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STOCHASTIC LOSS SYSTEMS: MODELS AND POLICIES*

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ABSTARCT

In a multi-server system, probability distributions and loss probabilities for customers arriving with k different priority categories are studied. Customers arrive in independent Poisson streams and their service times are exponentially distributed, with different rate for different priority. The non-queuing customers or the loss customers will be lost if the capacity is fully occupied. In these systems, particularly for higher priority customer the reduction of the loss probabilities is very essential to guarantee the quality of the service. Four different policies for k=2, high (emergency) and low (ordinary), were introduced utilising the fixed capacity of the system, producing different loss probabilities, by which the minimum can be selected. An example illustrating these results is also given.

INTRODUCTION

There are many real-life priority serving systems in which it is impractical, or even impossible, to pre-empt a low priority service in order to serve an arrival of high priority. When there are multiple servers, there is a probability that a high priority arrival may have to wait or lost because all the servers are busy. The wait or lost can be reduced by using a drop service (transfer) or reservation. In general, for a non-queuing system, when all the servers are busy, the low priority arrival is lost, while the high priority arrival will be served by dropping a low priority customer. In the case when all servers are busy with high priority, the arrival of high priority customer is lost.

A literature search showed that the concentration was on the multiple input streams, where it was first traced on the hospital application. Balintfy (1952) "(See Gross & Harris (1998)) considers a census-predictor model by formulating the system as a Markov process. Three different categories of patients (good, fair, and poor) are used to classify the patient's state. Blumberg (1961) considers only one patient type. Using the Poisson distribution for the demand process and deterministic service behaviour, he develops tables useful in studying the effects of various allocation policies under different conditions. Weckwerth (1966) employs a similar modelling

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approach and assumes that the negative exponential distribution satisfactorily characterises a patient's length of stay in the hospital (Esogbue and Singh 885).

Young (1962) was the first to discuss the cut-off model (where low priority customers are kept waiting if the number of servers busy when he arrives is at or above a specified cut-off level) to discusses two models for admissions and discharges.

Kolesar (1970), following the work of Young, developed a Markovian decision model for hospital admission scheduling. Describing the state of the system as the number of beds occupied at the start of a given period *t*, he introduced a linear programming model that exploits the Markovian structure to provide a basis for determining an optimal control policy for scheduling admissions (Esogbue and Singh 885). The utility of Kolesar's model to hospital administration has been severely questioned. The objection centres on the oversimplification of the system dynamics, the attempt to apply a Markovian decision model to a problem that is essentially non-Markovian, and the inherent difficulties involved in gathering realistic data.

It is argued by Taylor and Templeton (1980) that the cut-off model was formulated by Benn (1966) to solve a priority assignment problem in railroad transportation, and the results are published in Jaiswal (1968, pp. 204-214). The same model was later put forward by Shonick and Jackson (1973) to assist in finding how many hospital beds were needed to serve the emergency and regular patients. Cooper (1972) also considers the same model under the name priority reservation. McClain (1976) based on the model's assumptions examined the sensitivity of the predicted hospital census to several of the model's assumptions. Furthermore, Esogbue and Singh (1976) presented a version of the model based on the assumption that no customer wait but there are different service rates for the two priority classes.

This paper is focused on the serving systems with fixed multi-servers capacity and non-queuing customers. Customers' distribution and loss probabilities with k different priority categories and different serving rates are introduced in section 2. A particular system with two categories (k =2), high (emergency) and low (ordinary) priority, are also discussed and four different serving policies are introduced in section3.

In section four, a numerical illustrative example is presented to show the differences and preferences in relation to these policies.

2. MULTI-CATEGORY SYSTEM MODEL

Assume a system with *C* fixed capacities and k priority categories where k is the least important category. The customers with category *j,* (*j* = 1, 2, *…,* k), have a Poisson arrival rate of λ_j and exponential service time of rate μ_j .

