*Jordan Journal of Mathematics and Statistics (JJMS)* 2 (1), 2009, pp. 33-42

# SOME APLLICATIONS OF THE Co-HYPONORMAL OPERATOR *

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## ABSTRACT

The purpose of this article is to establish new characterizations of the co-hyponormal operator and some applications of the Fuglede-Putnam Theorem related to the co-hyponormal operator.

## 1. INTRODUCTION

The concept of co-hyponormal operator was introduced by P.R.Halmos [7]. In this paper we study some properties of such concept.

Our study takes advantage of the work of J.G.Stampfli [7,8], S.K.Berberian [9], R.J.Whitley [10], and P.R.Halmos [7] on the hyponormal.

Our work is a generalization of their work by changing some conditions on their theorems; such changes strengthen our results because the conditions that we discussed are weaker than their conditions.

### 2. Preliminary Results

The next definitions and lemmas give a brief description for the background on which the paper will build on. Let H be a separable infinite dimensional complex Hilbert space, and let B (H) denote the algebra of all bounded operators from H into H.

**Definition 2.1.** A mapping $\mathrm{A}:\mathrm{H}\to\mathrm{H}$ is called a linear operator if for all $x, y \in \mathrm{H}$ *and* $\alpha \in C$

(1) $\mathrm{A}(x+y)=\mathrm{A}(x)+\mathrm{A}(y)$ (2) $\mathrm{A}(\alpha x)=\alpha\mathrm{A}(x)$, [7].

Note that we write $\mathrm{A}x$ .instead of $\mathrm{A}(x)$.

**Definition 2.2.** The linear operator $\mathrm{A}:\mathrm{H}\to\mathrm{H}$ is said to be bounded if $\sup_{\|x\|\le 1}\|\mathrm{A}x\|<\infty$, [7].

**Definition 2.3.** Let $\mathrm{A}\in\mathrm{B(H)}$. The spectrum of $\mathrm{A}$, denoted by $\sigma(A)$, is $\sigma(\mathrm{A})=\{\lambda \in C : \mathrm{A}-\lambda I$ is not invertible}, [6].

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2000 **Mathematics Subject Classification:** Primary: 47B20, Secondary : 47B38.

**Keywords**: co-hyponormal, co-subnormal, Fuglede-Putnam Theorem.



**Received on:** Nov. 26, 2008 **Accepted on**: June 16, 2009

**Definition 2.4.** Let $A \in B(H)$. The point spectrum of $A$, $\sigma_p(A)$, is

$$\sigma_p(A) = \{\lambda \in C : Ker(A - \lambda I) \neq 0\}, \text{[6]}.$$

**Definition 2.5.** Let $A \in B(H)$. The approximate point spectrum of $A$, $\sigma_{ap}(A)$, is $\sigma_{ap}(A) = \{\lambda \in C :$ there is a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ and $\lim_{x \to \infty} \|(A - \lambda I)x_n\| = 0\}$,[6].

Note that $\sigma_p(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A)$.

**Lemma 2.6.** If $A \in B(H)$ and $\lambda \in C$, then the following statements are equivalent:

(a) $\lambda \notin \sigma_{ap}(A)$.

(b) $Ker(A - \lambda I) = \{0\}$ and $ran(A - \lambda I)$ is closed.

(c) There is a constant $c > 0$ such that $\|(A - \lambda I)x\| \geq c\|x\|$ for all $x$, [6].

**Definition 2.7.** Let $A \in B(H)$. The spectrum radius of $A$ is the number

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}, \text{[6]}$$

Note that $0 \leq r(A) \leq \|A\|$.

