STRONGLY REAL ELEMENTS IN SPORADIC GROUPS AND ALTERNATING GROUPS*

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ABSTRACT

We determine the elements in the Sporadic and the Alternating Groups which are strongly real. In the Alternating Groups every real element is strongly real, but this is not true in all Sporadic Groups. Main results for the Alternating groups are given in proposition 6 while the results for the Sporadic Groups are computed manually or by using GAP[5]. The Alternating and the Sporadic Groups in which every element is strongly real are A_n for (n = 5, 6, 10, 14), and J_1 and J_2.

1. INTRODUCTION

An element **x** is conjugate to an element **y** if there exists an element **g** in a group G such that $\mathbf{x=g^{-1}\ x\ g}$, and any element of order 2 is called an involution . An element **z** in a group G is called real if **z** is conjugate to $\mathbf{z}^{-1}$, and is called strongly real if it is the product of two involutions. Similarly the conjugacy class $\mathbf{z^G}$ is called real or strongly real if **z** is real or strongly real respectively. Clearly every strongly real element is real, but the converse is not true. For example, the quaternion group $\mathbf{Q_8}$ has the property that all its elements are real, but its only involution is central, which means that we cannot conjugate an element of order 4 to its inverse. In a recent paper [2], Tiep and Zalesski determine which finite simple groups have the property that every element is real. In particular, they show that the only Sporadic Groups or Alternating Groups with this property are J_1, J_2, A_5, A_6, A_{10} and A_{14} where J_1 and J_2 are the Yanko Sporadic Groups of orders 175560 and 604800 respectively [1]. A much more difficult question, raised in problem 14.82 of the Kourovka notebook [6], is which finite simple groups have the property that every element is strongly real. This problem is still open in general (but see [3], [4], for some cases, and related questions). In this paper we show that in fact all elements in the six groups J_1, J_2, A_5, A_6, A_{10} and A_{14} are strongly real. Moreover, we completely classify the strongly real elements in all the Alternating Groups and the Sporadic Groups. A well known software in Computing Algebra GAP [5] had been used to find all elements in the other 24 Sporadic Groups which are real but not strongly real.

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2. SPORADIC GROUPS

It is easy to check from the Atlas [1] that the only Sporadic Simple Groups in which every element is real are the Yanko Sporadic groups J_1 and J_2. The question then arises, is every element in these groups is strongly real? Now an element is strongly real if it is a product of two involutions, or equivalently, if it is in a subgroup of index 2 in a dihedral group. Thus we look for suitable dihedral groups inside the maximal subgroups of J_1 and J_2 .These are as follows; the two cases.

J_1 Case:

We need only to consider the maximal cyclic subgroups of the Yanko Sporadic groups J_1 of order 175560 with standard generators **a** and **b** given in [1] where **a** has order 2, **b** has order 3, **ab** has order 7 and **ababb** has order 19. Hence we are going to look for the elements 6A, 7A, 10A, 11A, 15A, and 19A. All notation are used from [1] , [8] as A<B means that A is a subgroup of B and 11:10 means F_{110} ; while : $D_{10} \times A_5 > D_{10} \times 2 = D_{20}$ means that the direct product of the dihedral group D_{20} which is the direct product of D_{10} and the cyclic group C_2 is a subgroup of the direct product of the dihedral group D_{10} and the Alternating group A_5 . These notation are widely used in [8] especially with the new versions of this electronic Atlas of Finite Group Representations. Using [5], [8] and other computational techniques such as looking closely to the powers of some elements, commutator, centralizer and other elements and orders that can be computed; One can determine the data given in Tables 1 & 2 .

Table 1 will give the maximal dihedral subgroup of J_1

6A: $D_6 \times D_{10} > D_6 \times 2 = D_{12}$.
7A: $7 : 6 > 7 : 2 = D_{14}$.
10A: $D_6 \times D_{10} > 2 \times D_{10} = D_{20}$.
11A: $11 : 10 > 11 : 2 = D_{22}$.
15A: $D_6 \times D_{10} > 15 : 2 = D_{30}$.
11A: $19 : 6 > 19 : 2 = D_{38}$.
Table 1: Maximal dihedral subgroups of J_1

J_2 Case:

The Yanko Sporadic groups J_2 of order 604800 with standard generators **a** and **b** given in [1] where **a** is in class 2B, **b** is in class 3B, **ab** has order 7 and **ababb** has order 12. Using the same procedure J_1 Case ; we need only to consider the maximal cyclic subgroups of J_2 and hence we are going to look for the elements 6B, 7A, 8A, 10AB, 10CD, 12A, and 15AB in J_2 . Using same notation which are used in [1] and [8] . As an example G:2 is the automorphism group of G , hence $L_3(2){:}2 > D_{12}$ means that the Dihedral group D_{12} is a subgroup of the automorphism group $L_3(2){:}2$.

Table 2 will give the maximal dihedral Subgroups of J_2.

