Jordan Journal of Mathematics and Statistics (JJMS) 2 (2), 2009, pp. 105-118

### ROBUST ESTIMATION OF THE SEASONAL AUTOCORRELATION OF THE PAR(1) MODEL \*

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**ABSTRACT:** In this article we are interested in the robust estimation of seasonal autocorrelation for the periodic autoregressive model of order one (PAR (1)). We used three estimators for the first – lag seasonal autocorrelation including the classical moment estimator beside two new proposed robust estimators. The effect of a single additive outlier contaminated in the time series is examined via bias and MSE. We have also studied the effects of some other factors on the quality of those estimators. The investigation is carried out using Monte – Carlo simulation. The results show that our proposed estimators are robust whereas the moment estimator is not. This conclusion is also assured via the bias and MSE of those estimators.

#### **1. Introduction**

The periodic autoregressive moving – average (PARMA) model is an extension of the ordinary Box – Jenkins ARMA model which is appropriate for modeling seasonal time series. A classical and complete reference of the ordinary ARMA models is Box et al. (1994). Denoting the number of seasons per year (period) as  $\omega$ , the varying orders PARMA model denoted by PARMA<sub> $\omega$ </sub>(p(v),q(v)) is written as:

$$(1-\phi_{l}(v)B-...-\phi_{p(v)}B^{p(v)})(X_{k,v}-\mu_{v}) = (1-\theta_{l}(v)B-...-\theta_{q(v)}B^{q(v)})q_{k,v}$$
(1.1)

where  $v=1,2,...,\omega$  denotes the season, k denotes the year,  $\{a_{k,v}\}\$  is a zero – mean white noise process with periodic variance  $\sigma^2(v)$ , p(v) and q(v) are, respectively, the AR and MA orders for season v,  $\mu_v$  is the mean of the  $v^{th}$  season and  $\phi_1(v),..., \phi_{p(v)}(v)$  and  $\theta_1(v),..., \theta_{q(v)}(v)$  are the AR and MA parameters of season v, respectively. For more details on PARMA models see Franses and Paap (2004).

<sup>2000</sup> Mathematics Subject Classification: Primary: 62M10; Secondary: 37M10, 62F35.

Keywords: Periodic autoregression, Robust estimation, Seasonal auto – correlations, Additive outlier, Monte – Carlo simulation.

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The periodic autoregressive (PAR) model is a special case of the PARMA model. In equation (1.1), setting q(v)=0 and p(v)=1 for each  $v=1,2,...,\omega$  we get the equation of the PAR<sub> $\omega$ </sub>(1) model. The PAR<sub> $\omega$ </sub>(1) model is of particular interest in this article. Although it could be the simplest PARMA model, it is however very important in practice. Beside its simplicity, its mathematical properties as the estimation of parameters and forecasting are straight forward. This model is also proved an efficient model to describe periodic autocorrelations among seasonal time series (Mcleod,1993). Assuming  $\omega=4$ , corresponding to the common quarterly time series, the zero – mean PAR<sub>4</sub>(1) model is written as:

$$X_{k,\nu} = \phi_1(\nu) X_{k,\nu-1} + a_{k,\nu}$$
(1.2)

where v=1,2,3,4 and k any integer. Note that the PAR<sub> $\omega$ </sub>(1) model, as in (1.2), consists of  $\omega$  equations with each equation as AR(1), so that (1.2) can be written as:

$$X_{k,1} = \phi_1(1)X_{(k-1),4} + a_{k,1}$$

$$X_{k,2} = \phi_1(2)X_{k,1} + a_{k,2}$$

$$X_{k,3} = \phi_1(3)X_{k,2} + a_{k,3}$$

$$X_{k,4} = \phi_1(4)X_{k,3} + a_{k,4}$$

In this model, the varying AR parameters as well as varying white noise variances makes this model suitable for modeling periodic autocorrelations.

In this article, we are interested in the robust estimation of the seasonal autocorrelation function (Seas. ACF) of the  $PAR_{\omega}(1)$  model. On one hand, the traditional estimation of PARMA models was studied by several authors including, for example, Pagano (1978), Vecchia (1985) and Basawa and Lund (1999). On the other hand, robust estimation in time series analysis is an area of interest for a long period of time. Fox (1972) could be the first who discussed the issue of outliers in time series data. Other references to this issue are, for example, Martin (1981) and Chang et al. (1988).