Suppose $A \in B(H_1, H_2)$, where $H_1, H_2$ are separable infinite dimensional complex Hilbert spaces. For each $y \in H_2$, the functional $f_y(x) = \langle Ax, y \rangle$ is a bounded linear functional on $H_1$. By the Riesz representation theorem, there exists a unique $y^*$ in $H_1$ such that for all $x \in H_1$

$$\langle Ax, y \rangle = f_y(x) = \langle x, y^* \rangle.$$

This gives rise to an operator $A^* : H_2 \to H_1$ defined by: $A^* y = y^*$.

Thus, for all $x \in H_1$, $\langle Ax, y \rangle = \langle y, A^* x \rangle$.

**Definition 2.8.** The operator $A^*$ is called the adjoint of $A$, [6].

Note that $A^* \in B(H_2, H_1)$ and $\|A^*\| = \|A\|$.

**Definition 2.9.** An operator $A \in B(H)$ is said to be subnormal if it has a normal extension i.e. $A \in B(H)$ is a subnormal if there exists a normal operator $N$ on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under N and the restriction of N to H coincides with A, [7].

**3. Main Results**

If $\mathrm{A} \in \mathrm{B(H)}$ is subnormal and $\mathrm{N} \in \mathrm{B(K)}$ is a normal extension of $\mathrm{A}$, then with respect to the decomposition $\mathrm{K} = \mathrm{H} \oplus \mathrm{H}^{\perp}$, $\mathrm{N}$ can be written as

$$\mathrm{N} = \begin{pmatrix} \mathrm{A}^* & 0 \\ R^* & S^* \end{pmatrix}, \text{ so } \mathrm{N}^* = \begin{pmatrix} \mathrm{A} & R \\ 0 & S \end{pmatrix}.$$

Since N is normal, it follows that

$$0 = \mathrm{N}^*\mathrm{N} - \mathrm{N}\mathrm{N}^* = \begin{pmatrix} \mathrm{AA}^* + RR^* & RS^* \\ SR^* & SS^* \end{pmatrix} - \begin{pmatrix} \mathrm{A}^*\mathrm{A} & \mathrm{A}^*R \\ R^*\mathrm{A} & R^*R + S^*S \end{pmatrix}.$$

This implies that $\mathrm{AA}^* - \mathrm{A}^*\mathrm{A} = -RR^*$ and hence $\mathrm{AA}^* - \mathrm{A}^*\mathrm{A} \geq 0$.

To achieve the aim of this article, we need the following definition and lemma.

**Definition 3.1.** An operator $\mathrm{A} \in \mathrm{B(H)}$ is called co-subnormal if its dual is subnormal.

**Definition 3.2.** An operator $\mathrm{A} \in \mathrm{B}(H)$ is called co-hyponormal if $\mathrm{AA}^* - \mathrm{A}^*\mathrm{A} \geq 0$.

It follows that every co-subnormal is co-hyponormal. The converse is not necessarily true. To see this, we need the following characterization of subnormal operator.

**Lemma 3.3.** Let $\mathrm{A} \in \mathrm{B}(H)$ then the following are equivalent:

a) $\mathrm{A}$ is subnormal.
b) There is a sequence $\{\mathrm{A}_n\}$ of normal operators such that $\mathrm{A}_n \to \mathrm{A}$ strongly.
c) For any integer $n \geq 0$ and every choice of vectors $x_0, x_1, \cdots, x_n$ in H, the matrix $[\langle \mathrm{A}^i x_j, \mathrm{A}^j x_i \rangle]$ is positive definite, [7].

**Example 3.4.** There exists a co-hyponormal operator that is not co-subnormal.

Let $e_0, e_1, \cdots$ be the standard basis of ℓ2(N ∪ {0}) and $\{\alpha_n\}_{n=0}^{\infty}$ a bounded increasing sequence in ℓ2(N ∪ {0}). Now define the weighted shift operator $S$ on ℓ2(N ∪ {0}) by : $Se_0 = 0$ , $Se_1 = \alpha e_{0\ 1}$ , $Se_n = \overline{\alpha_{n-1}} e_{n-1}$, $n \geq 2$ ,

then $S^* e_n = \alpha_n e_{n+1}$ .

