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HOMOTOPY ANALYSIS METHOD FOR DELAY-INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we implement the homotopy analysis method to solve delay-integro-differential equations. The convergence of the method is investigated. In the end, numerical experiments are presented to illustrate the computational effectiveness of the method.

1. INTRODUCTION

This paper considers an analytical solution to the delay-integro-differential equations (DIDEs)

$$\begin{aligned} y'(t) &= \lambda y(t) + \mu y(t-\tau) + \gamma \int_{t-\tau}^{t} K(t,s) y(s) ds, \qquad t \geq 0, \\ y(t) &= \varphi(t), \qquad t \leq 0, \end{aligned} \tag{1.1}$$

where λ, μ, γ are real numbers.

Delay-integro-differential equations have been studied by many authors. A motivating factor for the study of these equations is their application to the areas of science, engineering and technology. Many typical examples, such as stress-strain states of materials, motion of rigid bodies, aeroauto-elasticity problems and models of polymer crystallization, can be found in Kolmanovskii and Myshkis' monograph [9] and the references therein. Generally speaking, it is difficult to give the exact solutions of such equations. Recently, this class of equations have come to intrigue researchers in numerical computation and analysis [6, 8]. For example, Baker and Ford [3], Koto [10] dealt with the linear stability of numerical methods for VDIDEs. Baker and Ford [4, 5], Brunner [7] and Enright and Hu [9] studied the convergence of linear multistep methods, collocation methods and continuous Runge–Kutta methods, respectively. Zhang and Vandewalle [16, 17] investigated nonlinear stability of BDF methods, Runge–Kutta methods and general linear methods.

The homotopy analysis method (HAM) [1, 12, 13, 14, 15] is thoroughly used by many researchers to handle a wide variety of scientific and engineering applications. Awawdeh et al. [2] used HAM to solve the multi-pantograph delay differential equation

$$\begin{aligned} y'(t) &= \lambda y(t) + \sum_{i=1}^{k} \mu_i y(q_i t) + f(t), \qquad 0 < t < T, \\ y(0) &= \alpha, \end{aligned}$$

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where $0 < q_k < q_{k-1} < \cdots < q_1 < 1$.

In this paper, the homotopy analysis method (HAM) is applied to solve DIDEs of type (1.1). It is expected the proposed technique can be further applied to derive solutions for other types of DIDEs. Examples to illustrate the results are presented throughout the paper.

2. Solution Method

2.1. **Approach based on the HAM.** Consider the operator N defined according to Eq. (1.1) by

$$N[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t} - \lambda\phi(t;q) - \mu\phi(t-\tau;q) - \gamma\int_{t-\tau}^{t} K(t,s)\phi(s;q)ds, \qquad t \geq 0. \tag{2.1}$$

Let $y_0(t)$ be an initial guess of the exact solution $y(t)$. Also, $\hbar \neq 0$ an auxiliary parameter and L an auxiliary linear operator satisfies

$$L[f(x)] = 0 \quad \text{when } f(x) = 0.$$

All of $y_0(x)$, L and $\hbar$ will be chosen later with great freedom. Then we construct the HAM deformation equation in the following form:

$$(1-q)L[\phi(t;q) - y_0(t)] = q\hbar N[\phi(t;q)], \tag{2.2}$$

where $q \in [0,1]$ is an embedding parameter.

Obviously, when $q = 0$, Eq. (2.2) has the solution

$$\phi(t;0) = y_0(t), \tag{2.3}$$

and when $q = 1$, since $\hbar \neq 0$, Eq. (2.2) is equivalent to the original one (1.1), provided

$$\phi(t;1) = y(t). \tag{2.4}$$

Thus, according to (2.3) and (2.4), as the embedding parameter q increases from 0 to 1, $\phi(t;q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution $y(t)$. This kind of deformation $\phi(t;q)$ is totally determined by the so-called zeroth-order deformation equation (2.2).

Expanding $\phi(t;q)$ in Taylor's series with respect to q, we have

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m, \tag{2.5}$$

where

$$y_m(t) = D_m[\phi(t;q)] = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}.$$

D_m is called the mth-order homotopy-derivative of ϕ.