Moreover, the autocorrelation function (ACF) is a very important tool in time series modeling. For instance, in the ARMA context, the ACF plays a primary role in the identification and estimation phases (Box et al., 1994). For periodic time series, the ACF extends to the seasonal ACF which is defined in the next section. This function again play an important role in PARMA modeling (Franses and Paap, 2004).

In the next section, the theoretical and sample seasonal ACF is defined. Then the first lag periodic autocorrelation is explored for the  $PAR_{\omega}(1)$  model. Besides, three estimators for the first lag ordinary autocorrelation of the AR(1) model previously investigated by Berkoun et al. (2003) which include the moment estimator and two other robust estimators. Those estimators are then generalized to the

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 $PAR_{\omega}(1)$  model. Later on, those estimators are investigated using Monte – Carlo simulation.

#### 2. The Seasonal ACF of PAR(1) Model

The PAR model is not stationary in the ordinary sense. Alternatively, the PAR model is subject to a weaker type of stationarity named as periodic stationarity. The ordinary type of stationarity, known as second order stationarity requires constant mean and variance and that the autocovariances depend on time lag only (Cryer,1986). On the other hand, periodic stationarity requires that the mean and variances of the process are constants for each season and periodic with period  $\omega$ , and that the autocovariance depend on time lag and season only. The periodic stationarity conditions for various PARMA processes can be obtained by solving an eigen–value problem. This approach is achieved by transforming the PARMA model consisting of  $\omega$  equations into its corresponding  $\omega$  – variate vector ARMA model. For more details on this issue, see Ula and Smadi (1997). For instance, for the PAR<sub> $\omega$ </sub>(1) model, the periodic stationarity conditions is that

$$\left| \underset{v=l}{\overset{\omega}{\prod}} \phi_1(v) \right| < 1.$$
 (2.1)

Taking  $\omega=1$ ; the PAR<sub>4</sub>(1) model reduces to the ordinary zero – mean AR(1) model written as:

$$X_t = \phi X_{t-1} + a_t \,. \tag{2.2}$$

It is known that this model is stationary if  $|\phi| < 1$  which is the same as (2.1) for  $\omega = 1$ .

The ACF of AR(1) model is  $\rho_k = \phi^k$ , k=0,1,...Thus, the first order autocorrelation  $\rho_1$  is nothing but  $\phi$ . Berkoun et al. (2003) investigated robust estimation of  $\rho_1$  for the zero – mean AR(1) model. Assuming that the time series has a single additive outlier, they have investigated three estimators of  $\rho_1$ , namely:

$$\hat{\rho}_{1} = \frac{\sum_{t=2}^{n} Z_{t} Z_{t-1}}{\sum_{t=2}^{n} Z_{t-1}^{2}}$$
(2.3)

$$\widetilde{\rho}_{1} = Med\left\{\frac{Z_{2}}{Z_{1}}, \frac{Z_{3}}{Z_{2}}, \dots, \frac{Z_{n}}{Z_{n-1}}\right\}$$

$$(2.4)$$

$$\breve{\rho}_{1} = \frac{Med\{Z_{1}Z_{2}, Z_{2}Z_{3}, \dots, Z_{n-1}Z_{n}\}}{Med\{Z_{1}^{2}, Z_{2}^{2}, \dots, Z_{n-1}^{2}\}}$$
(2.5)

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where  $\hat{\rho}_1$  is the ordinary moment estimator of  $\rho_1$ ,  $\tilde{\rho}_1$  and  $\tilde{\rho}_1$  are two robust estimators of  $\rho_1$  that are originally proposed by Hurwicz (1950) and Haddad (2000), respectively. Berkoun et al. (2003) showed that the inference based on the moment estimator  $\hat{\rho}_1$  is highly sensitive to a single additive outlier. They also suggested replacing  $\hat{\rho}_1$  with the robust alternatives  $\tilde{\rho}_1$  and  $\tilde{\rho}_1$  for inference purposes in the presence or suspect of outliers in data. Now, for periodic stationary PARMA models, the seasonal autocorrelation function (Seasonal ACF) depends on the time lag and season only and is defined as:

$$\rho_j(\nu) = Cor(X_{k,\nu}, X_{k,\nu-j}) = \frac{\gamma_j(\nu)}{\sqrt{\gamma_o(\nu)\gamma_o(\nu-j)}},$$
(2.6)

where  $\gamma_j(v)$  denotes the seasonal autocovariance function and  $\gamma_o(v)$  denotes the variance of the process for season v and time lag j=0,1,... The moment estimator of  $\rho_j(v)$  is given by:

$$r_{j}(v) = \frac{c_{j}(v)}{\sqrt{c_{0}(v)c_{0}(v-j)}},$$
(2.7)

where  $c_i(v)$  denotes the sample seasonal autocovariance function, defined as:

$$c_{j}(v) = \frac{\sum_{k=0}^{n-1} (X_{k,v} - \bar{X}_{v})(X_{k,v-j} - \bar{X}_{v-j})}{n-1},$$
(2.8)

where  $\overline{X}_{\nu}$  is the sample mean of data in season v and n is the number of years of data (see, McLeod, 1994).

As far as the  $PAR_{\omega}(1)$  model is considered, it can be proved that the first lag autocorrelations are given by:

$$\rho_{1}(v) = \phi_{1}(v) \sqrt{\frac{\gamma_{0}(v-1)}{\gamma_{0}(v)}},$$
(2.9)

for v=1,2,..., $\omega$ . Note that in this case the first order autocorrelation are not the same as AR parameters but of similar sign and a function of them. For the computation of  $\rho_1(v)$ , given  $\phi_1(v)$  and  $\sigma^2(v)$ , (2.9) can be used along the fact that for the PAR<sub> $\omega$ </sub>(1) model

$$\gamma_0(v) = (\phi(v))^2 \gamma_0(v-1) + \sigma^2(v), v = 1, ..., \omega$$

This system of  $\omega$  equations can be written as:

$$A\Gamma_0 = \Sigma$$
,

where

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & -\phi_{l}^{2}(1) \\ -\phi_{l}^{2}(2) & 1 & 0 & \dots & 0 \\ 0 & -\phi_{l}^{2}(3) & 1 & \dots & 0 \\ \vdots & & & & & \\ \vdots & & & & & \\ 0 & \dots & -\phi_{l}^{2}(v-1) & 1 \end{bmatrix}, \quad \Gamma_{0} = \begin{bmatrix} \gamma_{0}(1) \\ \gamma_{0}(2) \\ \gamma_{0}(3) \\ \vdots \\ \gamma_{0}(\omega) \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma^{2}(1) \\ \sigma^{2}(2) \\ \sigma^{2}(3) \\ \vdots \\ \vdots \\ \sigma^{2}(\omega) \end{bmatrix}$$
(2.10)

So that  $\Gamma_0 = A^{-1}\Sigma$ .

For example, for the PAR<sub>4</sub>(1) model with  $\phi_1(1) = 0.8$ ,  $\phi_1(2) = 1.1$ ,  $\phi_1(3) = -0.7$  and  $\phi_1(4) = 0.3$  and  $\sigma^2(1) = ... = \sigma^2(4) = 1$ , we can easily show that  $\rho_1(1) = 0.6636$ ,  $\rho_1(2) = 0.8269$ ,  $\rho_1(3) = -0.7796$  and  $\rho_1(4) = 0.4320$ .

Shao (2007) investigated robust estimation of PAR models. He extends the results of Basawa (1985) towards PAR models. Shao focused on the estimation of the parameters of PAR models. In this article, we are interested in the estimation of the first order autocorrelation for the PAR<sub> $\omega$ </sub>(1) model. Beside the moment estimate of  $\rho_1(v)$  given in (2.7) with j=1 which we will denote as  $\hat{\rho}_1(v)$ , we propose in the next section two robust estimators for the PAR<sub> $\omega$ </sub>(1) model similar to  $\tilde{\rho}_1$  and  $\bar{\rho}_1$  for the AR(1) model given by (2.4) and (2.5), respectively.