So

$$SS^* = diag(|\alpha_0|^2, |\alpha_1|^2, \cdots),$$

$$S^*S = diag(0, |\alpha_0|^2, |\alpha_1|^2, \cdots)$$

Then

$$SS^* - S^*S = diag(|\alpha_0|^2, |\alpha_1|^2 - |\alpha_0|^2, \cdots) > 0.$$ Hence S is co-hyponormal.

To prove that S is not co-subnormal, it suffices to show that $S^*$ is not subnormal. Note that the matrix $[\langle S^{*i}e_j, S^{*j}e_i\rangle]$ $i, j = 0,1,2$ is not positive definite. In fact the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & \overline{\alpha_0} & \overline{\alpha_0\alpha_1} \\ \alpha_0 & |\alpha_1|^2 & \overline{\alpha_1}|\alpha_2|^2 \\ \alpha_0\alpha_1 & \alpha_1|\alpha_2|^2 & |\alpha_3|^2 \end{bmatrix}.$$

So

$$\det(\mathrm{A}) = |\alpha_0|^2|\alpha_1|^2\left[|\alpha_1|^2 - |\alpha_3|^2\right] + |\alpha_0|^2|\alpha_1|^2\left[|\alpha_2|^2 - |\alpha_1|^2\right] + |\alpha_1|^2|\alpha_2|^2\left[|\alpha_3|^2 - |\alpha_2|^2\right]$$

$$< \left[|\alpha_2|^2 - |\alpha_3|^2\right]\left[|\alpha_0|^2|\alpha_2|^2 + |\alpha_1|^2|\alpha_2|^2\right] < 0.$$

So $S^*$ is not subnormal, hence $S$ is not co-subnormal.

The following result gives some properties of co-hyponormal.

**Theorem 3.5.** Let $\mathrm{A} \in \mathrm{B}(H)$ be co-hyponormal, then

a) If $\mathrm{A}$ is invertible then so is $\mathrm{A}^{-1}$.

b) If $\lambda \in C$ then $\mathrm{A} - \lambda$ is co-hyponormal.

c) For all $x \in \mathrm{H}$, $\|\mathrm{A}x\|^2 \le \|\mathrm{A}^*x\|^2$.

d) If $\lambda \in \sigma_p(\mathrm{A}^*)$ and $x \in \mathrm{H}$ such that $A^*x = \overline{\lambda}x$ then $Ax = \lambda x$.

e) If $\mathrm{A}x = \lambda x$, $\mathrm{A}^*y = \overline{\mu}y$ and $\lambda \ne \mu$ then $\langle x, y\rangle = 0$.

**Proof:**

a) The proof uses the fact that if $Q$ is positive invertible and $Q \ge \mathrm{I}$, then $Q^{-1} \le \mathrm{I}$.

$$\mathit{Since}\ \ \mathrm{AA}^* \ge \mathrm{A}^*\mathrm{A} \Rightarrow \ (\mathrm{A}^*)^{-1}\mathrm{AA}^*\mathrm{A}^{-1} \ge \mathrm{I}$$

$$\Rightarrow ((\mathrm{A}^*)^{-1}\mathrm{AA}^*\mathrm{A}^{-1})^{-1} \le \mathrm{I}$$

$$\Rightarrow \ \mathrm{A}(\mathrm{A}^*)^{-1}\mathrm{A}^{-1}\mathrm{A}^* \le \mathrm{I}$$

$$\Rightarrow (\mathrm{A}^*)^{-1}\mathrm{A}^{-1} \le \mathrm{A}^{-1}(\mathrm{A}^*)^{-1}.$$

That is, $\mathrm{A}^{-1}$ is co-hyponormal.

