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GENERALIZED HANKEL-TYPE TRANSFORMATION FOR A CLASS OF TEMPERED ULTRADISTRIBUTIONS OF ROUMIEU-TYPE

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Abstract. As the third in a series of papers the present paper aims at investigating a theory supporting Hankel-type transformation on certain spaces of generalized functions. In an attempt to extend the transform to a space of generalized functions, namely, a space of tempered ultradistributions, an adequate definitions of classes of rapid descent ultradifferentiable functions is provided .In light of those definitions, various mappings involving various differential operators are shown to be continuous . Further, the generalized Hankel-type transform of Roumieu-type tempered (of slow growth) ultradistributions is obtained .

1. Introduction

The theory of ultradistributions is one of the generalizations of the theory of Schwartz' distributions. Since then in the recent past and even earlier it was extensively studied by many authors such as : Roumieu[13], [14], Komatsu [9], Beaurling [7], Carmichael, Pathak and Pilipović [12], Pathak[10], [11], Al-omari [2], [3], . . . ,to mention but few. The discussed spaces of tempered ultradistributions (ultradistributions of slow growth) which function as spaces of generalized functions are quite obviously include the Schwartz space S' of tempered distributions. For the sake of desired extensions of classical integral transforms to generalized functions researchers , along time of research, have reported two approaches which we find appropriate to be employed for tempered ultradistributions. The approach we specifically apply in this article consists in defining an appropriate testing function spaces $H^{\nu}_{\mu,a_i,A}, H^{\nu,b_j,B}_{\mu}, H^{\nu,b_j,B}_{\mu,a_i,A}$ of ultradifferentiable functions of rapid descents .Those test function spaces are shown to be closed with respect to the classical Hankel-type transform(see [1] and [6]). The corresponding transform of an ultradistribution f in the dual spaces is accordingly defined through the generalization of the Parseval's equation

$$\langle Tf, \phi\rangle = \langle f, T\phi\rangle$$

for all $\phi \in H^{\nu}_{\mu,a_i,A}, H^{\nu,b_j,B}_{\mu}, H^{\nu,b_j,B}_{\mu,a_i,A}$.

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Let μ be an arbitrary real parameter. The *Hankel-type* transformation is defined by

$$(1.1) \qquad (h_{\mu,\nu}f)(y) = y^{1+2\mu}\int_0^\infty \vartheta_{\mu,\nu}(xy)\, f(x)\, dx,\ \nu \geq -\frac{1}{2}$$

where $\vartheta_{\mu,y}(z) = z^{-\mu}J_\nu(z)\,, J_\nu(z)$ is the Bessel function of first kind of order ν.

For real values $\mu = -\frac{1}{2}$and $\mu = -\nu - 1$, the transformation (1.1) respectively includes the *Hankel transform* [15,p.127] and the celebrated *Schwartz -Hankel* transformation as well.

Verifying results spreads in two sections, namely, Section 3 and 4. In Section 3, we establish certain results for certain differential operators and further, make use of a new defined differential operator and accordingly prove new relevant theorem. Section 4, extend the *Hankel-type* transform to spaces of tempered ultradistributions of Roumieu-type we have recently discussed in [4, 5].

2. **Ultradifferentiable Functions of Rapid Descent**

It is assumed the reader is acquainted with the results reported in [4, 5]. The notation and terminology used in those papers will be continued. Sequences a_i and b_j, $i, j = 0, 1, 2, ...$, wherever they appear, are sequences of positive real numbers imposed by some of the following constraints:

$$(2.1) \qquad a_i^2 \leq a_{i-1}a_{i+1}, \forall i \in N,$$

$$(2.2) \qquad b_j^2 \leq b_{j-1}b_{j+1}, \forall j \in N,$$

There are constants S, S_1 and T, T_1 such that,

$$(2.3) \qquad a_i \leq ST^i \min_{0\leq k\leq i} a_k a_{i-k}, i \in N_\circ,$$

$$(2.4) \qquad b_j \leq S_1 T_1^j \min_{0\leq k\leq j} b_k b_{j-k}, j \in N_\circ.$$

From [5,pp.2], recall the consequences

$$(2.5) \qquad a_i a_k \leq a_\circ a_{i+k}, \forall i, k \in N_\circ$$

and

$$(2.6) \qquad b_i b_k \leq b_\circ b_{i+k}, \forall j, k \in N_\circ$$

Definition 2.1. *By $H^\nu_{\mu,a_i,A}$, we denote the set of all infinitely smooth functions on $(0, \infty)$ satisfying*

$$(2.7) \qquad \left| x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C_j^{\mu,\nu}(A+\alpha)^i a_i, i, j \in N_\circ$$

where $A, C_j^{\mu,\nu}$ are positive constants depending on ϕ and $a > 0$ being an arbitrary constant.

