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ON NIL – SEMI CLEAN RINGS *

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ABSTRACT*:* In this paper, the notions of semi-idempotent elements and nil-semiclean elements are introduced. We show that, if a ring R is a nil-semi clean ring, then the Jacobson radical, J(R), of R is nil. We also characterize some properties related to orthogonal idempotent elements e and f in a ring R. We prove that if R is a nil-semi clean ring and the idempotent elements of R are Central, then the Jacobson radical J(R) of R equals the set of all nilpotent elements of R. Finally, we show that if R is a commutative ring, then the polynomial ring R[x] is not nil-semi clean ring.

1. Introduction

A ring R is called nil-semi clean (nil-clean) if every element in R is nil-semi clean (nil-clean). Any finite field is an example of nil-semi clean ring. For further studies about clean and semi clean rings one may refer to the references [2], [3], [4], and [7].

***Definition* 1.1:** An element a in a ring R is called semi-idempotent if either a = 0 or the set $\{a^k; k \in N\}$ is finite and does not contain the element 0.

***Example 1.2*:**

- If F is a finite field, then all its elements are semi-idempotent.
- In any ring R every non nilpotent periodic element is semi-idempotent.
- Every semi-idempotent element is periodic.

***Definition 1.*3:** An element a in the ring R is called nil- semi clean (nil-clean) if a=s+n where s is semi-idempotent (idempotent) and n is nilpotent.

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***Definition 1.4*:** A ring R is called nil-semiclean (nil-clean) if every element of R is nil-semiclean (nil-clean). Throughout this paper the following notations will adopted

R: Associative ring with unity.

I_R: Set of all idempotents of R.

N_R: Set of all nilpotents of R.

SI_R: Set of all semi-idempotents of R.

P_R: Set of all periodics of R.

J(R): Jacobson radical of R.

U_R : Group of units in R.

2. Basic Properties

***Definition* 2.1:** A ring R is called semi commutative if for all elements a and b in A, there exist elements c and d in A such that a . b = c. a and b . a = a. d

***Lemma* 2.2 [1]:** If R is a semi commutative ring, then:

1- N_R is an ideal of R.
2- The Idempotents in R are central.

***Lemma* 2.3:** If R is a commutative (semi commutative) ring then, for a unit u and a nilpotent n we have: u + n is a unit.

Proof: since R is a commutative (semi commutative) then N_R is an ideal of R.

Now $u + n = u^{-1} (1 + u^{-1} n)$ and $u^{-1} n$ is nilpotent which implies $1 + u^{-1} n$ is unit. Hence u + n is unit

***Lemma* 2.4 [8]:** Every periodic element in R is clean.

Propositions 2.5:

1- The Homomorphic image of a nil-clean ring is a nil-clean.
2- If R is nil – clean, then R is clean.
3- If R is nil–semi clean then, R is semiclean.
4- If R is semi commutative and nil–semi clean, then R is clean.
5- If R is nil–clean, then R is nil –semiclean.

Proof:

1- The proof for this part is direct using homorphism properties.

2- Let a be any nil-clean element in R, then (a – 1) is also in R and it is, also, a nil-clean element. So (a – 1) can be expressed as, a - 1 = e + n where, e is idempotent and n is nilpotent. Thus, a = e + n + 1 where n is nilpotent. Therefore (n + 1) is a unit and a is clean.

3- Let a belong to R, then (a – 1) belongs to R and it is, also, a nil-semi clean element. So, a - 1 = p + n such that p is semi-idempotent and n is nilpotent. Then A = p + n +1. But n is nilpotent, then (n + 1) is a unit and since every semi-idempotent is periodic, then a is semiclean.

4- If a belongs to the nil – semi clean ring R, then a = p + n where p is semi-idempotent and n is nilpotent. By Lemma 2.3, every semi-idempotent is periodic therefore p is periodic and p is clean. Therefore, p = e + u where e is idempotent and u is unit. i.e. a = e + u + n. By Lemma 2.3, u + n is unit. so a is clean.

5- This follows, as I_R is a subset of SI_R

***Example 2.6*:**

1- The field of real numbers IR is clean, but it is not nil-semi clean. Indeed, SI_R={0, 1} and if r is any real number, which is nil-semi clean. Then, either r or r - 1 is a nilpotent. This is impossible for the case r = 5 for example.

2- If R = Z_3, then R is nil-semi clean But not nil-clean. Proof will be given at the end of the paper.

