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SOME PROPERTIES OF UNISERIALLY EMBEDDING OF SUBGROUPS OF p-GROUPS

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Abstract. This paper focuses attention on the study of the Question 3.1. of [1] and it can be considered as a continuation of the previously mentioned paper. A subgroup H of a p-group G is n-uniserial if for each $i = 1, ..., n$, there is a unique subgroup K_i such that $H \leq K_i$ and $|K_i : H| = p^i$. In case the subgroups of G containing H form a chain we say that H is uniserially embedded in G. We prove that if H is an n-uniserial subgroup of a cyclic p-group G, then H is uniserially embedded in G. We also show that if H is an n-uniserial subgroup of the p-group G such that $|G| \leq p^5$, then H is uniserially embedded in G and we determine that if H is a 1-uniserial subgroup of order p^2 in the p-group G of order p^5 and $C_G(H) = H$, then H is uniserially embedded in G.

1. Introduction

This paper focuses attention on the study of the Question 3.1. of [1]: what are the conditions on an n-uniserial subgroup for it to be uniserially embedded and it can be considered as a continuation of the previously mentioned paper. Let us say that a subgroup H of a p-group G is n-uniserial if for each $i = 1, ..., n$, there is a unique subgroup K_i such that $H \leq K_i$ and $|K_i : H| = p^i$. Thus H is uniserially embedded in K_n.
When does n-uniseriality imply uniserially embedded. The condition that H is 1-uniserial is equivalent to $N_G(H)/H$ being cyclic (p odd). If $|H| = p$ and G is not cyclic, this condition is equivalent to $N_G(H) = H \times T$ for a cyclic subgroup T of G. This situation was studied in [2], from which it follows that H is uniserially embedded if and only if $|T| = p$. Blackburn and He'thelyi in [1] discuss on the conditions of a 2-uniserial subgroup in the p-group G which it to be uniserially embedded in G. They proved the following theorems :

Theorem 1.1. [1] *Suppose that p is odd and that K is a 2-uniserial subgroup of order p in the p-group G. Then K is uniserially embedded in G.*

Theorem 1.2. [1] *For $p > 3$, let K be a 2-uniserial cyclic subgroup of the p-group G. Then K is uniserially embedded in G.*

Theorem 1.3. [1] *For p odd let K be a 2-uniserial elementary Abelian subgroup of order at most p^{p-1} of a p-group G. Then K is uniserially embedded in G.*

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In this paper, we study the conditions on an n-uniserial subgroup which it to be uniserially embedded. We also study the conditions on a 1-uniserial subgroup in the p-group G of order p^5 which it to be uniserially embedded in G.
Whenever possible we follow the notation and terminology of [5, 6]

2. Preliminaries

A subgroup H of a p-group G is n-uniserial if for each $i = 1, ..., n$, there is a unique subgroup K_i such that $H \leq K_i$ and $|K_i : H| = p^i$. Thus H is uniserially embedded in K_n.

Definition 2.1. [3]. *Let G be a finite p-group, where p is a prime number. A proper subgroup H of G is called soft if H is a maximal Abelian subgroup of G and is of index p in its normalizer. The chief properties of soft subgroups are given in* [3] *and* [4]*.*

Theorem 2.2. [2]. *Suppose that p is odd and that G is a p-group. Then H is 1-uniserial if and only if $N_G(H)/H$ is cyclic.*

Theorem 2.3. [1]. *Suppose that p is odd and that K is a 2-uniserial subgroup of order p in the p-group G. Then K is uniserially embedded in G.*

Theorem 2.4. [3]. *Let H be a soft subgroup in G. Then there is a unique maximal subgroup M of G which contains H.*

3. Uniserially embedding of subgroup

Lemma 3.1. *Let G be a finite cyclic p-group. Then every subgroup H of G of order n is unique.*

Proof. Straight forward.