An infinitely differentiable function $\phi(x), 0 < x < \infty$, belongs to $H_{\mu}^{\nu,b_j,B}$ if it possesses the property that

$$\left|x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C_j^{\mu,\nu} (B+\beta)^i b_j, i,j \in N_\circ \tag{2.8}$$

where $\beta > 0$,$C_j^{\mu,\nu}$ and B are positive constants depending on ϕ.

Similarly, $\phi \in H_{\mu,a_i,A}^{\nu,b_j,B}$ if $\phi \in C^\infty(0,\infty)$ and,

$$\left|x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C^{\mu,\nu} (A+\alpha)^i (B+\beta)^j a_i b_j, \tag{2.9}$$

holds for $i,j \in N_\circ$ and positive constants α, β depend on ϕ.

In view of Definition 2.1 we, on $H_{\mu,a_i,A}^{\nu}$, $H_{\mu}^{\nu,b_j,B}$, and $H_{\mu,a_i,A}^{\nu,b_j,B}$, respectively, define the norms

$$i_{j,\alpha}^{\mu,\nu}(\phi) = \sup \| x \in (0,\infty), i \in N_\circ \frac{\left|x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i a_i}, j \in N_\circ, \alpha = 1, \frac{1}{2}, ... \tag{2.10}$$

$$i_{i,\beta}^{\mu,\nu}(\phi) = \sup \| x \in (0,\infty), j \in N_\circ \frac{\left|x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(B+\beta)^j a_i}, i \in N_\circ, \beta = 1, \frac{1}{2}, ... \tag{2.11}$$

and

$$i_{\alpha,\beta}^{\mu,\nu}(\phi) = \sup \| x \in (0,\infty), i,j \in N_\circ \frac{\left|x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i (B+\beta)^i a_i b_j}, \alpha, \beta = 1, \frac{1}{2}, ... \tag{2.12}$$

Inspired in the work of Roumieu [13], the dual spaces $H_{\mu,a_i,A}^{\acute{\nu}}$, $H_{\mu}^{\acute{\nu},b_j,B}$, and $H_{\mu,a_i,A}^{\acute{\nu},b_j,B}$ are called *tempered ultradistributions* of *Roumieu-type* .The generalized *Hankel-type* transform of the classes on the duals is called *tempered ultradistributional Hankel-type transformation of Roumieu-type.*

3. Operations of $H_{\mu,a_i,A}^{\nu}$, $H_{\mu}^{\nu,b_j,B}$, AND $H_{\mu,a_i,A}^{\nu,b_j,B}$

We initiate the study with known differential operators [10,p.243]

$$N_{\mu,\nu} = x^{\nu-\mu} D x^{-\mu-\nu-1}, \tag{3.1}$$

and

$$M_{\mu,\nu} = x^{\mu-\nu} D x^{\mu+\nu+1} \tag{3.2}$$

Theorem 3.1. (i) *The mapping* $\phi \to N_{\mu,\nu}\phi$ *is a continuous linear mapping from* $H_{\mu,a_i,A}^{\nu}$ *into* $H_{-\mu,a_i,A}^{\nu}$.

(ii)*Let the sequence* (b_j) *satisfy* (2.4) . *Then, the mapping* $\phi \to N_{\mu,\nu}\phi$ *is a continuous linear mapping from* $H^{\nu,b_j,B}$ *into* $H_{-\mu}^{\nu,b_j,BT_1}$.

(iii) *If the sequence* (b_j) *satisfy* (2.4) *then the map* $\phi \to N_{\mu,\nu}\phi$ *is a continuous linear mapping of* $H_{\mu,a_i,A}^{\nu,b_j,B}$ *into* $H_{-\mu,a_i,A}^{\nu,b_j,BT_1}$.

Proof of (i) is a straightforward consequence of (2.7) and (3.1)and thus, details are avoided.