3- Let R = {m/n such that m, n $\in$ Z and n not divisible by 7} and G = {g, g^2, g^3} is a cyclic group of order 3. It is shown in [7] that the group ring RG is not clean but semiclean. So, RG is commutative and not clean so it is not nil-semi clean. Thus, RG is an example of a semi clean ring which is not nil-semiclean.

4- In any nil-clean ring R, the element 2 is nilpotent. Since R is a nil-clean ring, then we can write: 2 = e + n where e is in I_R and n is in N_R. This means that: 2-1 = e + n – 1. And so, 1 - e = n – 1 with n in N_R. Thus, n -1 is in U_R. So, 1 - e belongs to U_R and 1 - e is in I_R. But this means that 1 - e = 1 and e = 0. But 2 = e + n, so 2 = n in N_R. Therefore, 2 is nilpotent.

5- Z_3 is not nil-clean but it is nil-semi clean, since Z_3 is a finite field .

3. Main Results

***Theorem* 3.1**: If R is a nil-semi clean ring, then its Jacobson radical J(R) is a nil deal.

Proof: Let r be in J(R). Since R is a nil-semi clean ring, then r = s + n , where s is in SI_R and n in N_R. Let m be the least positive integer such that $n^m = 0$, So, $(r - s)^m = 0$. By the binomial expansion:

$$(r - s)^m = \sum_{t,y \in R}^{finite} try + s^m,$$

where y, t, are in R and we have:

$$s^m = -\sum_{t,y \in R}^{finite} try$$

Now y, t are in J(R), Since r in J(R) and J(R) ideal. So s^m is in J(R) and s is also in SI_R , Since s is in SI_R , then, s^m is periodic (as we will show later) there exists a positive integer k such that s^{km} is idempotent. s^m in J(R) means that s^{mk} is in J(R) so s^{mt} belongs to J(R) and to I_R . This means that $s^{mk} = 0$. But s is in SI_R, So s = 0 and r = s + n = n. Therefore, r is in N_R.

***Corollary* 3.2:** If R is semi commutative and nil-semi clean ring then J(R) =N_R .

***Lemma* 3.3 [5]:** If R is clean ring and I is right ideal of R not contained in J(R), then I contains non zero idempotent.

***Theorem* 3.4:** If R is a nil-clean ring and idempotent elements of R are central, then J(R) = N_R.

Proof: First we show that J(R) is in N_R. Since R is nil-clean ring, then R is nil-semi clean ring and the result follows.

To show that N_R is in J(R), let a be a non-zero element of N_R and suppose that a is not an element of J(R) . Consider I = aR the right ideal of R. It is clear that I is not in J(R) since (R is nil-clean ring, then R is clean ring). By *Lemma3.3*, there exists a nonzero idempotent in I = aR, say e, e = a r, for some r in R. Now,

$$(e\,a\,e)\,(e\,r\,e) = e\,a\,e\,r\,e = eare = eee = e$$

but e is a unit in eRe , So eae is one sided unit in eRe but idempotents of R are central, that is R and eRe are Dedekind finite which means that eae is in U_{eRe} and a is in N_R implies that eae is in N_{eRe}, then **eae** belongs to the intersection of U_{eRe} and N_{eRe}. This is a contradiction.

Theorem **3.5:** If e and f are two orthogonal idempotents in R, then:

1- For x in I_{eRe} and y in I_{fRf}, x +y is in I_R

2- For x in N_{eRe} and y in N_{fRf} , x +y is in N_R

3- For x in P_{eRe} and y in P_{fRf}, x +y is in P_R

4- For x in SI_{eRe} and y in SI_{fRf} , x +y is in SI_R

Proof:

1- Let x be an element of I_{eRe} and y an element of I_{fRf}, then x= ere and y=fsf for some r,s in R.

$$(x + y)^2 = (ere)^2 + (fsf)^2 + (ere)(fsf) + (fsf)(ere)$$
$$= ere + fsf + 0 + 0 \quad \text{(e, f, orthogonal)}$$
$$= x + y. \quad \text{So } x + y \text{ is in } I_R.$$

2- If x is in N_{eRe} and y is in N_{fRf} then $x^n = 0$, $y^m = 0$ for some positive integers m, n. Then: $(x + y)^{m+n} = x^{m+n} + y^{m+n} = x^n . x^m + y^m .y^n = 0$, where x . y = y . x = 0. So, x + y is in N_R.