Let $\rho_j = \dfrac{\lambda_j}{\mu_j}$ be the workload parameter for category *j*, and $P(n_1, n_2, ..., n_k)$ be the probability of n_j customers in the system from categories *j, j*=1, 2, …, k, where $n_j \geq 0, \sum_{j=1}^{k} n_j \leq C$ and = 0 otherwise . For simplicity we write P (0, 0, …, 0)= P_0 and $P(n_1, n_2, ..., n_k) = \mathrm{P}_{\boldsymbol{u}}$, where **u** is a vector with jth component n_j (*j* =1, 2, …, k). The steady state equations for this model are

$$\sum_{\mathrm{j}=1}^{k}\left(\lambda_{\mathrm{j}} + \mathrm{n}_{\mathrm{j}}\mu_{\mathrm{j}}\right)\mathrm{P}_{\boldsymbol{u}} = \sum_{\mathrm{j}=1}^{k}\left(\mathrm{n}_{\mathrm{j}} + 1\right)\mu_{\mathrm{j}}\mathrm{P}_{\boldsymbol{u}+\boldsymbol{v}} + \sum_{\mathrm{j}=1}^{k}\lambda_{\mathrm{j}}\mathrm{P}_{\boldsymbol{u}-\boldsymbol{v}}. \qquad (2.1)$$

Where **v** is a vector with all zero components except for a one at position *j* corresponding to priority *j.*

Theorem 2.1

When categories 1, 2, …, k have different service times, the probability distribution of the customers in the system is

$$P_u = \left(\prod_{\mathrm{j}=1}^{k}\frac{\rho_{\mathrm{j}}^{\mathrm{n_j}}}{\mathrm{n_j}!}\right)\mathrm{P}_0 \text{ Where, } \mathrm{P}_0 = \left[\sum_{\substack{(\mathrm{n}_1,\mathrm{n}_2,\ldots,\mathrm{n}_k)\\ \sum_{\mathrm{j}=1}^{k}\mathrm{n_j}\leq \mathrm{C}}}\prod_{\mathrm{j}=1}^{k}\frac{\rho_{\mathrm{j}}^{\mathrm{n_j}}}{\mathrm{n_j}!}\right]^{-1}. \qquad (2.2)$$

Proof:

To prove the theorem we need to show that (2.2) satisfies (2.1). Let $\mathrm{P}_{\boldsymbol{u}} = \mathrm{D}\prod_{\mathrm{j}=1}^{k}\frac{\rho_{\mathrm{j}}^{\mathrm{n_j}}}{\mathrm{n_j}!}$ then we show that D should be

$$\left[\sum_{\substack{(\mathrm{n}_1,\mathrm{n}_2,\ldots,\mathrm{n}_k)\\ \sum_{\mathrm{j}=1}^{k}\mathrm{n_j}\leq \mathrm{C}}}\prod_{\mathrm{j}=1}^{k}\frac{\rho_{\mathrm{j}}^{\mathrm{n_j}}}{\mathrm{n_j}!}\right]^{-1}$$

in order to satisfy the summability – to – one criterion. Let $\Re(\boldsymbol{u}) = \prod_{\mathrm{j}=1}^{k}\frac{\rho_{\mathrm{j}}^{\mathrm{n_j}}}{\mathrm{n_j}!}$, then substituting $\mathrm{P}_{\boldsymbol{u}} = \mathrm{D}\Re(\boldsymbol{u})$ into (2.1) gives

$$D\Re(\boldsymbol{u})\sum_{j=1}^{k}(\lambda_j + n_j\mu_j) = D\Re(\boldsymbol{u})\sum_{j=1}^{k}\lambda_j + D\Re(\boldsymbol{u})\sum_{j=1}^{k} n_j\mu_j .$$

Since the left-hand side is the same as the right hand-side, i.e. $P_{\boldsymbol{u}} = D\prod_{j=1}^{k}\frac{\rho_j^{n_j}}{n_j!}$. Now to evaluate *D*, we have:

$$\sum_{\substack{(n_1, n_2, \ldots, n_k) \\ \sum_{j=1}^{k} n_j \le C}} P_{\boldsymbol{u}} = 1.$$

Thus,

$$D = \left[\sum_{\substack{(n_1, \ldots\ldots, n_k) \\ \sum_{j=1}^{k} n_j \le C}} \prod_{j=1}^{k}\frac{\rho_j^{n_j}}{n_j!}\right]^{-1}.$$