Proof of (ii). Let $\phi \in H_{\mu}^{\nu,b_j,B}$. By virtue of (2.8) and (3.1) we have

$$\left|x^i\left(x^{-1}D\right)^j x^{\mu-\nu-1}\left(N_{\mu,\nu}\phi\right)(x)\right| \leq C_i^{\mu,\nu}\left(B+\beta\right)^{j+1} b_{j+1}$$

Employing (2.4) yields

$$\left|x^i\left(x^{-1}D\right)^j x^{\mu-\nu-1}\left(N_{\mu,\nu}\phi\right)(x)\right| \leq C_i^{\mu,\nu}\left(B+\beta\right)^{j+1} S_1T_1^{j+1}b_1b_j$$

Therefore,

$$\left|x^i\left(x^{-1}D\right)^j x^{\mu-\nu-1}\left(N_{\mu,\nu}\phi\right)(x)\right| \leq C_i^{\acute{\mu},\nu}\left(BT_1+\acute{\beta}\right)^j b_j$$

where $\acute{\beta}=T_1\beta$ and $C_i^{\acute{\mu},\nu}=C_i^{\mu,\nu}\left(B+\beta\right)S_1T_1b_1$.

This proves Part (**ii**).

Proof of (iii). Let $\phi \in H_{\mu,a_i,A}^{\nu,b_j,B}$. Allowing b_j satisfy (2.4) yields

$$\begin{aligned}\left|x^i\left(x^{-1}D\right)^j x^{\mu-\nu-1}\left(N_{\mu,\nu}\phi\right)(x)\right| &\leq C_{\alpha,\beta}^{\mu,\nu}\left(A+\alpha\right)^i\left(B+\beta\right)^{j+1}a_ib_{j+1}\\ &\leq C_{\alpha,\beta}^{\mu,\nu}\left(A+\alpha\right)^i\left(B+\beta\right)^j\left(B+\beta\right)a_iS_1T_1^{j+1}b_1b_j.\end{aligned}$$

Therefore, the relation can be formed as

$$\left|x^i\left(x^{-1}D\right)^j x^{\mu-\nu-1}\left(N_{\mu,\nu}\phi\right)(x)\right| \leq C_{\alpha,\beta}^{'\mu,\nu}\left(A+\alpha\right)^i\left(BT_1+\beta^{'}\right)^j a_ib_j$$

where $C_{\alpha,\beta}^{'\mu,\nu}=C_{\alpha,\beta}^{\mu,\nu}\left(B+\beta\right)S_1T_1b_1$ and $\beta^{'}=T_1\beta$.

This completes the proof of the theorem.

In an attempt to provide alternative differential operators which possess linearity among the test function spaces we define a differential operator $S_{\mu,\nu}$ through a motivation of the operators $N_{\mu,\nu}$ and $M_{\mu,\nu}$ defined by

$$S_{\mu,\nu}\phi=x^{-\mu-\nu}Dx^{\mu+\nu+1}\phi \tag{3.3}$$

The operator (3.3) is shown to possess the property of linearity of the rapid spaces and hence the corresponding duals as follows:

Theorem 3.2. *(i) Let the sequence (a_i) satisfy (2.3). Then the mapping*

$$\begin{aligned}H_{\mu,a_i,A}^{\nu} &\rightarrow H_{\mu,a_i,AT}^{\nu}\\ \phi &\rightarrow S_{\mu,\nu}\phi\end{aligned}$$

is a continuous linear map.

(ii) Let (b_i) satisfy (2.4). Then, the map $\phi \rightarrow S_{\mu,\nu}\phi$ is a continuous linear map of $H_{\mu}^{\nu,b_j,B}$ into H_{μ}^{ν,b_j,BT_1}.

(iii) Let a_i and b_i satisfy (2.3) and (2.4), respectively. The operation $\phi \rightarrow S_{\mu,\nu}\phi$ maps $H_{\mu,a_i,A}^{\nu,b_j,B}$ into $H_{\mu,a_1,AT}^{\nu,b_j,BT_1}$ continuously.