#### 3. Robust Estimation of the First Order Auto-Correlation

Assume that the stochastic process  $\{X_{k,\nu}\}$  follows the PAR<sub> $\omega$ </sub>(1) model. Let  $\{Z_{k,\nu}\}$  be the same process but contaminated with an additive outlier at year  $k_o$  and season  $v_o$ . That is,

$$Z_{k,v} = \begin{cases} X_{k,v}, & (k,v) \neq (\mathbf{k}_0, \mathbf{v}_0) \\ X_{k,v} + \Delta, & (k,v) = (\mathbf{k}_0, \mathbf{v}_0) \end{cases}$$

for which we are interested in estimating the first lag seasonal autocorrelation. Here, we generalize the three estimators in Berkoun et al. (2003) shown in equations (2.3), (2.4) and (2.5) for  $\rho_1(v)$  of the PAR<sub> $\omega$ </sub>(1) model, for  $v=1,...,\omega$ , as follows:

$$\hat{\rho}_{1}(v) = \frac{\sum_{k=1}^{n} (Z_{k,v} - \overline{Z}_{v})(Z_{k,v-1} - \overline{Z}_{v-1})}{\sqrt{\sum_{k=1}^{n} (Z_{k,v} - \overline{Z}_{v})^{2} \sum_{k=1}^{n} (Z_{k,v-1} - \overline{Z}_{v-1})^{2}}},$$
(3.1)

$$\widetilde{\rho}_{1}(v) = M_{k} d \left\{ \frac{Z_{k,v}^{*}}{Z_{k,v-1}^{*}} \right\} \sqrt{\frac{\frac{Med(Z_{k,v-1}^{*2})}{k}}{\frac{Med(Z_{k,v}^{*2})}{k}}}$$
(3.2)

and

$$\bar{\rho}_{1}(v) = \frac{Med\{Z_{k,v}^{*}Z_{k,v-1}^{*}\}}{\sqrt{Med\{Z_{k,v}^{*}\}}Med\{Z_{k,v-1}^{*}\}},$$
(3.3)

where  $\{Z_{k,\nu}^*\}$  is the seasonally median subtracted time series, that is

$$Z_{k,\nu}^* = Z_{k,\nu} - M_{i} d\{Z_{j,\nu}\}, k = 1, ..., n.$$

The moment estimator of  $\rho_1(v)$  given in (3.1) is nothing but (2.7) with j = 1. If the time series  $\{Z_{k,v}\}$  is zero – mean and non – periodic (i.e.,  $\omega = 1$ ) then  $c_1(v)$  in (2.8) reduces to  $c_1 = \frac{\sum Z_t Z_{t-1}}{n-1}$  and  $c_0(v) = c_0(v-1) = c_1 = \frac{\sum Z_t^2}{n-1}$ . Thus,  $r_1(v)$  in (2.7) reduces to the ordinary moment estimator of  $\rho_1$  in (2.3).

Similarly, it can easily be shown that the estimators in (3.2) and (3.3) reduce to (2.4) and (2.5) for the ordinary AR(1) model. In fact, (3.2) is developed based on (2.9) where  $\operatorname{Med}_{k}\left\{\frac{Z_{k,v}^{*}}{Z_{k,v-1}^{*}}\right\}$  estimates  $\phi_{1}(v)$  and  $\operatorname{Med}_{k}\left\{Z_{k,v}^{*2}\right\}$  estimates  $\gamma_{0}(v)$ . Also, (3.3) is developed based on the original definition of  $\rho_{1}(v)$  given by (2.6) for j = 1 with  $\operatorname{Med}_{k}\left\{Z_{k,v}^{*}Z_{k,v-1}^{*}\right\}$  estimating  $\gamma_{1}(v)$ .

Therefore, our main objective is to investigate the behaviors of the estimators  $\hat{\rho}_{i}(\nu)$ ,  $\tilde{\rho}_{i}(\nu)$  and  $\bar{\rho}_{i}(\nu)$  in the presence of a single additive outlier using Monte – Carlo simulation. In addition, we aim to study other factors that may affect the quality of those estimators like the magnitude and the place of the outlier and the realization length along some other factors.

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#### 4. Simulation Study

In order to investigate the behavior of the three estimators of  $\rho_1(v)$  for the  $PAR_{\omega}(1)$  model defined in equations (3.1) – (3.3) we will carry out a simulation study based on the following PAR models:

Model A: PAR<sub>4</sub>(1) with  $\phi$ 's : 0.8, 1.1, -0.7, 0.3.