Corollary 2.1

The probability of dropping a customer from priority category *j* out of the system is given by

$$\sum_{\substack{(n_1, \ldots, n_j, 0, \ldots, 0) \\ \sum_{i=1}^{j} n_i = C}} P_{\mathbf{u}} - \sum_{\substack{(n_1, \ldots, n_{j-1}, 0, \ldots, 0) \\ \sum_{i=1}^{j-1} n_i = C}} P_{\mathbf{u}} .$$

Proof:

Let T_j be the probability that a customer of category *j* is dropped out of the system, then

T_j = Pr {The system is full by categories 1, 2, …, *j* and $n_j > 0$}

= Pr {The system is full by categories 1, 2, …, *j* }

× Pr { $n_j > 0$ | system is full by categories 1, 2, …, *j*}

= Pr {The system is full by categories 1, 2, …,*j* - 1}

$$\times\left(1 - \frac{\Pr\{\text{The system is full by categories } 1, 2, \ldots, j\text{ -}1\}}{\Pr\{\text{The system is full by categories } 1, 2, \ldots, j\}}\right)$$

= Pr {The system is full by categories 1, 2, …, *j*}–

Pr {The system is full by categories 1, 2, …, *j* - 1}

$$= \sum_{\substack{(n_1, \ldots, n_j, 0, \ldots, 0) \\ \sum_{i=1}^{j} n_i = C}} P_{\mathbf{u}} - \sum_{\substack{(n_1, \ldots, n_{j-1}, 0, \ldots, 0) \\ \sum_{i=1}^{j-1} n_i = C}} P_{\mathbf{u}} .$$

Further, the proportion of customers of category *j* that are dropped from the system is given by

$$\left(\sum_{\substack{(n_1,\ldots,n_j,0,\ldots,0) \\ \sum_{i=1}^{j} n_i = C}} P_{\mathbf{u}} - \sum_{\substack{(n_1,\ldots,n_{j-1},0,\ldots,0) \\ \sum_{i=1}^{j-1} n_i = C}} P_{\mathbf{u}} \right) \frac{\sum_{i=1}^{\ell} \lambda_i}{\sum_{i=1}^{j} \lambda_i - \sum_{i=1}^{j-1} \lambda_i} .$$

Corollary 2.2

If $\mu_1 = \mu_2 = \ldots = \mu_\ell = \mu$ then

1) The probability distribution of the customers in the system is given by $P_n = \dfrac{\left(\sum_{j=1}^{k} \lambda_j\right)^n}{n!\mu^n} P_0$,

where $P_0 = \left[\sum_{n=0}^{C} \dfrac{\left(\sum_{j=1}^{k} \lambda_j\right)^n}{n!\mu^n} \right]^{-1}$ *, for* $0 \le n \le C$ (2.3)

2) The probability of dropping a customer from category *j* out of the system is given by

$$\Pr(\sum_{i=1}^{j} n_i = C) - \Pr(\sum_{i=1}^{j-1} n_i = C) .$$

Proofs:

1) Apply mathematical induction on k,

When k =2

Pr {*n* customers in the system} $= \sum_{\substack{(n_1, n_2) \\ n_1 + n_2 = n}} \dfrac{(\frac{\lambda_1}{\mu})^{n_1} (\frac{\lambda_2}{\mu})^{n_2}}{n_1! n_2!} P_0$

$$= \sum_{n_1=0}^{n} \frac{(\frac{\lambda_1}{\mu})^{n_1} (\frac{\lambda_2}{\mu})^{n-n_1}}{n_1!(n-n_1)!} P_0$$

$$= (\frac{\lambda_2}{\mu})^n \sum_{n_1=0}^{n} \frac{(\frac{\lambda_1}{\mu})^{n_1} (\frac{\lambda_2}{\mu})^{-n_1}}{n_1!(n-n_1)!} P_0$$

$$= (\frac{\lambda_2}{\mu})^n (\frac{1}{n!}) \sum_{n_1=0}^{n} \binom{n}{n_1} (\frac{\lambda_1}{\lambda_2})^{n_1} P_0$$

$$= (\frac{\lambda_2}{\mu})^n (\frac{1}{n!})(1 + \frac{\lambda_1}{\lambda_2})^n P_0$$

