Jordan Journal of Mathematics and Statistics (JJMS) 2009, 2(1), 1-10

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FRACTIONAL CALCULUS OPERATORS ASSOCIATED WITH A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS

G.MURUGUSUNDARAMOORTHY AND R.THEMANGANI

Abstract. Making use of fractional calculus operator, we define a new subclass of uniformly convex functions with negative coefficients. Characterization property, extreme points, distortion bounds, the results on Integral transform and sharp integral means inequalities for the function class are determined.

1. Introduction and Preliminaries

Denote by A the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z : \ |z| < 1.\}$. Denote by S the subclass of A consisting of functions normalized by $f(0) = 0 = f'(0) - 1$ which are univalent in $\mathbb{U}$ and ST and CV the subclasses of S of starlike and convex respectively. Also denote the subclass T consisting of functions in S which are univalent in $\mathbb{U}$ and of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

The class T was introduced and studied by Silverman [16].

A function $f(z)$ is uniformly convex (uniformly starlike) in $\triangle$ if $f(z)$ is in CV (ST) and has the property that for every circular arc γ contained in $\triangle$, with center ξ also in $\triangle$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by UCV and the class of uniformly starlike functions by UST (for details see [4, 5]). It is well known from [9, 14] that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

1991 *Mathematics Subject Classification.* Primary: 30C45; Secondary: 30C50.

Key words and phrases. Classes of Univalent functions, convex functions, starlike functions, uniformly convex functions, fractional derivatives, fractional integral, integral means.

Received : Aug. 31,2008 Accepted : June 16,2009.

In [14], Rønning introduced a new class of starlike functions related to UCV and defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \text{Re}\ \left\{\frac{zf'(z)}{f(z)}\right\}.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further, Rønning [15], generalized the class S_p by introducing a parameter $\gamma,\ -1 \leq \gamma < 1$,

$$f \in S_p(\gamma) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \text{Re}\ \left\{\frac{zf'(z)}{f(z)} - \gamma\right\}.$$

We recall the following definitions of the classical operators of fractional calculus, adopted for working in classes of analytic functions in complex plane by Owa [11] and Srivastava and Owa [18],etc.as follows.

Definition 1.1. [11] *Let the function $f(z)$ be analytic in a simply - connected region of the $z-$ plane containing the origin. The fractional integral of f of order μ is defined by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad \mu > 0, \tag{1.3}$$

where the multiplicity of $(z-\xi)^{1-\mu}$ is removed by requiring $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. [11] *The fractional derivatives of order μ, is defined for a function $f(z)$, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad 0 \leq \mu < 1, \tag{1.4}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\xi)^{-\mu}$ is removed as in Definition 1.1.

Definition 1.3. [11] *Under the hypothesis of Definition 1.2, the fractional derivative of order $n + \mu$ is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z), \quad (0 \leq \mu < 1\ ;\ n \in N_0). \tag{1.5}$$

With the aid of the above definitions, and their known extensions involving fractional derivative and fractional integrals, Srivastava and Owa [18] introduced the operator Ω^δ $(\delta \in \mathbb{R}; \delta \neq 2, 3, 4, \dots) : A \to A$ defined by

$$\Omega^\delta f(z) = \Gamma(2-\delta) z^\delta D_z^\delta f(z),$$

$$\Omega^\delta f(z) = z + \sum_{n=2}^{\infty} \Phi(n,\delta) a_n z^n \tag{1.6}$$

where

$$\Phi(n,\delta) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}. \tag{1.7}$$

Motivated by the earlier works of Altintas et al.[2, 3], Owa[12] , and Raina and Srivastava[13]) we define a subclass of uniformly convex functions based on fractional calculus operator .