Proof of (i). Let $\phi \in H^{\nu}_{\mu,a_i,A}$. With the aid of (3.3) we have

$$\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x) = \left(x^{-1}D\right)^{j} x^{-2\mu-2\nu-1}\left[\begin{array}{c} x^{2\mu+2\nu+2}Dx^{-\mu-\nu-1}\phi \\ +\left(2\mu+2\nu+2\right)x^{-\mu-\nu-1}\phi\left(x\right)x^{2\mu+2\nu+1}\end{array}\right]$$

Simple calculations yields

$$\begin{aligned}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x) &= \left(x^{-1}D\right)^{j} x^{2}\left(x^{-1}D\right)x^{-\mu-\nu-1}\phi(x) \\ &\quad +\left(2\mu+2\nu+2\right)\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x)\end{aligned}$$

That is,

$$\begin{aligned}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x) &= \left(x^{-1}D\right)^{j-1}\left(x^{-1}D\right)\left(x^{2}\left(x^{-1}D\right)x^{-\mu-\nu-1}\phi(x)\right) \\ &\quad +\left(2\mu+2\nu+2\right)\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) \\ &= 2\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) \\ &\quad +\left(x^{-1}D\right)^{j-1} x^{2}\left(x^{-1}D\right)^{2} x^{-\mu-\nu-1}\phi(x) \\ &\quad +\left(2\mu+2\nu+2\right)\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x)\end{aligned}$$

and hence the equation simplified to the form

$$\begin{aligned}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x) &= \left(2+2\mu+2\nu+2\right)\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) \\ &\quad +\left(x^{-1}D\right)^{j-1} x^{2}\left(x^{-1}D\right)^{2} x^{-\mu-\nu-1}\phi(x)\end{aligned}$$

Proceeding , j-times , as above, implies

$$\begin{aligned}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x) &= 2j\left(2\mu+2\nu+2\right)\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) \\ &\quad +x^{2}\left(x^{-1}D\right)^{j+1} x^{-\mu-\nu-1}\phi(x)\end{aligned}$$

Employing (2.3) yields

$$\begin{aligned}\left|x^{i}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x)\right| &\leq 2j\left(2\mu+2\nu+2\right)\left|x^{i}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x)\right| \\ &\quad +\left|x^{i+2}\left(x^{-1}D\right)^{j+1} x^{-\mu-\nu-1}\phi(x)\right| \qquad (3.4)\end{aligned}$$

.

Thus, we write

$$\begin{aligned}\left|x^{i}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x)\right| &\leq 2j\left(2\mu+2\nu+2\right)C^{\mu,\nu}_{j}\left(A+\alpha\right)^{i}a_{i} \\ &\quad +C^{\mu,\nu}_{j}\left(A+\alpha\right)^{i+2}a_{i+2} \\ &\leq 2j\left(2\mu+2\nu+2\right)C^{\mu,\nu}_{j}\left(A+\alpha\right)^{i}a_{i} \\ &\quad +C^{\mu,\nu}_{j}\left(A+\alpha\right)^{2}\left(A+\alpha\right)^{i}a_{i}ST^{i}T^{2}a_{2}.\end{aligned}$$

The inequality is, in short, put into the form

$$\left|x^{i}\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\left(S_{\mu,\nu}\phi\right)(x)\right| \leq C^{'\mu,\nu}_{j}\left(TA+a^{'}\right)^{i}a_{i},$$

where $a^{'} = \alpha T$ and $C_j^{'\mu,\nu}$ is certain constant.
Part (i) of the theorem is, therefore, proved.
Proof of part (ii). Let $\phi(x) \in H_{\mu}^{\nu,b_j,B}$. Employing (3.4) implies

$$\begin{aligned} \left| x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1} \left(S_{\mu,\nu}\phi\right)(x) \right| &\leq (2j)(2\mu+2\nu+2) C_j^{\mu,\nu} (B+\beta)^i b_j \\ &\quad + C_i^{\mu,\nu} (B+\beta)^{j+1} b_{j+1}. \\ &\leq (2j)(2\mu+2\nu+2) C_j^{\mu,\nu} (B+\beta)^i b_j \\ &\quad + C_i^{\mu,\nu} (B+\beta)^j (B+\beta) S_1 T_1^{j+1} b_1 b_j. \end{aligned}$$