Fortunately, the homotopy-series (2.5) contains an auxiliary parameter $\hbar$, and besides we have great freedom to choose the auxiliary linear operator L, as illustrated by Liao [12]. If the auxiliary linear parameter L and the nonzero auxiliary parameter $\hbar$ are properly chosen so that the power series (2.5) of $\phi(t;q)$ converges at $q = 1$. Then, we have under these assumptions the the so-called homotopy-series solution

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{2.6}$$

According to the fundamental theorems in calculus, each coefficient of the Taylor series of a function is unique. Thus, $y_m(t)$ is unique, and is determined by $\phi(t;q)$. Therefore, the governing equations and boundary conditions of $y_m(t)$ can be deduced from the zeroth-order deformation equation (2.2). For brevity, define the vectors

$$\overrightarrow{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \dots, y_n(t)\}.$$

Differentiating the zero-order deformation equation (2.2) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\begin{aligned} L[y_m(t) - \chi_m y_{m-1}(t)] &= \hbar \Re_m(\overrightarrow{y}_{m-1}(t)), \\ y_m(0) &= 0, \end{aligned} \tag{2.7}$$

where

$$\Re_m(\overrightarrow{y}_{m-1}(x)) = D_{m-1}(N[\phi]) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0} \tag{2.8}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

In this line we have that,

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = y'_{m-1}(t) - \lambda y_{m-1}(t) - \mu y_{m-1}(t-\tau) - \gamma \int_{t-\tau}^{t} K(t,s) y_{m-1}(s) ds \tag{2.9}$$

So, by means of symbolic computation software such as Mathematica, Maple, Matlab and so on, it is not difficult to get $\Re_m(\overrightarrow{y}_{m-1}(t))$ for large value of m.

The solutions of the high-order deformation equations (2.7) exist when the auxiliary linear operator L is invertible. By taking the inverse of the linear operator $L = \frac{d}{dt}$ in (2.7), we get for $m \geq 1$,

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar \int_0^t \Re_m(\overrightarrow{y}_{m-1}(\zeta)) d\zeta. \tag{2.10}$$

In this way, it is easily to obtain $y_m(t)$ one by one in the order $m = 1, 2, 3, ...$, and we have

$$y(t) = \sum_{m=0}^{M} y_m(t).$$

When $M \to \infty$, we get an accurate approximation of the original equation (1.1).

2.2. **Convergence analysis.**

Theorem 2.1. *If the series (2.6) converges, then it is the exact solution of the integral equation (1.1).*

Proof 1. If the series (2.6) converges, we can write

$$S(t) = \sum_{m=0}^{\infty} y_m(t),$$

and it holds that

$$\lim_{m\to\infty} y_m(t) = 0. \tag{2.11}$$

We can verify that

$$\begin{aligned}\sum_{m=1}^{n}[y_m(t) - \chi_m y_{m-1}(t)] &= y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) \\ &= y_n(t),\end{aligned}$$

which gives us, according to (2.11),

$$\sum_{m=1}^{\infty}[y_m(t) - \chi_m y_{m-1}(t)] = \lim_{n\to\infty} y_n(t) = 0. \tag{2.12}$$

Furthermore, using (2.12) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[y_m(t) - \chi_m y_{m-1}(t)] = L[\sum_{m=1}^{\infty}[y_m(t) - \chi_m y_{m-1}(t)]] = 0.$$

In this line, we can obtain that

$$\sum_{m=1}^{\infty} L[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(t)) = 0$$

which gives, since $h \neq 0$, that

$$\sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(t)) = 0. \tag{2.13}$$

Substituting $\Re_{m-1}(\overrightarrow{y}_{m-1}(x))$ into the above expression and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty} \Re_{m-1} &= \sum_{m=1}^{\infty}[y'_{m-1}(t) - \lambda y_{m-1}(t) - \mu y_{m-1}(t-\tau) - \gamma \int_{t-\tau}^{t} K(t,s) y_{m-1}(s)ds] \\ &= \sum_{m=0}^{\infty} y'_m - \lambda \sum_{m=0}^{\infty} y_m - \mu \sum_{m=0}^{\infty} y_m(t-\tau) - \gamma \int_{t-\tau}^{t} K(t,s) \sum_{m=0}^{\infty} y_m(s)ds \\ &= S'(t) - \lambda S(t) - \mu S(t-\tau) - \gamma \int_{t-\tau}^{t} K(t,s)S(s)ds \qquad (2.14)\end{aligned}$$

From (2.13) and (2.14), we have

$$S'(t) = \lambda S(t) + \mu S(t-\tau) + \gamma \int_{t-\tau}^{t} K(t,s)S(s)ds,$$

and so, $S(t)$ is the exact solution of (1.1). This completes the proof of the theorem.