Model B: PAR<sub>4</sub>(1) with  $\phi$ 's: 1.8, 0.9, -1.2, -0.51.

Model C:  $PAR_2(1)$  with  $\phi$ 's: 0.9, 0.8.

Model D:  $PAR_{12}(1)$  with  $\phi$ 's: 1.1, -0.8, 0.95, -1.2, 0.7, 0.9, -1.3, -0.7, 1.3, 0.5, 1.2, -1.8.

All of the PAR models above are chosen to be periodic stationary. In this study we will investigate several aspects during the comparison of various estimators of  $\rho_1(v)$ . First, we study the period length  $\omega$ . Notice that in models above  $\omega$  covers 2, 4 and 12. Next, we will study the behavior of estimators when the place of outlier is changed. The default time of outlier is chosen at t=13, which means that the outlier lies in season 1. We have simulated other cases with different locations of outliers. Besides, model B is chosen to examine the behavior of various estimators when the process is close to the non – stationarity region since  $\Pi\phi_i = 0.99$  is close to 1.

The simulations are carried out using a FORTRAN code written by the authors for realization lengths N=50,100,300. In each case, one thousand simulations are done, based on which the mean bias and MSE of various estimators of  $\rho_1(v)$ , defined by (4.1) – (4.3) are computed. Theoretical values of  $\rho_1(v)$  for various cases are computed using (2.9) and (2.10). The mean bias and MSE are computed as follows: Assume that for a specific PAR(1) model and specific season v, 1000 realizations each of length n years are simulated. Then we compute the exact correlation  $\rho_1(v)$  and then 1000 estimates of  $\rho_1(v)$  are computed, say:

$$\{(\rho_1^{*}(v))_1, (\rho_1^{*}(v))_2, ..., (\rho_1^{*}(v))_{1000}\}$$
. Then

Mean Bias = 
$$\frac{1}{1000} \sum_{k=1}^{1000} (\rho_1^*(v))_k - \rho_1(v)$$

and

MSE = 
$$\frac{1}{1000} \sum_{k=1}^{1000} \{(\rho_1^*(v))_k - \rho_1(v)\}^2$$

Tables (1) – (3) and Figures (1) – (4) present some selected results. In Figure (1) we compare between various estimators of  $\rho_1(v)$  for model (A) when there is no outlier and when an additive outlier is involved. In Table (1) we investigate the effect of constant variances of the white noise process against varying variances for model (A) for several values of realization length. Table (2) is based on model (B) in which we focus on the case were the parameters of the PAR(1) model is close to the non – stationarity region. Table (3) shows some similar results but for the PAR(1) model with period length 2. In Figures (2) and (3) we respectively study the mean bias and MSE of various estimators of  $\rho_1(v)$  for again model (A) but for various locations of the additive outlier. Finally, we investigate the case with period length 12 in Figure (4) which is based on model (D).



**Figure (1):** The mean bias and MSE for various estimates of  $\rho_1(v)$  for the PAR<sub>4</sub>(1) model (model A) with unit white noise variances and n = 300.

(•: No outliers,  $\blacksquare$ : With an additive outlier of  $\Delta = 50$  at t =13)