That is,

$$\left| x^i \left(x^{-1}\right)^j x^{-\mu-\nu-1} \left(S_{\mu,\nu}\phi\right)(x) \right| \leq C_i^{'\mu,\nu} \left(BT_1 + \beta^{'}\right)^j b_j,$$

where $\beta^{'} = \beta T_1$ and $C_i^{'\mu,\nu} = C_i^{\mu,\nu}(B+\beta) S_1 T_1 b_1 + (2j)(2\mu+2\nu+2) C_j^{\mu,\nu}$.
This proves Part(ii).
Proof of part (iii). Once again, using (3.4) and (2.9) and the fact that

$$2j \leq C_r (1+r)^j, r > 0,$$

implies

$$\begin{aligned} \left| x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1} \left(S_{\mu,\nu}\phi\right)(x) \right| &\leq C_r (1+r)^j (2\mu+2\nu+2) C^{\mu,\nu} (A+\alpha)^i (B+\beta)^j a_i b_j \\ &\quad + C^{\mu,\nu} (A+\alpha)^{i+2} (B+\beta)^{j+2} a_{i+2} b_{j+1} \end{aligned}$$

Invoked with (2.3) and (2.4) the above equation is reduced to the form

$$\begin{aligned} \left| x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1} \left(S_{\mu,\nu}\phi\right)(x) \right| &\leq C_r (1+r)^j (2\mu+2\nu+2) C^{\mu,\nu} (A+\alpha)^i (B+\beta)^j a_i b_j \\ &\quad + C^{\mu,\nu} (A+\alpha)^2 (B+\beta)(A+\alpha)^i (B+\beta)^j S T^{i+2} a_2 a_i S_1 T_1^{i+1} b_1 b_j \end{aligned}$$

That is,

$$\left| x^i \left(x^{-1}D\right)^j x^{-\mu-\nu-1} \left(S_{\mu,\nu}\phi\right)(x) \right| \leq C_{j,r}^{'\mu,\nu} \left(AT + \alpha^{'}\right)^i \left(BT_1 + \beta^{'}\right)^j a_i b_j,$$

where $\alpha^{'} = T\alpha$,$\beta^{'} = \beta T_1$ and
$C_{j,r}^{'\mu,\nu} = C_r (1+r)^j (2\mu+2\nu+2) C^{\mu,\nu} + C^{\mu,\nu} (A+\alpha)^2 (B+\beta) S T^2 a_2 S_1 T_1 b_1$.
This completes the proof of the theorem .

Theorem 3.3. *Let the sequence* (a_i) *satisfy* (2.3) *and* (b_i) *satisfy* (2.6), $n \in 2N$. *Then*

$$H_{\mu,a_i,A}^{\nu+2,b_j,B} \subset H_{\mu,a_i,AT^n}^{\nu,b_j,B}$$

Further, convergence in $H_{\mu,a_i,A}^{\nu+2,b_j,B}$ *implies convergence in* $H_{\mu,a_i,AT^n}^{\nu,b_j,B}$.

Proof. We prove the theorem for $n = 2$. Through simple calculations we obtain
$(x^{-1}D)^j x^{-\mu-\nu-1}\phi(x) = 2(x^{-1}D)^{j-1} x^{-\mu-\nu-3}\phi(x) + (x^{-1}D)^{j-3} x^2 (x^{-1}D) x^{-\mu-\nu-3}\phi(x)$.
But,

$$\left(x^{-1}D\right)^{j-3} x^2 \left(x^{-1}D\right) x^{-\mu-\nu-3}\phi(x) = \left(x^{-1}D\right)^{j-2}\left[x^2\left(x^{-1}D\right)^2 x^{-\mu-\nu-3}\phi(x) + 2\left(x^{-1}D\right)x^{-\mu-\nu-3}\phi(x)\right].$$

Thus,

$$\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) = 2\left(x^{-1}D\right)^{j-1} x^{-\mu-\nu-3}\phi(x) + \left(x^{-1}D\right)^{j-2}\left[\begin{array}{c} x^2\left(x^{-1}D\right)^2 x^{-\mu-\nu-3}\phi(x) \\ +2\left(x^{-1}D\right)x^{-\mu-\nu-3}\phi(x)\end{array}\right]$$

Simplifying the equation yields

$$\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) = 2.2\left(x^{-1}D\right)^{j-1} x^{-\mu-\nu-3}\phi(x) + \left(x^{-1}D\right)^{j-2} x^2\left(x^{-1}D\right)^2 x^{-\mu-\nu-3}\phi(x).$$