6B: $L_3(2):2 > D_{12}$.
7A: $L_3(2) : 2 > 7 : 6 > 7 : 2 = D_{14}$.
8A: $L_3(2) : 2 > D_{16}$.
10AB: $D_{10} \times A_5 > D_{10} \times 2 = D_{20}$.
10CD: $D_{10} \times A_5 > 2 \times A_5 > D_{20}$.
12A: $3.PGL_2(9) = 3.A_6 : 2 > (3 \times D_8):2 > D_{24}$.
15AB: $D_{10} \times A_5 > D_{10} \times D_6 \times D_{30}$.
Table 2: Maximal dihedral subgroups of J_2

Alternatively, we can check using the character table whether each element is a product of two involutions. For an element z in a conjugacy class Z is a product of elements x and y in classes X and Y respectively if and only if the structure constant:

$$\xi(X,Y,Z) = \frac{|G|}{|C_G(x)|\,|C_G(y)|}\sum_{\chi}\frac{\chi(x)\,\chi(y)\,\overline{\chi(z)}}{\chi(1)}$$

is non zero. Indeed $\xi(X,Y,Z)$ is well-known to be equal to the number of ways that z can be expressed as a product of xy with x in X and y in Y . Thus to determine if z is a product of two involutions, we only need to add up the structure constants $\xi(X,Y,Z)$ over classes X, Y of involutions, and we see whether the resulting integer is positive or zero. In the cases J_1 and J_2 they all are positive. We can use the same method in the other Sporadic Groups to determine which of the real classes are strongly real. We find that most of the real classes are strongly real, but there are some exceptions, listed in Table 3.

Table 3 will give the real classes which are not strongly real in Sporadic Groups

M11	all real elements are strongly real.
M12	all real elements are strongly real.
M22	8A.
M23	8A
HS	all real elements are strongly real.
J3	all real elements are strongly real.
M24	all real elements are strongly real.
McL	3A, 5A, 6A.
He	all real elements are strongly real.
Ru	all real elements are strongly real.

Suz	all real elements are strongly real.
ON	all real elements are strongly real.
Co3	all real elements are strongly real.
Co2	16B.
Fi22	all real elements are strongly real.
HN	8A .
Ly	all real elements are strongly real.
Th	8B.
Fi23	16AB, 22BC, 23AB.
Co1	all real elements are strongly real.
J4	all real elements are strongly real.
Fi_24	all real elements are strongly real.
BM	all real elements are strongly real.
M	8C, 8F, 24F, 24G, 24H, 24J, 32A, 32B, 40A, 48A.

Table 3: Real classes which are not strongly real in Sporadic Groups

The most surprising result is finding elements of small order like 8C and 8F in such a large group like the Monster which is real but not strongly real. . For completeness, we also list all the non real elements in the 26- Sporadic Groups.

Table 4 will give the complete list for all non real elements in the Sporadic Groups

M11	8AB, 11AB.
M12	11AB.
J1	None
M22	7AB.
J2	None
M23	7AB, 11AB, 14AB, 15AB, 23AB.
HS	14AB.
J3	19AB.
M24	7AB, 14AB, 15AB, 21AB, 23AB.
McL	7AB, 9AB, 11AB, 14AB, 15AB, 30AB.
He	7AB, 7DE, 14AB, 14CD, 21CD, 28AB.
Ru	16AB.
ON	31AB.
Co3	11AB, 22AB, 20AB, 23AB.
Co2	14AB, 15BC, 23BC, 30BC.
Fi22	11AB, 16AB, 18AB, 22AB.
HN	19AB, 35AB, 40AB.
Ly	11AB, 22AB, 33AB.

Fi23	16AB, 22BC, 23AB.
Co1	23AB, 39AB.
J4	7AB, 14AB, 14CD, 21AB, 28AB, 35AB, 42AB.
Fi_24	18GH, 23AB.
BM	23AB, 30GH, 31AB, 32CD, 46AB, 47AB.
M	23AB, 31AB, 39CD, 40CD, 44AB, 46AB, 46CD, 47AB, 56BC, 59AB, 62AB, 69AB, 71AB, 78BC, 87AB, 88AB, 92AB, 93AB, 94AB, 95AB, 104AB, 119AB.

Table 4: Non-real classes in Sporadic Groups

3. ALTERNATING GROUPS

Next we consider the Alternating groups An, which are simple for all n > 4. We can use the same method as above to prove that all elements in An for n = 5, 6, 10 and 14 are strongly real. However, by using a different method we can prove much more. Indeed, we prove that if n > 4, then every real element in An is strongly real. Recall that a conjugacy class C in Sn splits into two conjugacy classes in An if and only if its elements are a product of disjoint cycles of distinct odd lengths (including cycles of length 1).

Lemma 1 : If x is in a conjugacy class C, which is an A_n- class and an Sn-class, then x is a product of two involutions in An.

Proof : If x is in C, where C is an A_n - class and an S_n -class, then the cycle type of x includes either a cycle of even length, or, two cycles of the same odd length.

