# AN ALTERNATIVE PROOF
# OF THE FRIENDSHIP THEOREM∗

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## ABSTRACT

Several proofs of the friendship theorem are known. In this paper an alternative proof will be given for friendship theorem. The goal of this paper is to provide a proof which is perhaps more combinatorial.

## 1. INTRODUCTION

For our purposes a graph *G* is finite, undirected and has no loops or multiple edges. We denote the vertex set of *G* by $V(G)$ and the edge set of *G* by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\varepsilon(G)$, respectively. The cycle on *n* vertices is denoted by $C_n$ Let *G* be a graph and $u \in V(G)$. The degree of a vertex *u* in *G*, denoted by $d_G(u)$, is the number of edges of *G* incident to *u*. The neighbour set of a vertex *u* of *G* in a subgraph *H* of *G*, denoted by $N_H(u)$, consists of the vertices of *H* adjacent to *u*; we write $d_H(u) = |N_H(u)|$. Further, we define the non-neighbours $\overline{N}_G(u)$.

Let $G_1$ and $G_2$ be graphs. The Union $G_1 \cup G_2$ of $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and and edge set $E(G_1) \cup E(G_2)$ Two graphs $G_1$ and $G_2$ are disjoint if and only if $V(G_1) \cap V(G_2) = \phi$; $G_1$ and $G_2$ are edge disjoint if $E(G_1) \cap E(G_2) = \phi$. If $G_1$ and $G_2$ are disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs $G_1$ and $G_2$ is defined similarly, but in this case we need to assume $V(G_1) \cap V(G_2) \neq \phi$. The join $G \vee H$ of two disjoint graphs *G* and *H* is the graph obtained from $G + H$ by joining each vertex of *G* to each vertex of *H*. For vertex disjoint subgraphs $H_1$ and $H_2$ of *G* we let

$$E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\} \text{ and } \varepsilon(H_1, H_2) = |E(H_1, H_2)|.$$

## 2. The Friendship Theorem

The friendship theorem can be stated as follows: Suppose, in a group of at least three people we have the situation that any pair have precisely one common friend. Then there is always a person who is everybody's friend (Erdös, P., Rényi, A. & Sós, V. (1966)). The friendship theorem can be stated as follows:

**Theorem 2.1.** If *G* is a graph in which any two distinct vertices have exactly one common neighbour, then *G* has a vertex joined to all others.

Graphs satisfying the above property are called friendship graphs and such graphs are completely determined; they consist of edge disjoint triangles around a common vertex.

Several proofs of the friendship theorem are known. The first was due to Erdös, Rényi and Sós (1966). It is based on a theorem of Baer (1946) about polarities in finite projective planes.

A second proof is due to Wilf (1971). While this proof does not use Bear's theorem, it is based on computing the eigenvalues (and their multiplicities) of the square of the adjacency matrix of the graph argument.

A third proof was provided by Longyear and Parsons (1972). The proof is purely combinatorial, with no explicit reference to eigenvalues. But, in this proof Chvátal (1971) has observed, eigenvalues are involved indirectly, because the crucial step involves counting closed walks, and these numbers are the diagonal entries in powers of the adjacency matrix. The original application of this counting argument, by Ball (1948), was, moreover, an alternative proof and generalization of Baer's theorem.

Hammersley (1989) suggest that proofs avoiding eigenvalues exist, but they require complicated numerical arguments to eliminate regular graphs. Thus, in some sense, all known proofs of the friendship theorem rely on the eigenvalue techniques of Baer ( Bondy 1985). The goal of this paper is to provide a proof which is perhaps more combinatorial.

We now recall some notation and terminology. A complete graph is a graph with every pair of vertices adjacent. The complete graph on *n* vertices is denoted by $K_n$. Let $V^{/}$ be a non-empty subset of *V*(*G*), the subgraph of *G* induced by $V^{/}$, denoted by G[$V^{/}$], is the graph with vertex set $V^{/}$ and $E(G[V^{/}]) = \{uw \in E(G) : u, w \in V^{/}\}$. For a proper subgraph *H* of *G* we write *G*[*V*(*H*)] and *G* − *V*(*H*) simply as *G*[*H*] and *G* − *H* respectively. The minimum and maximum degrees of a graph *G* are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We consider graphs with the property:

**Property P:**    For every pair $x$ and $y$ of vertices of $G$, $\left|N_G(x) \cap N_G(w)\right| = 1$.

The class of graphs with $n$ vertices satisfying property $P$ is denoted by $\mathcal{G}(n; P)$. An example of a graph in $\mathcal{G}(n; P)$ is displayed in Figure 1.
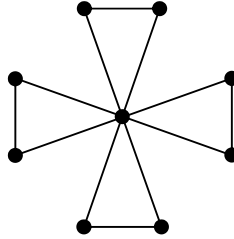


Figure 1

We will prove:

$$\mathcal{G}(n,P) = (K_1 \vee (\frac{n-1}{2})K_2)$$

Using graph theory, we establish the proof using the following important well know properties, the proofs of which use some rather straight forward ideas.