Proceeding , $j-$times, leads to the equation

$$\left(x^{-1}D\right)^{j} x^{-\mu-\nu-1}\phi(x) = 2j\left(x^{-1}D\right)^{j-1} x^{-\mu-\nu-3}\phi(x) + x^2\left(x^{-1}D\right)^{j} x^{-\mu-\nu-3}\phi(x) \tag{3.5}$$

Using (3.5) and the fact that $2j \le C_r(1+r)^j$, $r>0$,, implies

$$\frac{\left|x^i\left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i(B+\beta)^j a_i b_j} \le \frac{\left|x^i(2j)\left(x^{-1}D\right)^{j-1} x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i(B+\beta)^{j-1} a_i b_{j-1}}\cdot\frac{b_{j-1}}{b_j(B+\beta)} + \frac{\left|x^{i+2}\left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^{i+2}(B+\beta)^j a_{i+2} b_j}\cdot\frac{a_{i+2}(A+\alpha)^2}{a_i}.$$

Employing (2.6) implies

$$\frac{\left|x^i\left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i(B+\beta)^j a_i b_j} \le C_r(1+r)^j\frac{b_\circ}{b_1(B+\beta)}\cdot\frac{\left|x^i\left(x^{-1}D\right)^{j-1} x^{-\mu-\nu-1}\phi(x)\right|}{(A+\alpha)^i(B+\beta)^{j-1} a_i b_{j-1}} + \left((A+\alpha)^2 ST^{i+2}a_2\right)\frac{\left|x^{i+2}\left(x^{-1}D\right)^j x^{-\mu-\nu-3}\phi(x)\right|}{(A+\alpha)^{i+2}(B+\beta)^j a_{i+2} b_j}$$

But since

$$\frac{\left|x^i\left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{\left(T^2(A+\alpha)\right)^i(B+\beta)^j a_i b_j} \le \frac{\left|x^i\left(x^{-1}D\right)^j x^{-\mu-\nu-1}\phi(x)\right|}{\left(T(A+\alpha)\right)^i(B+\beta)^j a_i b_j}$$

we have

$$\frac{\left|x^{i}\left(x^{-1}D\right)^{j}x^{-\mu-\nu-1}\phi\left(x\right)\right|}{T^{2}\left(A+\alpha\right)^{i}\left(B+\beta\right)^{j}a_{i}b_{j}} \leq C_{r}\left(1+r\right)^{j}\frac{b_{\circ}}{b_{1}\left(B+\beta\right)}\cdot\frac{\left|x^{i}\left(x^{-1}D\right)^{j-1}x^{-\mu-\nu-1}\phi\left(x\right)\right|}{\left(A+\alpha\right)^{i}\left(B+\beta\right)^{j-1}a_{i}b_{j-1}} + \left(\left(A+\alpha\right)^{2}ST^{2}a_{2}\right)\frac{\left|x^{i+2}\left(x^{-1}D\right)^{j}x^{-\mu-\nu-3}\phi\left(x\right)\right|}{\left(A+\alpha\right)^{i+2}\left(B+\beta\right)^{j}a_{i+2}b_{j}}$$

Upon considering supremum over all $x\in(0,\infty)\, i,j\in N_{\circ}$, we have

$$\sup_{x,i,j}\frac{\left|x^{i}\left(x^{-1}D\right)^{j}x^{-\mu-\nu-1}\phi\left(x\right)\right|}{T^{2}\left(A+\alpha\right)^{i}\left(B+\beta\right)^{j}a_{i}b_{j}} \leq H_{r}^{\beta}i_{\alpha,\beta}^{\mu,\nu+2}\left(\phi\right)+\left(A+\alpha\right)^{2}ST^{2}ai_{2}i_{\alpha,\beta}^{\mu,\nu+2}\left(\phi\right),$$

for some constant H_r^β .This establishes the theorem for $n=2$.
Induction on n completes the proof of the theorem. □