3- Let x be in P_{eRe} and y in P_{fRf}. There exist integers m, n p and q with m>n and p>q, such that $x^m = x^n$, $y^p = y^q$. To show that $x^{n(m-n)}$ and $y^{q(p-q)}$ are idempotents in R, note that:

$$x^n = x^m = x^{m-n+n} = x^{m-n} x^n = x^{m-n} . x^m$$
$$= x^{2m-n} = x^{2m-n+n-n} = x^{2(m-n)+n} = \ldots = x^{k(m-n)+n} \quad \text{for all } k = 1,2,3, \ldots$$

When k = n we get $x^n = x^{n(m-n)+n}$ which means that:

$$[x^{n(m-n)}]^2 = x^{n(m-n)} x^{n(m-n)}$$
$$= x^{n(m-n) + n(m-n) + n - n}$$
$$= x^{n(m-n) + n} x^{n(m-n) - n}$$
$$= x^n . x^{n(m-n) - n}$$
$$= x^{n(m-n)}$$

That is $x^{n(m-n)}$ and $y^{q(p-q)}$ are idempotents in R. Consider the positive integer

$t = 2n(m - n)q(p - q)$, then: $(x + y)^{2t} = x^{2t} + y^{2t}$ for $xy = yx = 0$

$x^{2t} = x^{4n(m-n)q(p-q)} = [x^{2n(m-n)}]^{2q(p-q)}$

$= [x^{n(m-n)}]^{2q(p-q)} = x^t$.

Similarly, $y^{2t} = y^t$ $(x + y)^{2t} = x^t + y^t = (x + y)^t$. That is, x + y is in P_R.

4- If x is in SI_{eRe} and y in SI_{fRf}, then x is in P_{eRe} and y is in P_{fRf}. By 3 above, x+y is in P_R. It is sufficient for x+y not to be nilpotent, to show x + y in SI_R. For that, suppose that $(x+y)^t = 0$ for some positive integer t, then $(x+y)^t = x^t + y^t = 0$. So $x^t = -y^t$ which means that $x^{t+1} = x(-y^t) = -xy\,y^{t-1} = 0$. This contradicts the fact that x is in SI_{eRe} and that x + y is not nilpotent. And so x + y is in SI_R.

***Theorem* 3.6:** Let R be a ring and e a central idempotent in R. Then the following holds:

1- If eRe and $_fR_f$ are nil-clean rings, then R is nil-clean ring

2- If eRe and $_fR_f$ are nil-semiclean rings, then R is nil-semiclean ring, where f = 1–e.

Proof: e central idempotent, then R = eRe + fRf. If x is in R, then x = a + b where a is in eRe and b is in fRf.

1- If eRe and fRf are nil-clean rings, then
$a = e_1 + n_1$, $b = e_2 + n_2$ where e_1 belongs to I_{eRe} and e_2 is in I_{fRf} and n_1 is in N_{eRe}, n_2 belongs to N_{fRf}. Now $x = a + b = e_1 + n_1 + e_2 + n_2 = e_1 + e_2 + n_1 + n_2$. From theorem 3.5 $e_1 + e_2$ is in I_R and $n_1 + n_2$ is in N_R. Therefore, x is nil-clean element, and R is nil-clean ring.

If eRe and fRf are nil-semiclean rings, then
$a = e_1 + n_1$, $b = e_2 + n_2$ such that e_1 belongs to SI_{eRe}, e_2 in SI_{fRf} and n_1 in N_{eRe}, n_2 in N_{fRf}. Now $x = a + b = e_1 + n_1 + e_2 + n_2 = e_1 + e_2 + n_1 + n_2$. From Theorem 3.5: $e_1 + e_2$ is in SI_R and $n_1 + n_2$ belongs to N_R. So, x is nil-semiclean element, and R is nil-semiclean ring.

***Theorem* 3.7:** If x is nil-clean element in a ring R such that x = e + n, en = ne, where $e \in I_R$ and $n \in N_R$, then x is a clean element.

Proof: If x = e + n and en = ne , where $e \in I_R$ and $n \in N_R$, then x can be written as:

$x = 1 - e + 2e - 1 + n$. It is clear that $1 - e \in I_R$.