		(a) $\sigma_{\nu}^2 = 1,1,1,1$			(b) $\sigma_{\nu}^2 = 10,100,1,4$		
N	Season	$\hat{\rho}_1(v)$	$\widetilde{ ho}_1(v)$	$\breve{ ho}_1(v)$	$\hat{\rho}_1(v)$	$\widetilde{ ho}_1(v)$	$\breve{ ho}_1(v)$
50	1	- 0.5451 (0.3171)	- 0.0611 (0.0332)	- 0.1592 (0.0706)	- 0.0499 (0.0049)	- 0.0435 (0.0224)	- 0.1089 (0.0419)
	2	- 0.6701 (0.4678)	- 0.0036 (0.0239)	- 0.1037 (0.0429)	- 0.0720 (0.0146)	- 0.0195 (0.0466)	- 0.0888 (0.0315)
	3	0.5943 (0.3540)	0.0037 (0.0251)	0.1077 (0.0425)	0.0415 (0.0017)	- 0.0008 (0.0003)	0.0000 (0.0000)
	4	- 0.3420 (0.1177)	- 0.0229 (0.0398)	- 0.1484 (0.0515)	- 0.4353 (0.1895)	- 0.0011 (0.0086)	- 0.0125 (0.0071)
100	1	- 0.4924 (0.2515)	- 0.0282 (0.0164)	- 0.1486 (0.0407)	- 0.0276 (0.0017)	- 0.0222 (0.0109)	- 0.0863 (0.0210)
	2	- 0.6159 (0.3878)	- 0.0011 (0.0118)	- 0.0959 (0.0240)	- 0.0674 (0.0089)	- 0.0082 (0.0075)	- 0.0921 (0.0194)
	3	0.5310 (0.2826)	- 0.0024 (0.0134)	0.1020 (0.0284)	0.0408 (0.0017)	- 0.0003 (0.0002)	0.0001 (0.0000)
	4	- 0.3126 (0.0984)	- 0.0085 (0.0178)	- 0.1432 (0.0366)	- 0.4352 (0.1894)	- 0.0028 (0.0045)	- 0.0121 (0.0048)
300	1	- 0.3866 (0.1523)	- 0.0100 (0.0053)	- 0.1288 (0.0226)	- 0.0097 (0.0003)	- 0.0043 (0.0032)	- 0.0722 (0.0099)
	2	- 0.5000 (0.2522)	- 0.0022 (0.0035)	- 0.0888 (0.0128)	- 0.0704 (0.0064)	- 0.0039 (0.0075)	- 0.1004 (0.0138)
	3	0.4156 (0.1731)	0.0031 (0.0038)	0.1006 (0.0157)	0.0399 (0.0016)	0.0002 (0.0001)	- 0.0001 (0.0001)
	4	- 0.2646 (0.0704)	- 0.0062 (0.0063)	- 0.1413 (0.0260)	- 0.4348 (0.1890)	- 0.0004 (0.0014)	- 0.0117 (0.0019)

**Table (1):** The mean bias and MSE (given in parentheses) of estimates of  $\rho_1(v)$  for the PAR<sub>4</sub>(1) model (model A) with additive outlier  $\Delta = 50$  at time t = 13

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**Table (2):** The mean bias and MSE (given in parentheses) of estimates of  $\rho_1(v)$  for the PAR<sub>4</sub>(1) model (model B) with unit white noise variances and additive outlier  $\Delta = 50$  at time t = 13

	(a) N = 50			(b) N = 300		
Season	$\hat{\rho}_1(v)$	$\widetilde{\rho}_1(v)$	$\check{\rho}_1(v)$	$\hat{\rho}_1(v)$	$\widetilde{\rho}_1(v)$	$\check{\rho}_1(v)$
1	- 0.3766	- 0.0396	- 0.0569	- 0.0341	- 0.0057	- 0.0093
	(0.1684)	(0.0111)	(0.0146)	(0.0015)	(0.0009)	(0.0009)
2	- 0.4068	- 0.0050	- 0.0630	- 0.1489	- 0.0005	- 0.0130
	(0.1853)	(0.0107)	(0.0158)	(0.0224)	(0.0010)	(0.0010)
3	0.5304	0.0086	0.0264	0.3910	0.0021	0.0054
	(0.2866)	(0.0079)	(0.0075)	(0.1530)	(0.0008)	(0.0007)
4	0.5288	0.0521	0.0866	0.4122	0.0084	0.0176
	(0.2844)	(0.0178)	(0.0263)	(0.1700)	(0.0012)	(0.0019)

**Table (3):** The mean bias and MSE (given in parentheses) of estimates of  $\rho_1(v)$  for the PAR<sub>2</sub>(1) model (model C) with unit white noise variances and additive outlier  $\Delta = 50$  at time t = 13