$$= (\frac{\lambda_1 + \lambda_2}{\mu})^n (\frac{1}{n!}) P_0 .$$

Now suppose (2.3) is true for k. We need to prove it for k +1, we have

$$\Pr \{n \text{ customers in the system}\} = \sum_{\substack{(n_1, n_2, \ldots, n_{k+1}) \\ n_1 + n_2 + \ldots + n_{k+1} = n}} \prod_{j=1}^{k+1} \frac{\rho_j^{n_j}}{n_j\,!} P_0$$

$$= \sum_{n_{k+1}=0}^{n} \prod_{\substack{j=1 \\ (n_1, \ldots, n_k) \\ n_1 + n_2 + \ldots + n_k = n - n_{k+1}}}^{k} \frac{\rho_j^{n_j} \rho_{\ell+1}^{n_{k+1}}}{n_j\,! n_{k+1}\,!} P_0$$

$$= \sum_{n_{k+1}=0}^{n} \frac{\left(\sum_{j=1}^{\ell} \lambda_j \right)^{.n - n_{k+1}} \rho_{k+1}^{n_{k+1}}}{(n - n_{k+1})! n_{k+1}\,!} P_0$$

$$= \left(\frac{1}{\mu^n\, n!} \right) \sum_{n_{k+1}=0}^{n} \binom{n}{n_{k+1}} \left(\sum_{j=1}^{k} \lambda_j \right)^{n - n_{k+1}} \lambda^{n_{k+1}}{}_{k+1} P_0$$

$$= \left(\frac{\left(\sum_{j=1}^{k} \lambda_j \right)^n}{\mu^n\, n!} \right) \left(1 + \frac{\lambda_{k+1}}{\sum_{j=1}^{k} \lambda_j} \right)^n P_0$$

$$= \frac{\left(\sum_{j=1}^{k+1} \lambda_j \right)^n}{\mu^n\, n!} P_0 .$$

2) Let T_j be the probability that a customer in category *j* is dropped out of the system then,

$$
\begin{aligned}
T_j &= \Pr\{(\sum_{i=1}^{j} n_i = C) \text{ and } n_i > 0\} \\
&= \Pr(\sum_{i=1}^{j} n_i = C) \times \Pr(n_i > 0 / \sum_{i=1}^{j} n_i = C) \\
&= \Pr(\sum_{i=1}^{j} n_i = C) \times (1 - \frac{\Pr(n_i = 0)}{\Pr(\sum_{i=1}^{j} n_i = C)}) \\
&= \Pr(\sum_{i=1}^{j} n_i = C) - \Pr(\sum_{i=1}^{j-1} n_i = C).
\end{aligned}
$$

Further, the proportion of dropping a customer from category j out of the system is given by

$$
\left(\Pr(\sum_{i=1}^{j} n_i = C) - \Pr(\sum_{i=1}^{j-1} n_i = C) \right) \frac{\sum_{i=1}^{j} \lambda_i}{\sum_{i=1}^{j} \lambda_i - \sum_{i=1}^{j-1} \lambda_i} .
$$

3. TWO CATEGORY MODEL SYSTEM

In this section only two priority categories (high and low) are considered, each arriving according to a Poisson process with rates λ_1 and λ_2 for high and low respectively. The total arrival rate is $\lambda = \lambda_1 + \lambda_2$. Service times are exponentially distributed, with mean service times $1/\mu_1$ and $1/\mu_2$ for high and low priority respectively. The workload parameters $\rho_j = \frac{\lambda_j}{\mu_j} \ (j = 1, 2)$ and $\rho = \rho_1 + \rho_2$.

Suppose there are C servers and the service rate for the two priorities are different. No customers of either priority are allowed to queue. The loss probabilities and customer's probability distributions are defined in relation to the following four different policies.

3.1 Policy one

Both priorities will have the same chance of being served providing the capacity C is not full. In the case of all servers in use (fully occupied capacity), the low priority arrival is always lost, while the high priority arrival will be served by dropping the low priority already being served in the system. When all C servers are occupied by high priority customers, arrivals from either category are lost. Using the same method in theorem 2.1 we have the following theorem.