b) $(\mathrm{A}^* - \overline{\lambda})(\mathrm{A} - \lambda) = \mathrm{A}^*\mathrm{A} - \lambda\mathrm{A}^* - \overline{\lambda}\mathrm{A} + |\lambda|^2$

$$\le \mathrm{AA}^* - \lambda\mathrm{A}^* - \overline{\lambda}\mathrm{A} + |\lambda|^2$$

$$= (\mathrm{A} - \lambda)(\mathrm{A}^* - \overline{\lambda}).$$

So $\mathrm{A}$ is co-hyponormal.

c) For all $x \in \mathrm{H}$, we have $\langle (\mathrm{AA}^* - \mathrm{A}^*\mathrm{A})x, x \rangle \le 0$

$$\Rightarrow \langle \mathrm{AA}^* x, x \rangle \le \langle \mathrm{A}^*\mathrm{A}x, x \rangle$$

$$\Rightarrow \|\mathrm{A}x\|^2 \le \|\mathrm{A}^* x\|^2.$$

d) Since $\|\mathrm{A}x - \lambda x\| \le \|\mathrm{A}^* x - \bar{\lambda} x\| = 0$, $\mathrm{A}x = \lambda x$.

e) Now $\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle \mathrm{A}x, y \rangle = \langle x, \mathrm{A}^* y \rangle = \mu \langle x, y \rangle$. But $\lambda \ne \mu$ hence $\langle x, y \rangle = 0$.

**Theorem 3.6.** If $\mathrm{A} \in \mathrm{B}(H)$ is co-hyponormal, then $\|A^n\| = \|\mathrm{A}\|^n$, and so $r(\mathrm{A}) = \|\mathrm{A}\|$.

**Proof:** If $x \in \mathrm{H}$ *and* $n \ge 1$, then for B= $\mathrm{A}^*$ we have

$$\|B^n x\|^2 = \langle B^n x, B^n x \rangle = \langle \mathrm{B}^*\mathrm{B}^n x, B^{n-1} x \rangle$$

$$\le \|B^* B^n x\| \|B^{n-1} x\|$$

$$\le \|B^* B^{*n} x\| \|B^{n-1} x\|.$$

Hence $\|B^n x\|^2 \le \|(B^*)^{n+1}\| \|\mathrm{B}^{n-1}\|$.

Now, suppose $\|B^k\| = \|B\|^k \quad for \quad 1 \le k \le n.$

Then $\|B\|^{2n} = \|B^n\|^2 \le \|B^{*n+1}\| \|B^{n-1}\| \Rightarrow \|B\|^{n+1} \le \|B^{n+1}\|$

But $\|B^{n+1}\| \le \|B\|^{n+1}$, therefore $\|B^n\| = \|B\|^n$, *for all n.*

Now $r(\mathrm{B}) = \lim_{n \to \infty} \|B^n\|^{\frac{1}{n}}$, so we have $r(B) = \|B\|$. But $\|\mathrm{A}\| = \|\mathrm{A}^*\|$, and so $r(\mathrm{A}) = r(\mathrm{A}^*)$.

**Theorem 3.7**. Let $\mathrm{T} = \mathrm{A} + i\mathrm{B}$ be the Cartesian decomposition of $\mathrm{T}$ with $\mathrm{AB}$ co-hyponormal. If $\mathrm{A}$ or B is positive, then $\mathrm{T}$ is normal.

**Proof:** First assume that $\mathrm{A} \ge 0$ and let $Q = \mathrm{A}B$, then $QA = AQ^*$. Now by Fuglede-Putnam Theorem for co-hyponormal, we have $Q^*\mathrm{A} = \mathrm{A}Q$, i.e., $BA^2 = A^2 B$.

But $\mathrm{A}$ is positive, so $\mathrm{AB} = \mathrm{BA}$ (i.e., $\mathrm{T}$ is normal).

Now, if B is positive, then apply the same arguments to $-i\mathrm{T} = \mathrm{B} - i\mathrm{A}$.