Theorem 3.2. *Suppose that p is a prime number, then any subgroup of a finite cyclic p-group G is uniserially embedded in G.*

Proof. Let K be an n-uniserial subgroup of G. Hence for each $i = 1, ...n$, there is a unique subgroup K_i such that $K \leq K_i$ and $|K_i : K| = p^i$. Let $K \leq T_1 \leq ... \leq T_m \leq G$ be a chain of subgroups G containing K and $|T_i : K| = p^i$(i=1,...,m). By Lemma 3.1, K is a m-uniserial subgroup in G. Since G is a cyclic p-group, therefore $m \leq n$ and for each i=1,...,m, $T_i = K_i$. Thus K is uniserially embedded in G.

Theorem 3.3. *Suppose that p is a prime number and K is a 2-uniserial subgroup of order p^α in p-group G of order at most p^5. Then K is uniserially embedded in G.*

Proof. If $\alpha = 1$, then by Theorem 1.1, the proof is complete. Let $\alpha \geq 2$ and H_1, H_2 be the unique subgroups of orders $p^{\alpha+1}, p^{\alpha+2}$ respectively containing K. So $|H_1| = p^{\alpha+1}$ and $|H_2| = p^{\alpha+2}$. Thus $p^{\alpha+2} \leq |G| \leq p^5$, so $4 \leq \alpha + 2 \leq 5$. If $\alpha + 2 = 5$, then $\alpha = 3$ and $H_2 = G$. In this case, let $K \leq T_1 \leq ...T_m \leq G$ be a chain of subgroups of G such that $|T_i : H| = p^i$(i=1,...,m). Therefore $m \leq 2$. Thus $T_1 = H_1$ and $T_2 = H_2 = G$, that is, K is uniserially embedded in G. If $\alpha + 2 = 4$, then $|K| = p^2, |H_1| = p^3$ and $|H_2| = p^4$. Let $K \leq K_1 \leq K_2 \leq ... \leq K_m \leq G$ be a chain of subgroups of G containing K and $|K_i : K| = p^i$(i=1,...,m). So $|K_m| = p^{m+2}$. Thus $m \leq 3$, and $G = K_3$. So $K \leq K_1 \leq K_2 \leq K_3 = G$ such that $|K_i : K| = p^i$(i=1,2,3). Since K is a 2-uniserial, then $K_1 = H_1$ and $K_2 = H_2$, that is, K is uniserially embedded in G and the proof is completed.

Theorem 3.4. *Suppose that p is a prime number and that $n > 1$. If H is a non-trivial n-uniserial subgroup in the p-group G of order at most p^5, then H is uniserially embedded in G.*

Proof. Since $|G| \leq p^5$, $n \leq 4$. If $n = 2$, by Theorem 3.3, the proof is complete. Let $n \geq 3$ and $|H| = p^\alpha$ such that $\alpha \geq 1$. If n=3, then there are unique subgroups K_1, K_2 and K_3 such that $H \leq K_i$ and $|K_i : H| = p^i$(i=1,2,3), so $|K_3| = p^{\alpha+3}$. Thus $1 \leq \alpha \leq 2$ and $|G| \geq p^4$. If $\alpha = 1$, then $|K_3| = p^4$. If $|G| = p^4$, then the proof is complete. Let $|G| = p^5$ and $H \leq T_1 \leq T_2 \leq ... \leq T_m \leq G$ be a chain of subgroups of G such that $|T_i : H| = p^i$(i=1,...,m). So $|T_m| = p^{m+1}$. Thus $m \leq 4$ and $|T_4| = p^5$, that is, $T_4 = G$. Since H is a 3-uniserial, hence $T_i = K_i, i = 1, 2, 3$. Thus H is uniserially embedded in G. If $\alpha = 2$, then $|K_3| = p^5$, H is uniserially embedded in G, clearly. Let n=4. Thus for each i=1,...,4, there is a subgroup K_i such that $H \leq K_i$ and $|K_i : H| = p^i$. So $|K_4| = p^{\alpha+4}$, hence $\alpha = 1$, clearly, H is uniserially embedded in G.