Theorem 3.1.1

The probability distribution of the customers (high and low priorities) in the system is given by

$$P(n_1, n_2) = \frac{\rho_1^{n_1} \rho_2^{n_2}}{n_1! n_2!} P(0,0) \text{ ,}$$

Where $P(0,0) = (\sum_{\substack{(n_1, n_2) \\ n_1 + n_2 \le C}} \frac{\rho_1^{n_1} \rho_2^{n_2}}{n_1! n_2!})^{-1}$.

From the above theorem we have the following observations:

1. The probability of losing a low priority customer is given by

$$\sum_{\substack{(n_1, n_2) \\ n_1 + n_2 = C}} P(n_1, n_2) \text{ .}$$

2. The probability of dropping a low priority customer out of the system is given by

$$\sum_{\substack{(n_1, n_2) \\ n_1 + n_2 = C}} P(n_1, n_2) - P(C, 0) \text{ .}$$

3. The probability of losing a high priority customer is given by

$$P(C, 0) = \frac{\rho_1^C}{C!} P(0,0) \text{ .}$$

3.2 Policy two

In this policy we divide the capacity *C* into two subsystems C_1 and C_2 for high and low priority respectively, the division can follow any desirable (optimal) proportion. A low priority customer is lost when C_2 is full, and it is dropped when a high priority customer arrives while C_1 is full. A high priority customer is lost when C_1 and C_2 are fully occupied by high priority customers. Using the same method in theorem 2.1 we have the following theorem.

Theorem 3.2.1

1. The probability distribution of the high priority customers in the high priority subsystem is given by

$$P(n) = (\frac{\rho_1^n}{n!}) P_0, \, n = 0, 1, 2, ..., C_1 \text{ ,}$$

Where $P_0 = (\sum_{n=0}^{C_1} \frac{\rho_1^n}{n!})^{-1}$.

2. The probability distribution of the customers, low and high priorities, in the low priority subsystem is given by

$$P(n_1,n_2)=\frac{\rho_1^{n_1}\rho_2^{n_2}}{n_1!n_2!}P(0,0),$$

Where $\rho_1=\frac{\lambda_1'}{\mu_1}$, $\rho_2=\frac{\lambda_2}{\mu_2}, \lambda_1'=\lambda_1 P(C_1)$, and $P(0,0)=(\sum_{\substack{(n_1,n_2)\\ n_1+n_2\le C_2}}\frac{\rho_1^{n_1}\rho_2^{n_2}}{n_1!n_2!})^{-1}$.

Corollary 3.2.1

The probability of dropping a low priority customer out of the low priority subsystem is given by

$$(\frac{\rho_1^{C_1}}{C_1!}P_0)(1-\frac{P(C_2,0)}{\sum_{n_1+n_2=C_2}P(n_1,n_2)})$$

Where, $P_0=\left(\sum_{n=0}^{C_1}\frac{\rho_1^n}{n!}\right)^{-1}$.

Proof:

Pr {dropping a low priority customer} = Pr {The high priority subsystem is full} $\times$ Pr{there is at least one low priority customer in the low priority subsystem | low priority subsystem is full} $=(\frac{\rho_1^{C_1}}{C_1!}p_0)$ [1-Pr {low priority subsystem is full by high priority customers} / Pr {low priority subsystem is full}]

$$=(\frac{\rho_1^{C_1}}{C_1!}p_0)\left(1-\frac{P(C_2,0)}{\sum_{n_1+n_2=C_2}P(n_1,n_2)}\right).$$

Further, the probability of losing a high priority customer is given by

$$\frac{\rho_1^{C_1}}{C_1!}P_0P(C_2,0).$$

3.3 Policy three

Having C_1 and C_2 defined as in policy two, we consider a cutoff level *k* for a high priority in the low priority subsystem (C_2). Since the serving times for high and low priorities are different, and the arrival at and prior to the cutoff level *k* could be high or low priority, we need to ensure that the total serving time in the low priority subsystem (C_2) remains constant. This situation arises only when the recently exited customer's priority differs to the arriving one. In this case we always give the exited customer's priority service time to the arriving one. Here, there is no serving beyond the cutoff level *k*. In the case of serving after *k* level the exited customer's serving time will be allocated to the replaced (or transferred) customer being served after cutoff level *k*.