**Theorem 3.8.** Let $\mathrm{A}, \mathrm{B}, \mathrm{X} \in \mathrm{B(H)}$ such that $\mathrm{A}, \mathrm{B}^*$ are co-hyponormal and $\mathrm{X}$ is invertible. If $\mathrm{AX} = \mathrm{XB}$, then there exists a unitary $U$ *such that* $\mathrm{A}U = U\mathrm{B}$ and hence $\mathrm{A}, B$ are normal.

**Proof:** Since $\mathrm{AX} = \mathrm{XB}$, it follows by Fuglede-Putnam Theorem that $\mathrm{A}^*\mathrm{X} = \mathrm{XB}^*$, and so $\mathrm{X}^*\mathrm{A} = \mathrm{BX}^*$. Hence $\mathrm{BX}^*\mathrm{X} = \mathrm{X}^*\mathrm{AX} = \mathrm{X}^*\mathrm{XB}.$

Let $\mathrm{X} = U\mathrm{P}$ be the polar decomposition of $\mathrm{X}$. Since $\mathrm{X}$ is invertible, it follows that $\mathrm{P}$ is invertible and $U$ is unitary.

Since $\mathrm{BP}^2 = \mathrm{P}^2\mathrm{B}$ *and* $\mathrm{P} \geq 0$, *it follows that* $\mathrm{BP} = \mathrm{PB}.$ Thus $\mathrm{A}U\mathrm{P} = U\mathrm{PB}$ implies $\mathrm{A}U\mathrm{P} = U\mathrm{BP}$, since $\mathrm{P}$ is invertible, we have $\mathrm{A}U = U\mathrm{B}.$

Now, $\mathrm{A}$ *and* $\mathrm{B}$ are unitarily equivalent. So $\mathrm{A}^*, \mathrm{B}$ are co-hyponormal. Hence $\mathrm{A}, \mathrm{B}$ are normal.

**Theorem 3.9.** Let $\mathrm{T} = \mathrm{A} + i\mathrm{B}$ be the Cartesian decomposition of $\mathrm{T}$ with $\mathrm{AB}$ co-hyponormal. If $\mathrm{T}$ is co-hyponormal, then $\mathrm{T}$ is normal.

**Proof:** If $Q = \mathrm{AB}$ then $Q\mathrm{A} = \mathrm{A}Q^*$, so by Fuglede-Putnam Theorem for co-hyponormal we have $Q^*\mathrm{A} = \mathrm{A}Q$ $(i.e.\ \mathrm{BA}^2 = \mathrm{A}^2\mathrm{B})$.

Now $\mathrm{TT}^* - \mathrm{T}^*\mathrm{T} = 2i(\mathrm{AB} - \mathrm{BA}) \geq 0.$ Let $\mathrm{Y} = 2i(\mathrm{AB} - \mathrm{BA}).$ Then $\mathrm{YA} = 2i(\mathrm{A}^2\mathrm{B} - \mathrm{ABA}) = -\mathrm{AY}$. Thus, $YA^2$ = (YA) A= -AYA=$A^2$ Y. But Y is positive so YA=AY=0, and so A (AB-BA)=(A$\mathrm{B} - \mathrm{BA})\mathrm{A} = 0$.

Hence

$$\sigma(\mathrm{AB} - \mathrm{BA}) = 0.$$

But $\mathrm{AB} - \mathrm{BA}$ is skew-Hermitian, so it is normal.

Thus,

$\mathrm{AB} = \mathrm{BA}$ (i.e., $\mathrm{T}$ is normal).

**Theorem 3.10**. *Let* $\mathrm{T} = \mathrm{A} + i\mathrm{B} \in \mathrm{B(H)}$ be the Cartesian decomposition of $\mathrm{T}$. If $\mathrm{T}$ is co-hyponormal and $\mathrm{Re\,T}$ *or* $\mathrm{Im\,T}$ is compact, then $\mathrm{T}$ is normal. Consequently, a compact co-hyponormal operator must be normal.