For $0 \leq \gamma < 1$, $0 \leq \alpha < 2$ and $0 \leq \beta < 2$, we let $UCV(\alpha,\beta,\gamma)$ be the class of functions $f \in S$ satisfying the inequality

$$\text{Re}\left\{\frac{\Omega^\alpha f(z)}{\Omega^\beta f(z)} - \gamma\right\} > \left|\frac{\Omega^\alpha f(z)}{\Omega^\beta f(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{1.8}$$

We also let $TUCV(\alpha,\beta,\gamma) = UCV(\alpha,\beta,\gamma) \cap T$.

The main object of the present paper is to investigate some coefficient estimates, extreme points, distortion bounds and results on integral transforms for the subclass $TUCV(\alpha,\beta,\gamma)$. We also obtain integral means inequality for higher order fractional derivative and fractional integrals of functions belonging to this class.

2. Main Results

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the class $TUCV(\alpha,\beta,\gamma)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $UCV(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty} \frac{[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{1-\gamma}|a_n| \leq 1, \tag{2.1}$$

$0 \leq \alpha < 2,\ 0 \leq \beta < 2$ *and* $0 \leq \gamma < 1$.

Theorem 2.2. *A function $f(z)$ of the form (1.2) is in the class $TUCV(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} [2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]\, a_n \leq 1-\gamma,$$

$0 \leq \alpha < 2,\ 0 \leq \beta < 2$ *and* $0 \leq \gamma < 1$.

The proofs of Theorem 2.1 and Theorem 2.2, are much akin to the proofs of theorems given in [10], hence we omit the details.

Corollary 2.3. *Let a function $f(z)$ defined by (1.2) belong to the class $TUCV(\alpha,\beta,\gamma)$. Then*

$$a_n \leq \frac{1-\gamma}{[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}, \quad n \geq 2.$$

The equality is attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\gamma}{[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]} z^n, \quad n \geq 2. \tag{2.2}$$

Theorem 2.4. *(Extreme Points) Let the function $f(z)$ defined by (1.2) satisfy (1.8). Define $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{B_n}$ $(n = 2,3,\dots)$, where $B_n = \frac{[2\Phi(n,\alpha)-(1+\gamma)\Phi(n,\beta)]}{(1-\gamma)}$.*

Then $f \in TUCV(\alpha,\beta,\gamma)$ *if and only if* $f(z)$ *can be expressed as* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, *where* $\sum_{n=1}^{\infty} \lambda_n = 1$ *and* $\lambda_n \geq 0$.

Proof. Suppose $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{\lambda_n}{B_n} B_n = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1$. Thus $f \in TUCV(\alpha,\beta,\gamma)$.

Conversely, suppose $f \in TUCV(\alpha,\beta,\gamma)$. Since $|a_n| \leq \frac{1}{B_n}$, $(n = 2, 3, \dots)$, we may set $\lambda_n = B_n|a_n|$, $(n = 2, 3, \dots)$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$.
This completes the proof. □

Theorem 2.5. *Let* $0 \leq \alpha < 2$, $\beta \leq \alpha$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in TUCV(\alpha,\beta,\gamma)$, *then*

$$|z| - \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]}\ |z|^2 \leq |f(z)| \leq |z| + \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]}\ |z|^2$$

for $z \in \mathbb{U}$.

Proof. Let $0 \leq \alpha < 2$, $\beta \leq \alpha$ and $0 \leq \gamma < 1$ and let $f(z) \in TUCV(\alpha,\beta,\gamma)$. Then, by virtue of Theorem 2.2, we obtain

$$[2\Phi(2,\alpha) - (1+\gamma)\Phi(2,\beta)] \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} [2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)] a_n \leq 1-\gamma$$

where $\Phi(n,\alpha)$ and $\Phi(n,\beta)$ are given by (1.7). This readily yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\gamma}{[2\Phi(2,\alpha)-(1+\gamma)\Phi(2,\beta)]} = \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]} \tag{2.3}$$

Using (1.2) and (2.3) we obtain

$$\begin{aligned} |f(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]}\ |z|^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]}\ |z|^2 \end{aligned}$$

which completes the proof of Theorem 2.5. □

Using the definitions of fractional integral operator and fractional derivative operator, we obtain distortion bounds for the class $TUCV(\alpha,\beta,\gamma)$.