Case 1 : a cycle $(x_1, x_2, \dots, x_{2k})$ of even length or a product of even cycles . For each even cycle of the form $(x_1, x_2, \dots, x_{2k})$; We choose the involution **a** which inverts x to be either

$(x_2, x_{2k})(x_3, x_{2k-1}) \dots (x_{k-1}, x_{k+1})$

which is a product of k − 2 transpositions, or

$(x_1, x_{2k})(x_2, x_{2k-1}) \dots (x_{k-2}, x_{k+1})(x_{k-1}, x_k)$

which is equal to the product of (k−1) transpositions. We can make sure that **a** is an even permutation by choosing the correct one of these two cases. Hence the product of even permutations is an even permutation .

Case 2 : Two cycles

$(x_1, x_2, \dots, x_{2k-1})$ and $(y_1, y_2, \dots, y_{2k-1})$

We can choose

$(x_2, x_{2k-1})(y_2, y_{2k-1})(x_3, x_{2k-2})(y_3, y_{2k-2}) \dots, (x_{k-1}, x_k)(y_{k-1}, y_k)$

Or $(x_1, y_1)(x_2, y_{2k-1})(y_2, x_{2k-1}) \dots (x_{k-1}, y_k)(y_{k-1}, x_k)$.

After we have fixed the action of all the other cycles, we can choose its action on these cycles, so that it is an even permutation.

Lemma 2 : If x in a conjugacy class C, and C is an A_n - class but not an S_n -class, then the following are equivalent

1. x is a product of two involutions in A_n .
2. x is real.
3. number of cycles in x is n mod 4.

Proof

1 $\Rightarrow$ **2** : If x = ab where both a and b are involutions then $x^{-1} = (ab)^{-1} = b^{-1}a^{-1}$ = ba; this implies that $x^{-1}a = b$; therefore $ax^{-1}a = ab = x$. Hence x is real.

2 $\Rightarrow$ **3** : We know that the cycle type of x is cycles of distinct odd lengths; therefore to conjugate x to x^{-1}, we need to reverse each cycle; to do this we must fix exactly one point in each cycle; therefore if the number of cycles in x is k; then **a** which is our element to conjugate x to x^{-1} has exactly k fixed points and is an involution. This implies that a has n − k moved points; which implies (n − k)/2 transpositions. Hence a is in An if and only if (n−k)/2 is even if and only if n−k is divisible by 4 if and only if n is congruent to k mod 4.

3 $\Rightarrow$ **1** : If n is congruent to k mod 4 where k is the number of cycles in x then let **a** be an involution in S_n conjugating x to x^{-1}. By the above argument **a** is a product of (n − k)/2 transpositions, so is in A_n. Now $a^{-1}xa = x^{-1}$ and $a^2 = 1$ implies $axa = x^{-1}$ and therefore xaxa = 1; i.e. xa is an involution and x^{-1} = axa = ab where b = xa is an involution in A_n. Hence x =ab which is a product of two involutions.

Corollary 3 : If n > 4, every real element of A_n is strongly real.

Proof Immediate from **Lemma 1**, and the equivalence of 1 and 2 in **Lemma 2**.

Corollary 4 : If n > 4, an element of A_n is non real if and only if it is the product of k disjoint cycles of distinct odd lengths where $n \equiv (k + 2) \bmod 4$.

Proof : Immediate from **Lemma 1** and the equivalence of 2 and 3 in **Lemma 2.**

Lemma 5 : If n > 6 and $n \neq 10, 14$, then there exists C an A_n class not an S_n class where x is in C but is not real in A_n.

Proof:

If n is congruent to 0 mod 4 and n > 3 then take x to be an element of cycle type (n−1).1. This has cycles of distinct odd lengths. By Lemma 2, x is not real since n is not congruent to 2 mod 4. Similarly, if n is congruent to 1 mod 4 and n > 8 ; take x of cycle type (n − 4).3.1, which implies that n is not congruent to k mod 4 as k = 3. If $n \equiv 2 \bmod 4$ and n > 14, take x of cycle type (n − 9).5.3.1. If $n \equiv 3 \bmod 4$ and n > 3, take x of cycle type n. These cases together deal with all n > 4 except for n = 5, 6, 10, 14.

Proposition 6 : If n=5, 6, 10, 14 then every element of A_n is a product of two involutions; i.e. every element is strongly real.

Proof : By **Lemma 1**, we need only to consider elements whose cycles have distinct odd lengths. By **Lemma 2**, we need only to show that the number k of cycles is congruent to n mod4.

1. If n = 5, the only possibility for a cycle type is 5, so k = 1, and 1 is congruent to 5 mod 4.
2. If n = 6, the only possibility for a cycle type is 5.1, so k = 2, and 2 is congruent to 6 mod 4.
3. If n = 10, the only possible cycle types are 9.1 or 7.3, so k = 2, and 2 is congruent to 10 mod 4.
4. If n = 14, the only possible cycle types are 9.5, 11.3 or 13.1, so k = 2, and 2 is congruent 14 mod 4.

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