Suppose n_1 and n_2 are the number of high and low priority customers in the low priority subsystem and assume $P(n_1, n_2) =$ Pr {n_1 high priority and n_2 low priority in the low priority subsystem}, ($n_1 + n_2 < k; = 0$ otherwise), and P_m $=$ Pr {m servers are busy in the low priority subsystem | n_1 servers are serving high priority customer in the low priority subsystem}($m = k,...,C_2$; $n_1 = 0,1,...,k; = 0$ otherwise). Then the steady state equations become:

$$P(n_1, n_2)(\lambda + n_1\mu_1 + n_2\mu_2) = \lambda_1' P(n_1 - 1, n_2) + \lambda\lambda_2 P(n_1, n_2 - 1) + (n_1 + 1)\mu_1 \\ \times P(n_1 + 1, n_2) + (n_2 + 1)\mu_2 P(n_1, n_2 + 1), \quad \text{if } n_1 + n_2 < k \tag{2.4}$$

For given n_1 $(n_1 = 0,1,..., k)$

$$P_k(\lambda_2 + N_k) = \lambda P_{k-1} + N_{k+1} P_{k+1} \qquad \text{if } m = k \tag{2.5}$$

$$P_m(N_m + \lambda_2) = \lambda_2 P_{m-1} + N_{m+1} P_{m+1} \qquad \text{if } k < m < C_2 \tag{2.6}$$

$$N_{C_2} P_{C_2} = \lambda_2 P_{C_2 - 1} \qquad \text{if } m = C_2 \tag{2.7}$$

Where

$$N_m = n_1\mu_1 + (m - n_1)\mu_2$$

$$P_n' = \frac{\rho_1^{n_1}}{n!} P_0'' \text{ and } P_0'' = \left(\sum_{n=0}^{C_1} \frac{\rho_1^n}{n!}\right)^{-1}.$$

$$\lambda = \lambda_1' + \lambda_2, \rho_2 = \frac{\lambda_2}{\mu_1}, \rho_1' = \frac{\lambda_1'}{\mu_2} \lambda_1' = \lambda_1 P_{C_1}', \text{ and } N_m' = n_1 + (m - n_1)\left(\frac{\mu_2}{\mu_1}\right).$$

The solution of equation (2.4) is given by

$$P(n_1, n_2) = \frac{\rho_1'^{n_1} \rho_2^{n_2}}{n_1! n_2!} P_0 \qquad \text{if } 0 \le n_1 + n_2 < k \tag{2.8}$$

Now, equations (2.5), (2.6), and (2.7) become:

$$(\rho_2 + N_k') P_k = \rho P_{k-1} + N_{k+1}' P_{k+1} \qquad \text{if } k = m$$

$$(N_m' + \rho_2) P_m = \rho_2 P_{m-1} + N_{m+1}' P_{m+1} \qquad \text{if } k < m < C_2$$

$$N'_{C_2} P_{C_2} = \rho_2 P_{C_2 - 1} \qquad \text{if } \ m = C_2$$

The above set of equation is true for any given n_1, this can be written in a matrix form as follows,

$\mathbf{AP}^* = \beta\ \mathbf{e}$, where $\beta = \rho P_{k-1}$, $\mathbf{e} = [1,0,0,...,0]$, and $\mathbf{P}^* = [P_k, P_{k+1}, ..., P_{C_2}]$. Hence

$$\mathbf{A} = \begin{bmatrix} \rho_2 + N'_k & -N'_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho_2 & N'_{k+1} + \rho_2 & -N'_{k+2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho_2 & N'_{k+2} + \rho_2 & -N'_{k+3} & 0 & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho_2 & N'_{C_2 -1} + \rho_2 & -N'_{C_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\rho_2 & N'_{C_2} \end{bmatrix}$$

It follows that:

$$P_{C_2 - 1} = \frac{N'_{C_2}}{\rho_2} P_{C_2},$$

$$P_{C_2 - 2} = \frac{N'_{C_2} N'_{C_2 - 1}}{\rho_2^2} P_{C_2}$$

$$P_{C_2 - j} = \frac{N'_{C_2} ... N'_{C_2 - j + 1}}{\rho_2^j} P_{C_2}, \quad 1 \le j \le C_2 - k - 1 \qquad (2.9)$$