**Lemma 2.1.** Let $\mathcal{G} \in G(n; P)$. Then $G$ is regular or $\Delta(G) = n - 1$.

**Proof:**    Suppose $\Delta(G) < n - 1$.    Consider a vertex $x$ in $G$.    Let $N_G(x)$ and $\overline{N}_G(x)$ denote the neighbours and non-neighbours respectively of $x$.    Let $N_G(x) = \{x_1, x_2, ..., x_t\}$. Since $G$ has property $P$, each $x_i$ is joined to exactly one vertex in $N_G(x)$. Consequently, $H = G[N_G(x)]$ is a 1-regular graph with $t$ vertices. Hence, $t$ is even. In fact, every vertex of $G$ must have even degree. Without loss of generality, we let $x_i x_{i+1} \in E(G)$ for $i = 1, 3, 5, ..., t - 1$. Now, consider a vertex $w \in \overline{N}_G(x)$. For $\left|N_G(x) \cap N_G(w)\right| = 1$, $w$ must be joined to exactly one vertex, say $x_1$, of $N(x)$.
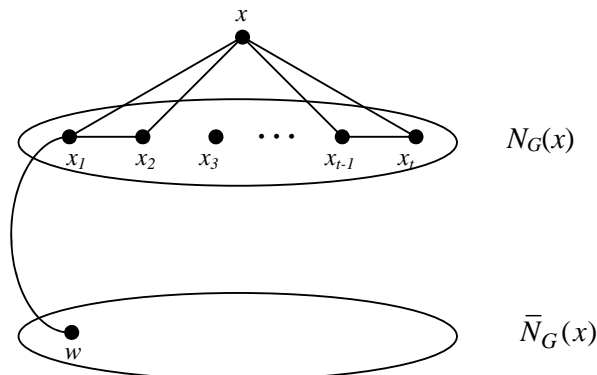


Figure 2

For $i \geq 3$, let $z_i \in N_G(x_i) \cap N_G(w)$. Clearly, $z_i \in \overline{N}_G(x)$. Hence, $w$ is joined to different vertices $z_3, z_4, \ldots z_t$ of $\overline{N}_G(x)$, and thus $d_G(w) \geq 1 + t - 2 = t - 1$. Since $d_G(w)$ must be even, $d_G(w) \geq t$. In particular, for the case when $t = \Delta$ we have $d_G(w) = \Delta$. In fact, every vertex in $\overline{N}_G(x)$ has degree $\Delta$. Since each vertex in $N_G(x)$ is at distance 2 from some vertex in $\overline{N}_G(x)$ it follows that every vertex in $N_G(x)$ has degree $\Delta$. Consequently, $G$ is regular. This completes the proof.

**Lemma 2.2.** (Füredi (1996)) Let $G$ be a graph on $n = q^2 + q + 1$ vertices with $N_G(x) \cap N_G(y) \leq 1$ for every pair $x, y \in V(G)$. Then for $q \geq 15$

$$\varepsilon(G) \leq \frac{1}{2} q(q+1)^2$$

with equality holding if and only if $q$ is a prime power greater than 13.

**Lemma 2.3.** Let $G$ be a graph on $n$ vertices with $N_G(x) \cap N_G(y) \leq 1$ for every pair $x$, $y \in V(G)$, such that $G$ contains no cycle of length 4. Then

$$\varepsilon(G) < \frac{n}{4}(1 + \sqrt{4n-3})$$

.

**Proof:** Let $G$ be a graph containing no cycle of length 4. Let $v_1, v_2, \ldots, v_n$ be the vertices of $G$. Now, clearly, one can select from the set $N(v_i)$ of vertices joined by an edge to $v_i$, $\binom{d(v_i)}{2}$ pairs, and no pair $(v_i, v_j)$ can be contained in both $N(v_k)$ and $N(v_l)$ with $k \neq l$ because otherwise $v_i, v_j, v_k, v_l$ would be a cycle of length 4 contained in $G$. Thus we must have,

$$\sum_{i=1}^{n} \binom{d(v_i)}{2} \leq \binom{n}{2}$$

Now, we have,

$$(\sum_{i=1}^{n} d(v_i))^2 \leq n \sum_{i=1}^{n} (d(v_i))^2$$

and

$$(\sum_{i=1}^{n} d(v_i))^2 - n\sum_{i=1}^{n}(d(v_i)) \qquad \leq 2n\sum_{i=1}^{n}\binom{d(v_i)}{2}$$

$$\leq 2n\binom{n}{2}$$

$$= 2n(\frac{n^2 - n}{2})$$

$$= n^3 - n^2 .$$

Because, $\sum_{i=1}^{n} d(v_i) = 2\varepsilon(G)$, we have $4(\varepsilon(G))^2 - 2n\varepsilon(G) \leq n^3 - n^2$ which

implies $\varepsilon(G) \leq \dfrac{n}{4}(1 + \sqrt{4n - 3})$.