**Proof:** Assume $\mathrm{Re\,T} = \mathrm{A}$ is compact. Since $\mathrm{A}$ is compact and self-adjoint then there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for H consisting of eigen-vectors of B, say $\mathrm{B}\varphi_j = b_j\varphi_j$. where $b_{j's}$ are the eigen-values of B.

Now, $\mathrm{TT}^* - \mathrm{T}^*\mathrm{T} = 2i(\mathrm{BA} - \mathrm{AB}) \geq 0.$

$$\sum_{n=1}^{\infty}\left\langle(\mathrm{TT}^* - \mathrm{T}^*\mathrm{T})\varphi_n, \varphi_n\right\rangle = 2i\sum_{n=1}^{\infty}\left\langle(\mathrm{BA} - \mathrm{AB})\varphi_n, \varphi_n\right\rangle$$

$$= 2i\sum_{n=1}^{\infty}\left[\left\langle \mathrm{BA}\varphi_n, \varphi_n\right\rangle - \left\langle \mathrm{AB}\varphi_n, \varphi_n\right\rangle\right]$$

$$= 2i\sum_{n=1}^{\infty}\left[b_n\left\langle \mathrm{A}\varphi_n, \varphi_n\right\rangle - b_n\left\langle \varphi_n, \mathrm{A}\varphi_n\right\rangle\right] = 0 .$$

Since $\left\langle(\mathrm{TT}^* - \mathrm{T}^*\mathrm{T})\varphi_n, \varphi_n\right\rangle \geq 0, \quad \forall n$, it follows that $\left\langle(\mathrm{TT}^* - \mathrm{T}^*\mathrm{T})\varphi_n, \varphi_n\right\rangle = 0.$

Let $Q = \mathrm{TT}^* - \mathrm{T}^*\mathrm{T}$. Then $\left\langle Q\varphi_n, \varphi_n\right\rangle = 0 \Rightarrow \left\langle Q^{\frac{1}{2}}\varphi_n, Q^{\frac{1}{2}}\varphi_n\right\rangle = 0 \Rightarrow Q^{\frac{1}{2}}\varphi_n = 0$

$\Rightarrow Q\varphi_n = 0, \ \forall n.$

But

$\left|\left\langle Q\,\varphi_j, \varphi_i\right\rangle\right|^2 \leq \left\langle Q\varphi_i, \varphi_i\right\rangle\left\langle Q\varphi_j, \varphi_j\right\rangle = 0$. Hence $\left\langle(\mathrm{TT}^* - \mathrm{T}^*\mathrm{T})\varphi_n, \varphi_n\right\rangle = 0$ for all n.

So $\mathrm{T}^*\mathrm{T} = \mathrm{TT}^*$ as required.

Note that every power of a normal operator is normal. For co-hyponormal operators the fact is different. The following theorem shows that if $\mathrm{A}$ is co-hyponormal then $\mathrm{A}^2$ may be not.

**Theorem 3.11.** If $U$ is the unilateral shift and $\mathrm{A} = U + 3U^*$, then $\mathrm{A}$ is co-hyponormal but $\mathrm{A}^2$ is not.

**Proof:** The proof that $\mathrm{A}$ is co-hyponormal can be done in (at least) two ways, each of which is illuminating. Algebraically:

$$\mathrm{AA}^* = (U + 3U^*)(U^* + 3U) = UU^* + 3U^2 + 3U^{*2} + 9\mathrm{I}$$

$$\mathrm{A}^*\mathrm{A} = (U^* + 3U)(U + 3U^*) = \mathrm{I} + 3U^{*2} + 3U^2 + 9UU^* .$$

Therefore

$$AA^* - A^*A = 9(I - UU^*) \geq 0 \ .$$

Numerically: since

$$A = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0\ddots \end{bmatrix} \text{ and } A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0\ddots \end{bmatrix}$$

it follows that if $x = (\alpha_0, \alpha_1, \alpha_2, \cdots)$

$$Ax = (3\alpha_1, \alpha_0 + 3\alpha_2, \alpha_1 + 3\alpha_3, \cdots)$$

and

$$A^*x = (\alpha_1, 3\alpha_0 + \alpha_2, 3\alpha_1 + \alpha_3, \cdots).$$

Then $\|Ax\| \leq \|A^*x\|$. Hence $A$ is co-hyponormal.