Note that we have great freedom to choose the value of the auxiliary parameter $\hbar$. Mathematically the value of $y(t)$ at any finite order of approximation is dependent upon the auxiliary parameter $\hbar$, because the zeroth and high-order deformation equations contain $\hbar$. Let $R_\hbar$ be the set of all values of $\hbar$ which ensures the convergence of the HAM series solution (2.6) of $y(t)$. According to Theorem 1, all of these series solutions must converge to the solution of the original equations (1.1). Let $\hbar$ be the variable of the horizontal axis and the limit of the series solution (2.6) of $y(t)$ be the variable of vertical axis. Plot the curve $y(t)$ *vs* $\hbar$, where $y(t)$ denotes the limit of the series (2.6). Because the limit of all convergent series solutions (2.6) is the same, there exists a horizontal line segment above the region $\hbar \in R_\hbar$. So, by plotting the curve $y(t)$ *vs* $\hbar$ at a high enough order approximation, one can find an approximation of the set $R_\hbar$ (for more details see [12]).

3. Applications

In this section, the validity of the proposed approach is illustrated by two examples.

3.1. **Example 1.**

First we consider the following delay integro-differential equation

$$\begin{aligned} y'(t) &= y(t-1) + \int_{t-1}^{t} y(s)ds, && t \geq 0, \\ y(t) &= e^t, && t \leq 0, \end{aligned} \tag{3.1}$$

which has the exact solution $y(t) = e^t$. We use the set of base functions

$$\{t^n : n \geq 1,\ n \in \mathbb{N}\},$$

in order to represent $y(t)$,

$$y(t) = \sum_{k=1}^{\infty} b_k t^k, \tag{3.2}$$

where b_k is a coefficient to be determined later. According to (2.2), the zeroth-order deformation equation can be given by

$$(1-q)L[\phi(t;q) - y_0(t)] = q\hbar\left(\frac{\partial \phi(t;q)}{\partial t} - \phi(t-1;q) + \int_{t-1}^{t} \phi(s;q)ds\right).$$

Under the rule of solution expression denoted by (3.2), we can choose the initial guess of $y(t)$ as follows:

$$y_0(t) = 1,$$

and we choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t},$$

with the property

$$L[C] = 0,$$

where C is an integral constant. Hence, the mth-order deformation equation, $m \geq 1$, can be given by

$$\begin{aligned} y_m(t) &= \chi_m y_{m-1}(t) + \hbar \int_0^t (y'_{m-1}(\tau) - y_{m-1}(\tau - 1) - \int_{\tau-1}^{\tau} y_{m-1}(s)ds)d\tau. \\ y_m(0) &= 0 \end{aligned}$$

Consequently, the HAM series solution is

$$y(t) = y_0(t) + \sum_{m=1}^{K} y_m(t), \tag{3.3}$$

where K is the number of terms. To investigate the influence of $\hbar$ on the convergent of the solution series (3.2), we plot the so-called $\hbar$-curve of $y(0.4)$ as shown in Fig. 1.

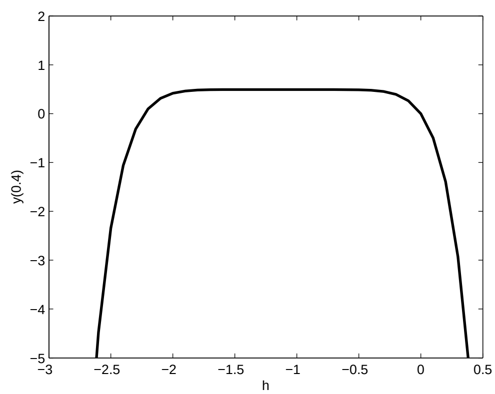

FIGURE 1. The curve $y(0.4)$ vs $\hbar$ the 5th order of approximation for Example 1.

According to this $\hbar$-curve, it is easy to conclude that $-2 \leq \hbar \leq 0$ is the valid region of $\hbar$, which corresponds to the line segments nearly parallel to the horizontal axis. A proper value of $\hbar = -0.5$ is taken and then the twenty terms from the series solution expression b HAM is plotted in Figure 2.