Equality is possible only if $\sqrt{4n - 3} = 3 + 4m$ with $m$ a positive integer and this is same as $\sqrt{4n - 3} = 2k + 1$ for some odd integer $k$. Therefore, $n = k^2 + k + 1$.

$$\frac{n}{4}(1 + \sqrt{4n - 3}) = \frac{(k^2 + k + 1)}{4}(2k + 2)$$

$$= \frac{(k^2 + k + 1)(k + 1)}{2}$$

$$= \frac{1}{2}(k(k + 1) + 1)(k + 1)$$

$$= \frac{1}{2}k(k + 1)^2 + \frac{1}{2}(k + 1)$$

$$> \frac{1}{2}k(k + 1)^2 .$$

Which contradicts Lemma 2.2. Thus, equality in lemma 2.3 is not possible. This completes the proof.

**Lemma 2.4.** Let $n > 3$ and $G \in \mathcal{G}(n; P)$. Then $G$ is not regular.

**Proof:** Suppose to the contrary that $G$ is a $k$-regular graph. The situation is depicted in Figure 3.
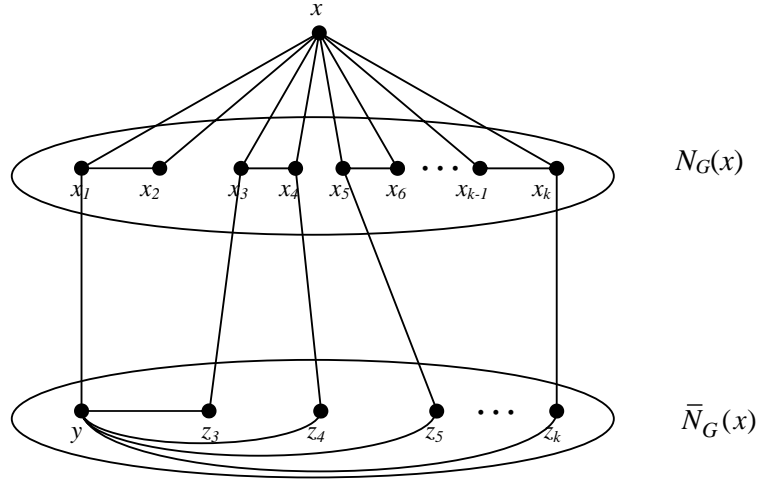
Figure 3

Consider the vertex $y \in \overline{N}_G(x)$   We may assume that $x_i = N_G(y) \cap N_G(x)$ and for $i = 3, 4, \ldots, k$, $N_G(y) \cap N_G(x_i) = z_i$ .Since $\left| N_G(x) \cap N_G(x_i) \right| = 1$ for every $i$ and $d_G(x_i) = k$ for each $i, 1 \le i \le k$ .

$$\varepsilon(x_i, \overline{N}_G(x)) = k - 2 .$$

Further, $\varepsilon(y, N_G(x) = 1$ for every vertex $y \in \overline{N}_G(x)$ and $\left| N_G(x) \cap N_G(y) \right| = 1$.

Hence,

$$\left| \overline{N}_G(x) \right| = k(k-2)$$

and so,

$$n = 1 + k + k(k-2) = k^2 - k + 1.$$

Now,

$$\varepsilon(G) = \frac{k(k^2 - k + 1)}{2}.$$

By Lemma 2.3, we have,

$$\varepsilon(G) = \frac{k(k^2 - k + 1)}{2} < \frac{k^2 - k + 1}{4}(1 + \sqrt{4k^2 - 4k + 1})$$

that is

$$k < \frac{1}{2}(1 + 2k - 1) = k$$

a contradiction.  This proves that $G$ cannot be regular.  This completes the proof.

**Theorem 2.2.**  Let $G \in \mathcal{G}(n; P)$.  Then

$$\mathcal{G}(n, P) = K_1 \vee (\frac{n-1}{2})K_2$$

**Proof:**  By Lemma 2.4, $G$ has a vertex, say $x$, of degree $n - 1$:  The proof of Lemma 2.1 gives that $G - x$ is a 1-regular graph.  Hence,

$$\mathcal{G}(n, P) = K_1 \vee (\frac{n-1}{2})K_2 \quad .$$ This completes the proof of the theorem.

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