Theorem 3.5. *Suppose that H is a 1-uniserial subgroup in the p-group G of order p^α. If $|H| = p^\beta$ and $\alpha - 2 \leq \beta \leq \alpha - 1$, then H is uniserially embedded in G.*

Proof. Let $\beta = \alpha - 1$. Since H is a 1-uniserial subgroup in G, hence there is a unique subgroup K such that $H \leq K$ and $|K : H| = p$. So $|H| = p^{\alpha-1}$ and $|K| = p^\alpha$, thus G=K. Let $\beta = \alpha - 2$. Since K is unique and the only subgroup containing K is G itself, so $H < K < G$ and hence H is uniserially embedded in G and the proof is completed.

Theorem 3.6. *Suppose that p is odd and H is a 1-uniserial subgroup of order $p^{\alpha-3}$ in the p-group G of order p^α. Assume that one of the following conditions is satisfied:*
(a) G is Abelian.
(b) There is a subgroup K of G such that $H \leq K$, K is not normal in G and $|K : H| = p$. Then H is uniserially embedded in G.

Proof. Let G be Abelian group. Since H is a 1-uniserial subgroup in G, there is a unique subgroup K of G such that $H \leq K$ and $|K : H| = p$. So $|K| = p^{\alpha-2}$, thus $H, K \trianglerighteq G$. By Theorem 2.2, G/H is cyclic. So G/K is cyclic and by Lemma 3.1, G/K has a unique subgroup of order p. Let $M/K \leq G/K$ and $|M/K| = p$. So $|M| = p^{\alpha-1}$, and thus G has a unique subgroup of order $p^{\alpha-1}$. Therefore H is uniserially embedded in G. Let (b) hold. Thus there is a unique subgroup T such that $H \leq T$ and $|T : H| = p$, hence T=K, and therefore T is not normal subgroup in G. Since $T < N_G(T) < G$, hence $|N_G(K)| = p^{\alpha-1}$, so $N_G(K)$ is maximal in G. Let $H \leq S_1 \leq ... \leq S_m \leq G$ be a chain of subgroups of G such that $|S_i : H| = p^i$(i=1,...,m), therefore $|S_m| = p^{m+\alpha-3}$, and thus $m \leq 3$. Therefore $|S_3| = p^\alpha$, that is, $G = S_3$. Since H is a 1-uniserial, hence $S_1 = T$.

T is a maximal subgroup in the nilpotent group S_2. Thus $T \trianglerighteq S_2$. So $S_2 \leq N_G(T)$. Since $|N_G(T)| = |S_2| = p^{\alpha-1}$, $S_2 = N_G(T)$, therefore for any chain of subgroups of G, $H \leq S_1 \leq ... \leq S_m \leq G$ such that $|S_i : H| = p^i$(i=1,...,m), $m = 3, S_1 = T$ and $S_2 = N_G(T)$, that is, H is uniserially embedded in G.

Theorem 3.7. *Suppose that p is a prime number and H is a 1-uniserial subgroup of order p^2 in the p-group G of order p^5. If $C_G(H) = H$, then H is uniserially embedded in G.*

Proof. Since $H \cong Z_{p^2}$ or $H \cong Z_p \times Z_p$ and $H < N_G(H)$, $|N_G(H)/H| = p^\alpha$ such that $1 \leq \alpha \leq 3$. We know that $N_G(H)/C_G(H) \hookrightarrow Aut(H)$, so $|N_G(H)/C_G(H)| = p$. Since H is Abelian, self-centralizer and $|N_G(H) : H| = p$, hence H is soft subgroup of G, by Theorem 2.4, there is a unique maximal subgroup M of G containing H. Let $H \leq T \leq S \leq G$ be a chain of G such that $|T| = p^3$ and $|S| = p^4$. Since H is a 1-uniserial in G, then T is unique. Also S is a maximal subgroup of G containing H. Thus S=M, that is, H is uniserially embedded in G and the proof is completed.

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