N	Season	$\hat{\rho}_1(v)$	$\widetilde{\rho}_1(v)$	$\breve{ ho}_1(v)$
50	1	- 0.6530 (0.4459)	- 0.0801 (0.0283)	- 0.1364 (0.0558)
	2	- 0.6445 (0.4341)	- 0.0369 (0.0315)	- 0.1417 (0.0581)
100	1	- 0.5642 (0.3276)	- 0.0328 (0.0123)	- 0.1093 (0.0294)
	2	- 0.5695 (0.3327)	- 0.0185 (0.0128)	- 0.1091 (0.0303)
300	1	- 0.3868 (0.1523)	- 0.0118 (0.0039)	- 0.0842 (0.0127)
	2	- 0.4316 (0.1883)	- 0.0071 (0.0040)	- 0.0872 (0.0137)



**Figure (2):** The mean bias for the PAR<sub>4</sub>(1) model (model A) with unit white noise variances and n = 300,  $\Delta = 50$ . The place of additive outlier (a) t = 13, (b) t = 14, (c) t = 15, (d) t = 16.  $(\bullet: \hat{\rho}_1(v), \bullet: \tilde{\rho}_1(v), \bullet: \tilde{\rho}_1(v))$ 



**Figure (3):** The MSE for the PAR<sub>4</sub>(1) model (model A) with unit white noise variances and n = 300,  $\Delta = 50$ . The place of additive outlier (a) t = 13, (b) t = 14, (c) t = 15, (d) t = 16. ( $\bullet$ :  $\hat{\rho}_1(v)$ ,  $\bullet$ :  $\tilde{\rho}_1(v)$ ,  $\bullet$ :  $\tilde{\rho}_1(v)$ )



**Figure (4):** The mean bias and MSE for the  $PAR_{12}(1)$  model (Model D) and unit white noise variances = 1, n = 300,  $\Delta$  = 50 and place of additive outlier at time t = 13. (a) Mean bias, (b) MSE.

 $(\bullet: \hat{\rho}_1(v), \bullet: \tilde{\rho}_1(v), \bullet: \tilde{\rho}_1(v))$ 

### 5. Discussion and Conclusion

In view of the simulation results above, we can see in all tables and figures above that the moment estimator of  $\rho_1(v)$ ,  $\hat{\rho}_1(v)$ , is affected by the outlier while the other two estimators were apparently robust. This fact is true via bias and MSE. For instance, in Figure (1) we can see that for the two robust estimators  $\tilde{\rho}_1(v)$  and  $\tilde{\rho}_1(v)$ , the bias and MSE were not affected by the presence of outliers while  $\hat{\rho}_1(v)$  is clearly affected as in Figure (1 – 1b).

We also conclude that, in general, for all estimators the mean bias and MSE decrease as the realization length increases. The best of the two robust estimators in almost all cases was  $\tilde{\rho}_1(v)$ , defined by (3.2), in view of bias and MSE. Changing the white noise variances makes no significant changes on the behavior of  $\tilde{\rho}_1(v)$  and  $\tilde{\rho}_1(v)$ , as can be seen in Table (1), whereas  $\hat{\rho}_1(v)$  is slightly affected by this change. If the parameters of the PAR(1) model is close to the non – stationarity region, then all estimators seem not affected by this issue as shown in Table (2).

As for the period length  $\omega$ , we can see in Tables (1) and (2) in which  $\omega = 4$  and in Table (3) and Figure (4) which correspond, respectively to  $\omega = 2$  and 12 that the conclusions above regarding the three estimators are in general still valid.

However, we may conclude from Figure (4), in which  $\omega = 12$ , that the effect of the single additive outlier which is contaminated in the first season is apparent in the bias and MSE of  $\hat{\rho}_1(v)$  for all seasons v = 1, ..., 12. The place of outlier has no significant change on our conclusions as depicted in Figures (2) and (3).

Finally, we should point out that our conclusions are initially valid for the selected models and cases. However, it is clear that the moment estimator of  $\rho_1(v)$  was not robust in the presence of outliers, as expected. We should also emphasize that it is not our objective to neglect outlying observations. But, in this article, we draw attention to the bad effect of such observations on the estimation of the ACF which may then affect other phases of model building as the identification and estimation of proper models. In addition, in this study we have considered the case of a single additive outlier. Further studies may examine other types and numbers of outliers on the autocorrelations of PAR(1) model or other PARMA models.

#### Acknowledgement

We would like to thank an anonymous referee for his valuable comments that improved this article.

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