3.2. **Example 2.** As a second example, we consider the delay integro-differential equation

$$\begin{aligned} y'(t) &= y(t) + 2y(t - \frac{1}{2}) + \int_{t-\frac{1}{2}}^{t} ts\ y(s)ds, \qquad t \geq 0, \\ y(t) &= \varphi(t), \qquad t \leq 0, \end{aligned}$$

where $\varphi(t)$ is chosen so that the exact solution is $y(t) = \sin t$. We use the set of base functions

$$\{t^n :\ n \in \mathbb{N}\},$$

to represent $y(t)$,

$$y(t) = \sum_{k=1}^{\infty} b_k t^k,$$

where b_k is a coefficient to be determined. As an initial approximation of $y(t)$ we choose

$$y_0(t) = 0.$$

and we select the the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t},$$

with property

$$L[C] = 0,$$

where C is an integral constant. Hence, that the mth order deformation equation is

$$y_m(t) = (\chi_m + \hbar) y_{m-1}(t) - \hbar(y_{m-1}(t) + 2y_{m-1}(t - \frac{1}{2}) + \int_{t-\frac{1}{2}}^{t} ts\ y_{m-1}(s)ds),$$

subject to the initial condition

$$y_m(0) = 0.$$

Now we successfully obtain $y_1(t), y_2(t), \ldots, y_m(t)$. In order to find range of admissible values of $\hbar$, the $\hbar$-curve is plotted in Figure. 3 for 8th-order approximation. Figure 4 presents a comparison of the numerical solution of 10th-order HAM approximation and the exact solution.

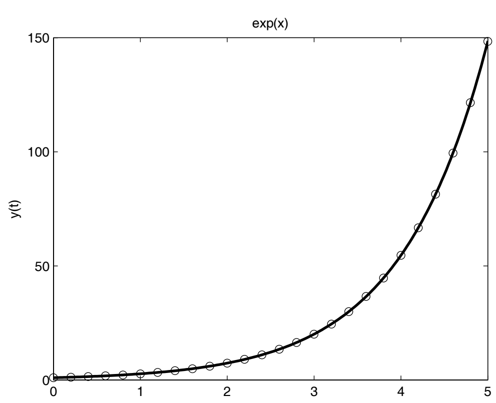

FIGURE 2. Comparison of the exact solution for Example 1 to the numerical solution. Hollow dots: 20th-order HAM approximation; continued solid : exact solution.

4. Conclusion

Solving delay-integro-differential equations lack of analytical or closed form solutions. Based on the fact, this study has focused on developing a procedure to obtain an explicit analytical solution concerning the delay-integro-differential equations. A series solution is evaluated in a very fast convergence rate where the accuracy is improved by increasing the number of terms considered. Shortly, from now on, HAM can be used as a powerful solver for delay-integro-differential equations of type (1.1). However, it is difficult to extend the present research to more general nonlinear DIDEs, such as the case with many independent constant delays and the case with time-depending delays. We hope to leave these opening problems to the future work.

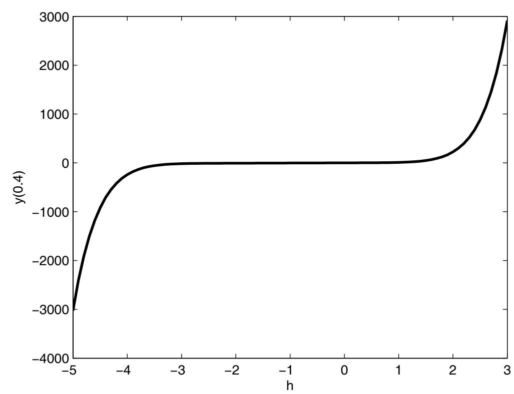

FIGURE 3. The curve $y(0.4)$ vs $\hbar$ the 8th order of approximation for Example 2.

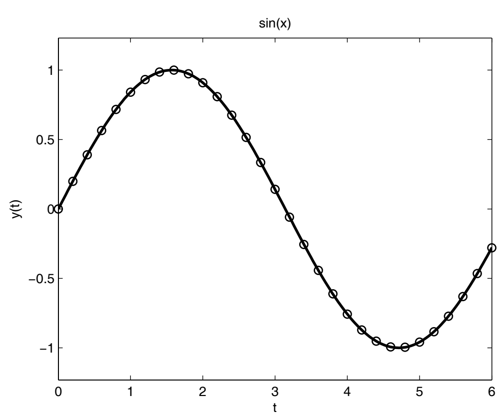

FIGURE 4. Comparison of the exact solution for Example 2 to the numerical solution. Hollow dots: 20th-order HAM approximation with $\hbar = -0.5$; continued solid : exact solution.

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