Next, we show that $A^2$ is not co-hyponormal. If $x = e_0 - 3e_2 = (1, 0, -3, 0, \cdots)$,

$$\Rightarrow Ax = (U + 3U^*)(e_0 - 3e_2) = -8e_1 - 3e_3$$

$$\Rightarrow \quad \|Ax\|^2 = 73$$

$$A^2x = (U + 3U^*)(-8e_2 - 3e_4) = -24e_0 - 17e_2 - 3e_4$$

$$\Rightarrow \|A^2x\|^2 = 874$$

$$A^*x = -9e_3$$

$$\Rightarrow \quad \|A^*x\|^2 = 81$$

$$A^{*2}x = -9e_2 - 27e_4$$

$$\Rightarrow \quad \|A^{*2}x\|^2 = 810.$$

But $\|A^2x\|^2 \geq \|A^{*2}x\|^2$. Therefore, $A^2$ is not co-hyponormal.

**Theorem 3.12.** If A is co-hyponormal and right invertible, and then A need not be invertible. Moreover, if A is left invertible, then its invertible.

**Proof:** Consider A = $S_l$, where $S_l$ is the left shift operator then A is co-hyponormal and $AA^{-1} = I$ but $AA^{-1} \neq I$.

Now, assume A is co-hyponormal and BA=I for some $B \in B(H)$ then $A^*B^*A^* = A^*$ so $A^*(B^*A^* - I) = 0$. Since A is co-hyponormal, we have $A(B^*A^* - I) = 0$.

So $AB - I = BA(A^*B^* - I)$

$= B(A(A^*B^* - I))$

$= B0 = 0.$

Hence $AB = I.$

**Theorem 3.13.** If $A \in B(H)$ is co-hyponormal, then $\sigma_{ap}(A) = \sigma(A).$

**Proof:** If $\lambda \notin \sigma(A)$, then $A - \lambda I$ is invertible, so $ran(A - \lambda I)$ is dense in H. Hence $A - \lambda I$ is bounded below, i.e. there exists an $\alpha$ such that $\|(A - \lambda I)x\| \geq \alpha \|x\|$ for all $x \in H$. Therefore, $\lambda \notin \sigma_{ap}(A)$ i.e. $\sigma_{ap}(A) \subset \sigma(A).$

Conversely, assume that $\lambda \notin \sigma_{ap}(A)$, then there is an $\alpha > 0$ such that $\|(A - \lambda I)x\| \geq \alpha \|x\|$ for all $x \in H.$

Since $A - \lambda$ is co- hyponormal, we also have $\|(A - \lambda I)^* x\| = \|(A^* - \overline{\lambda} I)x\| \geq \|(A - \lambda I)x\| \geq \alpha \|x\|$ for all $x \in H$, and hence $\ker(A^* - \overline{\lambda}) = \{0\}.$

Consequently,

$$cl(ran(A - \lambda I)) = \left[\ker(A^* - \overline{\lambda} I)\right]^{\perp} = \{0\}^{\perp} = H.$$

Thus $A - \lambda I$ is bounded below, hence $A - \lambda I$ is invertible, i.e. $\lambda \notin \sigma(A).$ Therefore, $\sigma(A) \subset \sigma_{ap}(A).$ This completes the proof.

**Acknowledgement:** The authors would like to express their sincere thanks to the referee for kind suggestions.

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