Theorem 2.6. *Let the function $f(z)$ defined by (1.2) be in the class $TUCV(\alpha,\beta,\gamma)$. Then we have*

$$\left|D_z^{-\mu}(\Omega^\alpha f(z))\right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{2(2-\beta)(1-\gamma)|z|}{(2+\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

and

$$\left|D_z^{-\mu}(\Omega^\alpha f(z))\right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{2(2-\beta)(1-\gamma)|z|}{(2+\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

for $0 \leq \alpha < 2$, $\beta \leq \alpha$ *and* $0 \leq \gamma < 1$; $z \in \mathbb{U}$.

Proof. Define the function $F(z)$ by

$$\begin{aligned} F(z) &= \Gamma(2+\mu)z^{-\mu}D_z^{-\mu}(\Omega^\alpha f(z)) \\ &= \Gamma(2+\mu)z^{-\mu}D_z^{-\mu}[\Gamma(2-\mu)z^\alpha D_z^\alpha f(z)] \\ &= z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n+1)\Gamma(2-\alpha)\Gamma(2+\mu)}{\Gamma(n+1-\alpha)\Gamma(n+1+\mu)}z^n a_n \\ &= z-\sum_{n=2}^{\infty}\Psi(n)a_n z^n \end{aligned}$$

where

$$\Psi(n)=\frac{\Gamma(n+1)\Gamma(n+1)\Gamma(2-\alpha)\Gamma(2+\mu)}{\Gamma(n+1-\alpha)\Gamma(n+1+\mu)} \qquad (n\geq 2).$$

It is easy to see that

$$0<\Psi(n)\leq\Psi(2)=\frac{4}{(2-\alpha)(2+\mu)}. \tag{2.4}$$

Using Theorem 2.2, we have,

$$\frac{4}{2-\alpha}-(1+\gamma)\frac{2}{2-\beta}\sum_{n=2}^{\infty}a_n \leq \sum_{n=2}^{\infty}\left[2\Phi(n,\alpha)-(1+\gamma)\Phi(n,\beta)\right]a_n \leq 1-\gamma$$

$$\sum_{n=2}^{\infty}a_n \leq \frac{(2-\alpha)(2-\beta)(1-\gamma)}{2[2(2-\beta)-(1+\gamma)(2-\alpha)]}. \tag{2.5}$$

Using (2.4) and (2.5) we can see that,

$$\begin{aligned} |F(z)| &\geq |z|-\Psi(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geq |z|-\frac{2(2-\beta)(1-\gamma)}{(2+\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}|z|^2 \end{aligned}$$

and

$$\begin{aligned}|F(z)| &\leq |z| + \Psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2(2-\beta)(1-\gamma)}{(2+\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}|z|^2.\end{aligned}$$

Which proves the assertion of Theorem 2.6. □

Theorem 2.7. *Let the function $f(z)$ defined by (1.2) be in the class $TUCV(\alpha, \beta, \gamma)$. Then we have*

$$|D_z^{\mu}(\Omega^{\alpha} f(z))| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 - \frac{2(2-\beta)(1-\gamma)|z|}{(2-\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

and

$$|D_z^{\mu}(\Omega^{\alpha} f(z))| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 + \frac{2(2-\beta)(1-\gamma)|z|}{(2-\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

for $0 \leq \alpha < 2,\ \beta \leq \alpha,\ 0 \leq \gamma < 1$ *and* $0 \leq \mu < 1;\ z \in \mathbb{U}$.

