz.
\end{aligned} \tag{3.27}$$

Recalling (3.24) and let $m\to+\infty$, by Fatou's lemma we deduce that

.

$$\begin{aligned}
&\int_{Q_T} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i|^{p_i-2}\nabla u_i\right).(\nabla u_i^n-\nabla u_i)dz \\
&\le A\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n.\nabla u_i + |\nabla u_i^m|^{p_i-2}\nabla u_i.\nabla u_i^n\right)dz \\
&+\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|\left|\left(f_i(x,u_1^{n-1},u_2^{n-1})-f_i(x,u_1,u_2)\right)\right|dz \\
&= J_1+J_2
\end{aligned} \tag{3.28}$$

And we use (3.4) and (3.5) to estimate J_1 and J_2 as below. For J_1, by Hölder inequality we have

$$\begin{aligned} J_1 \ &\leq \ A\left(\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|^{p_i}\,|\nabla u_i|^{p_i}\,dz\right)^{\frac{1}{p_i}}\left(\int_{Q_T}|\nabla u_i^n|^{p_i})dz\right)^{\frac{p_i-1}{p_i}} \\ &\quad +A\left(\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|^{\frac{p_i-1}{p_i}}\,|\nabla u_i|^{p_i}\,dz\right)^{\frac{p_i-1}{p_i}}\left(\int_{Q_T}|\nabla u_i^n|^{p_i})dz\right)^{\frac{1}{p_i}} \\ &\leq \ C\left(\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|^{p_i}\,|\nabla u_i|^{p_i}\,dz\right)^{\frac{1}{p_i}}+ \\ &\quad +C\left(\int_{Q_T} |\sinh(\lambda(u_i^n-u_i))|^{p_i'}\,|\nabla u_i|^{p_i}\,dz\right)^{\frac{1}{p_i'}}. \end{aligned} \tag{3.29}$$

Since $|\sinh(\lambda(u_i^n-u_i))|$ is uniformly bounded for all Lebesgue dominated convergence theorem, in view of (3.14) and (3.19) we assert that J_1+J_2 tends to zero when $n\to\infty$.

Then

$$\lim_{n\to+\infty}\int_{Q_T}\left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n-|\nabla u_i|^{p_i-2}\nabla u_i\right).(\nabla u_i^n-\nabla u_i)\ dz=0. \tag{3.30}$$

With the similar process to (3.23), it follows that

$$\lim_{n\to+\infty}\int_{Q_T}\left(|\nabla u_i^n|^{p_i}-|\nabla u_i|^{p_i}\right)\ dz=0, \tag{3.31}$$

which implies that

$$\begin{gathered} \mathcal{A}_i u_i^n \to \mathcal{A}_i u_i \text{ strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)); \\ f_i(.,u_1^n,u_2^n) \to f_i(.,u_1^n,u_2^n) \text{ strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)); \\ a_i(x)\min\{|\nabla u_i^n|^{p_i},n\} \to a_i(x)\,|\nabla u_i|^{p_i} \text{ strongly in } \quad \text{in } L^1(Q_T). \end{gathered}$$

Thus

$$\frac{\partial u_i^n}{\partial t}\to\frac{\partial u_i}{\partial t}\quad\text{strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega))+L^1(Q_T).$$

Therefore

$$\begin{aligned} \frac{\partial u_i}{\partial t} \ &\in \ L^{p_i'}(0,T_0;W_0^{-1,p_i'}(\Omega))+L^1(Q_T); \\ \frac{\partial u_i}{\partial t}-\mathcal{A}_i u_i \ &= \ a_i(x)\,|\nabla u_i|^{p_i}+f_i(x,u_1,u_2). \end{aligned}$$

As $L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega))+L^1(Q_T)\subset L^1(0,T;H^{-s}(\Omega))$ for large enough, then u_i^n converges strongly to u_i in $C(0,T;H^{-s}(\Omega))$ and

$$u_i^n(x,0)\to u_i(x,0) \text{ strongly in } H^{-s}(\Omega)$$

implies $u_i(x,0)=u_{i0}(x)$.

The proof of Theorem 3.1 is completed.

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