To show that 2e-1 + n is unit, we proceed as follow:

$$(2e-1)\ (e2-1)=4e + 1 - 2e - 2e=1.$$

So 2e-1 is unit and (2e-1), n commute and 2e-1 + n is unit. Thus, x is clean.

***Theorem* 3.8:** If R is a commutative ring , then the polynomial ring R[x] is not nil-semi clean.

Proof: Since x is a nil-semiclean element, then it can be expressed as x = p + n, where $p \in SI_{R[x]}$ and n is nilpotent. But $SI_{R[x]} = SI_R$, this implies that $p \in R$ which means that (x - p) is nilpotent in R[x]. Hence the contradiction.

***Theorem* 3.9:** Let R be a ring such that IR = {0,1}. If R is nil- semi clean ring, then each element of R is unit or sum of two units .

Proof: let R be a nil- semi clean ring, and $a \in R$. So $a-1 \in R$ and nil- semi clean, i.e. a -1= s + n ; $s \in SI_R$, $n \in N_R$ since $s \in SI_R \subseteq P_R$, there exist $k \in N$ such that s^k is idempotent belong to $I_R = \{0,1\}$, i.e., $s^k = 0$ or $s^k = 1$, But s semi-idempotent ,Thus s=0 or $s^k = 1$. Now if s = 0, then a-1=n , a=n+1. Since n is nilpotent then a=n+1 is unit . If $s^k = 1$, then s unit and a=s + n+1, so a is the sum of two units.

***Theorem* 3.10:** Let R be a ring and e is an idempotent of R such that

$$e R(1-e) +(1-e)Re \subseteq J(R).$$

if eRe and (1-e) R (1-e) are semi clean rings, then R is semi clean ring.

Proof: Let $x \in R$. x can be written as: x = a+c+d+b; a = ehe, b = f kf where $h, k \in R$, **c**$\in$ e R(1-e) and **d**$\in$ (1-e)Re .Since eRe and fRf are semi clean rings, then

$$a = e_1 +u_1 ,\ b= e_2 +u_2 ;\ e_1 \in P_{eRe} ,\ e_2 \in P_{fRf} ,\ u_1 \in U_{eRe} ,\ u_2 \in U_{fRf} .$$

it is known $e_1+e_2 \in P_R$ and $u_1+u_2 \in U_R$ and by our assomption we get $c+d \in J(R)$. Therefore:

$$(c + d) + (u_1 +u_2) \in U_R .$$

Now $x = p +u$; $p = e_1+e_2 \in P_R$, $u= (c+d) + (u_1 +u_2) \in U_R$

i.e x is semiclean element and since x is arbitrary we find R is semiclean ring .

***Theorem* 3.11:** Let a be an element of unitary ring R such that:

$a = s + u$; $us=su$, $s\in SI_R$, $u \in U_R$ then:

a -$Ann_r(a) \subseteq Rs$

b-$Ann_l(a) \subseteq sR$

Proof : Given $x \in Ann_r(a)$, Now $xa = 0$ and So , $xa = 0$, $x(s + u) = 0$,
and this implies that $xs + xu = 0$ and $-xs = xu \Rightarrow -xsu^{-1} = x$.
Since $us = su$, $u^{-1}s = su^{-1}$, then $x = -xsu^{-1} = -xu^{-1}s \in Rs$. Other part similarly.

***Theorem* 3.12:** Let a be an element of unitary ring R such that:

$a = s + n$; $ns = sn$, $s\in SI_R$, $n \in N_R$, then

$Ann_r(a) \subseteq R(1-s)$

$Ann\,l(a) \subseteq (1-s)R$

Proof : Given that $x \in Ann_r(a)$. Now :

$xa = 0 \Rightarrow x(s + n) = 0$

$xs + xn = 0$

$-xs = xn$ add x for 2 sides

$-xs + x = xn + x$

$x(-s + 1) = x(n + 1)$

we have n s = s n which implies:

$$(n+1)(-s + 1) = n(-s + 1) + (-s + 1)$$
$$= -ns + n + (-s + 1)$$
$$= (-s + 1)n + (-s + 1)$$
$$= (-s + 1)(n + 1)$$

Therefore:

$(n+1)(-s + 1) = (-s + 1)(n + 1)$ which implies that

$(n+1)^{-1}(-s + 1) = (-s + 1)(n + 1)^{-1}$ $(n \in N_R \Rightarrow n+1 \in U_R)$

thus

$x = x(n + 1)^{-1}(-s + 1) \in R(1-s)$.

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