Theorem 2.8. *Let the function $f(z)$ defined by (1.2) be in the class $TUCV(\alpha, \beta, \gamma)$. Then we have*

$$\left|D_z^{1+\mu}(\Omega^{\alpha} f(z))\right| \geq \frac{|z|^{-\mu}}{\Gamma(1-\mu)} \left\{1 - \frac{2(2-\beta)(1-\gamma)|z|}{(1-\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

and

$$\left|D_z^{1+\mu}(\Omega^{\alpha} f(z))\right| \leq \frac{|z|^{-\mu}}{\Gamma(1-\mu)} \left\{1 + \frac{2(2-\beta)(1-\gamma)|z|}{(1-\mu)[2(2-\beta)-(1+\gamma)(2-\alpha)]}\right\}$$

for $0 \leq \alpha < 2,\ \beta \leq \alpha,\ 0 \leq \gamma < 1$ *and* $0 \leq \mu < 1;\ z \in \mathbb{U}\backslash\{0\}$.

3. Integral Means for the Class

Recently Ahuja and Jahangiri [1], Kim and Choi [6], Murugusundaramoorthy and Thomas Rosy [10] and Silverman [17] had obtained sharp integral means inequalities for univalent functions with negative coefficients. In this section we obtain some results on integral means inequalities.

We recall first the concept of subordination between analytic functions and a subordination theorem of Littlewood [8] .

Given two functions $f(z)$ and $g(z)$ which are analytic in Δ with $f(0) = g(0)$. The function $f(z)$ is said to subordinate to $g(z)$ in Δ if there exists a function $w(z)$ analytic in Δ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \Delta$ such that $f(z) = g(w(z)),\ z \in \Delta$. We denote the subordination by $f(z) \prec g(z)$.

Theorem 3.1. [8] *If the functions $f(z)$ and $g(z)$ are analytic in Δ with $g(z) \prec f(z)$ then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f(\eta e^{i\theta})\right|^{\eta} d\theta, \quad \eta > 0, \;\; 0 < r < 1. \tag{3.1}$$

Theorem 3.2. *Let $\eta > 0$ and $f_2(z)$ is defined by (2.2). If $f(z) \in TUCV(\alpha, \beta, \gamma)$ then for $z = re^{i\theta}$ and $0 < r < 1$*

$$\int_0^{2\pi} \left|D_z^{1+\delta} f(z)\right|^{\gamma} d\theta \leq \int_0^{2\pi} \left|D_z^{1+\delta} f_2(z)\right|^{\gamma} d\theta, \quad \gamma > 0, \;\; 0 \leq \delta < 1. \tag{3.2}$$

Proof. By virtue of the fractional derivative formula given and defined in Definition (1.3), we find from (1.2) that

$$\begin{aligned} D_z^{1+\delta} f(z) &= \frac{z^{-\delta}}{\Gamma(1-\delta)} \left[1 - \sum_{n=2}^{\infty} \frac{\Gamma(1-\delta)\Gamma(n+1)}{\Gamma(n-\delta)} a_n z^{n-1}\right] \\ &= \frac{z^{-\delta}}{\Gamma(1-\delta)} \left[1 - \sum_{n=2}^{\infty} \Gamma(1-\delta) \frac{n!}{(n-2)!} \Psi(n) a_n z^{n-1}\right] \end{aligned}$$

where

$$\Psi(n) = \frac{\Gamma(n-1)}{\Gamma(n-\delta)}, \quad (0 \leq \delta < 1; \;\; n \geq 2).$$

Since $\Psi(n)$ is decreasing function of n we have

$$0 < \Psi(n) \leq \Psi(2) = \frac{1}{\Gamma(2-\delta)} \quad (0 \leq \delta < 1; n \geq 2).$$

Similarly

$$D_z^{1+\delta} f_2(z) = \frac{z^{-\delta}}{\Gamma(1-\delta)} \left[1 - \frac{\Gamma(1-\delta)\Gamma(3)}{B_2 \Gamma(2-\delta)} z\right]$$

where B_2 as in Theorem 2.2. For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} \Gamma(1-\delta) \frac{n!}{(n-2)!} \Psi(n) a_n z^{n-1}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|1 - \frac{\Gamma(1-\delta)\Gamma(3)}{B_2 \Gamma(2-\delta)} z\right|^{\eta} d\theta,$$