4. **Hankel-Type Transformation of Tempered Ultradistributions.**

This section is devoted for the investigation of the transformation (1.1) on the obtained spaces. The generalized Hankel-type transformation on the duals of Roumieu-type tempered ultradistributions is defined as the adjoint oparator. For this end recall the equations $[8, p.16]$

$$D_{x}\left(x^{\mu+\nu}\vartheta_{\mu,\upsilon}\left(xy\right)\right)=x^{\mu+\nu-1}\vartheta_{\mu-1,\upsilon-1}\left(xy\right), \tag{4.1}$$

and

$$\left(x^{-1}D_{x}\right)x^{\mu-\nu}\vartheta_{\mu,\upsilon}\left(xy\right)=-y^{2}x^{\mu-\nu}\vartheta_{\mu+1,\upsilon+1}\left(xy\right). \tag{4.2}$$

From those equations we derive equations which we use in the sequel

$$\left(x^{-1}D_{x}\right)^{i}\left(x^{\mu+\nu}\vartheta_{\mu,\upsilon}\left(xy\right)\right)=x^{\mu+\nu-2i}\vartheta_{\mu-i,\upsilon-i} \tag{4.3}$$

and

$$\left(x^{-1}D_{x}\right)^{i}\left(x^{\mu-\nu}\vartheta_{\mu,\upsilon}\left(xy\right)\right)=\left(-1\right)^{j}y^{2i}x^{\mu-\nu}\vartheta_{\mu+1,\upsilon+1}\left(xy\right). \tag{4.4}$$

Theorem 4.1. *Let μ and ν be real numbers, $\nu\geq-\frac{1}{2}$ and μ arbitrary. Let the sequence (a_i) satisfy (2.3) and $a_i\leq a_{i+1}\forall i>0$. The transform (1.1) is an automorphism from $H^{\nu}_{\mu,a_i,A}\rightarrow H^{\nu}_{\mu,a_i,AT}$.*

Proof. Using (4.4) leads to

$$\left|y^{i}\left(y^{-1}D_{y}\right)^{j}y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)\left(x\right)\right|=\left|\left(-1\right)^{j}y^{\mu-\nu+i}\int_{0}^{\infty}x^{2j}\vartheta_{\mu+j,\upsilon+j}\left(xy\right)\phi\left(x\right)dx\right|$$

Right hand side part of the equation

$$\left|(-1)^j y^{\mu-\nu+i}\int_0^\infty x^{2j}\vartheta_{\mu+j,\upsilon+j}(xy)\,\phi(x)\,dx\right| = \left|(-1)^j y^{\mu-\nu+i}\int_0^\infty x\left(x^{\mu+\nu+2j}\vartheta_{\mu+j+i,\upsilon+j+i}(xy)\right)x^{-\mu-\nu-1}\phi(x)\,dx\right|.$$

Hence,

$$\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| = \left|(-1)^j y^{\mu-\nu+i}\int_0^\infty x\left(x^{\mu+\nu+2j}\vartheta_{\mu+j+i,\upsilon+j+i}(xy)\right)x^{-\mu-\nu-1}\phi(x)\,dx\right|$$

Equation (4.3), integrating by parts,i-times, and employing the fact

$$\left|(xy)^{\mu-\nu+i}\,\vartheta_{\mu+j-i,\upsilon+j-i}(xy)\right| \leq C_{i,j,\mu,\nu}$$

where $C_{i,j,\mu,\nu}$ is certain constant, lead to

$$\begin{aligned}\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| &\leq \\ \int_0^\infty \left|(xy)^{\mu-\nu+i}\,\vartheta_{\mu+i+j,\upsilon+i+j}(xy)\,x^{2\nu+i+2j+1}\left(x^{-1}D\right)^i x^{-\mu-\nu-1}\phi(x)\right|dx \\ \leq C_{i,j,\mu,\nu}\int_0^\infty\left|x^{2\nu+i+2j+1}\left(x^{-1}D\right)^i x^{-\mu-\nu-1}\phi(x)\right|dx\end{aligned} \tag{4.5}$$

For real numbers $p > 2\nu+i+2j+1$, (4.5) can be expressed as

$$\begin{aligned}\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| &\leq C_{i,j,\mu,\nu}\int_0^\infty\left|x^p\left(x^{-1}D\right)^i x^{-\mu-\nu-1}\phi(x)\right|dx \\ &\leq C_{i,j,\mu,\nu}\sum_{k=0}^{p+2}\left|x^k\left(x^{-1}D\right)^i x^{-\mu-\nu-1}\phi(x)\right| \\ &\leq C_{i,j,\mu,\nu}\sum_{k=0}^{p+2}C_i^{\mu,\nu}(A+\alpha)^k a_k \\ &\leq C'_{i,j,\mu,\nu}(A+\alpha)^{p+2}\sum_{k=0}^{p+2}a_k \\ &\leq C'_{i,j,\mu,\nu}(p+3)(A+\alpha)^2 ST^2 a_2\left((A+\alpha)T\right)^p a_p\end{aligned}$$

for some constant $C'_{i,j,\mu,\nu}$. Therefore,

$$\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| \leq C'_{i,j,\mu,\nu}(p+3)(A+\alpha)^2 ST^2 a_2\left((A+\alpha)T\right)^p a_p \tag{4.6}$$