$\eta > 0, \;\; 0 \leq \delta < 1$. Thus by Theorem 3.1 and applying Theorem 2.2 it sufficies to show that

$$1 - \sum_{n=2}^{\infty} \Gamma(1-\delta) \frac{n!}{(n-2)!} \Psi(n) a_n z^{n-1} \prec 1 - \frac{\Gamma(1-\delta)\Gamma(3)}{B_2 \Gamma(2-\delta)} w(z)$$

where

$$\begin{aligned} |w(z)| &= \left| \frac{B_2\Gamma(2-\delta)}{\Gamma(1-\delta)} \sum_{n=2}^{\infty} \frac{\Gamma(1-\delta)n!}{(n-2)!} \Psi(n) a_n z^{n-1} \right| \\ &= \left| B_2 \sum_{n=2}^{\infty} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} B_n a_n < 1 \end{aligned}$$

by means of the hypothesis of Theorem. In light of the above inequality we have the subordination (3.2) which evidently proves the theorem. □

4. Integral Transform of the class $TUCV(\alpha, \beta, \gamma)$

For $f \in \mathbb{A}$ we define the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where λ is a real valued, non-negative weight function normalized so that $\int_0^1 \lambda(t)dt = 1$. Since special cases of $\lambda(t)$ are particularly interesting such as $\lambda(t) = (1+c)t^c$, $c > -1$, for which V_λ is known as the Bernardi operator, and

$$\lambda(t) = \frac{(c+1)^\delta}{\lambda(\delta)} t^c \left(\log \frac{1}{t} \right)^{\delta-1}, \quad c > -1, \quad \delta \geq 0$$

which gives the Komatu operator(see [7]).

First we show that the class $TUCV(\alpha, \beta, \gamma)$ is closed under $V_\lambda(f)$.

Theorem 4.1. *Let $f \in TUCV(\alpha, \beta, \gamma)$. Then $V_\lambda(f) \in TUCV(\alpha, \beta, \gamma)$.*

Proof. By definition, we have

$$\begin{aligned} V_\lambda(f)(z) &= \frac{(c+1)^\delta}{\lambda(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\lambda(\delta)} \lim_{r\to 0^+} \left[\int_r^1 t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \right]. \end{aligned}$$

A simple calculation gives

$$V_\lambda(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{(1-\gamma)} \left(\frac{c+1}{c+n} \right)^\delta a_n \leq 1. \tag{4.1}$$

On the other hand by Theorem 2.2, $f(z) \in TUCV(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{(1-\gamma)} a_n \leq 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (4.1) holds and the proof is complete. □

The above theorem yields the following two special cases.

Theorem 4.2. *If $f(z)$ is starlike of order γ then $V_\lambda f(z)$ is also starlike of order α.*

Theorem 4.3. *If $f(z)$ is convex of order γ then $V_\lambda f(z)$ is also convex of order α.*

Theorem 4.4. *Let $f \in TUCV(\alpha, \beta, \gamma)$. Then $V_\lambda f(z)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}}.$$

Proof. It is sufficient to prove

$$\left| \frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)} - 1 \right| < 1 - \xi. \tag{4.2}$$

For the left hand side of (4.2) we have

$$\left| \frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)} - 1 \right| = \left| \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}} \right|$$
$$\leq \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}.$$

The last expression is less than $1-\xi$ since,

$$|z|^{n-1} < \left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{(n-\xi)(1-\gamma)}.$$

Therefore, the proof is complete. □

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

Theorem 4.5. *Let $f \in TUCV(\alpha, \beta, \gamma)$. Then $V_\lambda f(z)$ is convex of order $0 \leq \xi < 1$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)[2\Phi(n,\alpha) - (1+\gamma)\Phi(n,\beta)]}{n(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}}.$$

Acknowledgements: The authors would like to thank the referee for his insightful suggestions.

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School of Science and Humanities, V I T University,, Vellore - 632014,T.N.,India.
E-mail address: gmsmoorthy@yahoo.com