Assuming $\zeta_{i,j,\mu,\nu} = C'_{i,j,\mu,\nu}(p+3)(A+\alpha)^2 ST^2 a_2$ and $a' = \alpha T$ implies

$$h_{\mu,\nu} \in H^{\nu}_{\mu,a_i,AT}$$

This completes the proof of the theorem. □

Theorem 4.2. *(i) If* $\phi \in H^{\nu,b_j,B}_{\mu,a_i,A}$ *,* $a_i \leq a_{i+1}, i \geq 0$ *and (2.3) holds. Then*

$$h_{\mu,\nu}\phi \in H^{\nu,b_j,B}_{\mu,a_i,AT}$$

(ii)If $\phi \in H^{\nu,b_j,B}_{\mu}$ *, then* $h_{\mu,\nu}\phi \in H^{\nu,b_j,B}_{\mu}$ *.*

Proof. Let $\phi \in H^{\nu,b_j,B}_{\mu,a_i,A}$. Then, (4.6) ,(2.3) and, the assumption $a_i \leq a_{i+1}, i \geq 0$, yields

$$\begin{aligned}\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| &\leq C_{i,j,\mu,\nu}\sum_{k=0}^{p+2}C^{\mu,\nu}\left(A+\alpha\right)^k\left(B+\beta\right)^i a_k b_i \\ &\leq (p+3)\,C_{i,j,\mu,\nu}C^{\mu,\nu}\left(A+\alpha\right)^2 ST^2 a_2\left(\left(A+\alpha\right)T\right)^p a_p\left(B+\beta\right)^i b_i\end{aligned}$$

which can be put into the form

$$\left|y^i\left(y^{-1}D_y\right)^j y^{-\mu-\nu-1}\left(h_{\mu,\upsilon}\phi\right)(x)\right| \leq \eta_{i,j,\mu,\upsilon}\left(AT+\alpha^{'}\right)^p\left(B+\beta\right)^i a_p b_i,$$

where $\eta_{i,j,\mu,\upsilon} = (p+3)\,C_{i,j,\mu,\nu}C^{\mu,\nu}\left(A+\alpha\right)^2 ST^2 a_2$ and $\alpha^{'} = \alpha T$ This proves Part (i) of the theorem.Employing(4.6), Proof of Part (ii) follows similarly. Thus, the theorem is, therefore, completely proved. □

Theorem 4.1 and 4.2 suggest the following theorem to be stated from which we define the generalized Hankel- type intregral transformation of tempered ultradistributions of Roumieu-type $f \in H'^{\nu}_{\mu,a_i,A}, {}^{\prime\nu,b_j,B}_{\mu}$ or $H'^{\nu,b_j,B}_{\mu,a_i,A}$ as the adjoint mapping.

Theorem 4.3. *(i)Let* $f \in {}'^{\nu}_{\mu,a_i,A}$ *and* $\phi \in H^{\nu}_{\mu,a_i,A}$ *.Then* $\left\langle h^{\prime}_{\mu,\nu}f,\phi\right\rangle = \left\langle f, h_{\mu,\nu}\phi\right\rangle$

(ii)Let $f \in H'^{\nu,b_j,B}_{\mu}\left(H'^{\nu,b_j,B}_{\mu,a_i,A}\right)$ *.Then* $\left\langle h^{\prime}_{\mu,\nu}f,\phi\right\rangle = \left\langle f, h_{\mu,\nu}\phi\right\rangle$ *,*

for all $\phi \in H^{\nu,b_j,B}_{\mu}\left(H^{\nu,b_j,B}_{\mu,a_i,A}\right)$

Proof. The proof is a straightforward consequence of Theorem 4.1 and 4.2 □

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