And

$$(\rho_2 + N'_k) P_k - N'_{k+1}\ P_{k+1} = \rho P_{k-1} \qquad (2.10)$$

Substituting P_{k+1} from (2.9) into (2.10) we obtain

$$(\rho_2 + N'_k) P_k - \frac{N'_{C_2} ... N'_{k+1}}{\rho_2^{C_2 - k - 1}} P_{C_2} = \rho P_{k-1}$$

Hence, if we apply equation (2.8) on P_k and P_{k-1} we obtain

$$(\rho_2 + N'_k) \sum_{\substack{(n_1, n_2) \\ n_1 + n_2 = k}} \frac{\rho_1'^{n_1} \rho_2^{n_2}}{n_1! n_2!} P_0 - \frac{N'_{C_2} ... N'_{k+1}}{\rho_2^{C_2 - k - 1}} P_{C_2} = \rho \sum_{\substack{(n_1, n_2) \\ n_1 + n_2 = k-1}} \frac{\rho_1'^{n_1} \rho_2^{n_2}}{n_1! n_2!} P_0$$

Hence

$$P_{C_2} = \frac{\rho_2^{C_2-k-1}}{N'_{C_2}...N'_{k+1}}[(\rho_2 + N'_k)\sum_{\substack{(n_1,n_2)\\ n_1+n_2=k}} \frac{\rho_1'^{n_1}\rho_2^{n_2}}{n_1!n_2!}P_0 - \rho \sum_{\substack{(n_1,n_2)\\ n_1+n_2=k-1}} \frac{\rho_1^{n_1}\rho_2^{n_2}}{n_1!n_2!}P_0]$$

And

$$P_m = \frac{N'_{C_2}...N'_{m+1}}{\rho_1^{C_2-m}}P_{C_2}, \; m = k+1,...,C_2-1 .$$

Now we have the following theorem.

Theorem 3.3.1

The probability distribution of the customers in the low priority subsystem is given by

$$P(n_1,n_2) = \frac{\rho_1'^{n_1}\rho_2^{n_2}}{n_1!n_2!}P_0 \qquad \text{if } n_1+n_2<k$$

$$P(m) = \sum_{n_1=0}^{k} \frac{N'_{C_2}....N'_{m+1}}{\rho_2^{C_2-m}}P_{C_2}(\frac{\rho_1^{n_1}}{n_1!}P'_0) \qquad \text{if } m = k,...,C_2$$

Where

$$P'_0 = (\sum_{n=0}^{k}\frac{\rho_1^n}{n!})^{-1}, \text{ and } P_0 \text{ given by } \sum_{\substack{(n_1,n_2)\\ n_1+n_2<k}} \frac{\rho_1'^{n_1}\rho_2^{n_2}}{n_1!n_2!} + \sum_{m=k}^{C_2}P(m) = 1$$

Proof:

For $m \geq k$, we have

Pr {n_2 servers busy in low priority subsystem}= $\sum_{n_1=0}^{k}\Pr\{m$ servers busy in low priority subsystem \ n_1 busy by high priority in low priority subsystem}× Pr {n_1 high priority in low priority subsystem}

$$= \sum_{n_1=0}^{k} \frac{N'_{C_2}....N'_{m+1}}{\rho_2^{C_2-m}}P_{C_2}(\frac{\rho_1^{n_1}}{n_1!}P'_0) \qquad \text{if } m = k,...,C_2 .$$

From the above theorem we have the following observations:

1. The probability of losing a high priority customer is given by

$$(P'_{C_1})\sum_{m=k}^{C_2}P(m)$$

   Where $P'_{C_1} = \frac{\rho_1^{C_1}}{C_1!}P_0''$ and $P_0'' = (\sum_{n=0}^{C_1}\frac{\rho_1^n}{n!})^{-1}$.

2. The probability of losing a low priority customer is given by

$$\sum_{n_1=0}^{k} N'_{C_2}\left(\frac{\rho_1^{n_1}}{n_1!}P'_0\right)\left(\frac{\rho_2^{C_2-k-1}}{N'_{C_2}\ldots N'_{k+1}}[(\rho_2+N'_k)\sum_{\substack{(n_1,n_2)\\ n_1+n_2=k}}\frac{\rho_1'^{n_1}\rho_2^{n_2}}{n_1!n_2!}P_0-\rho\sum_{\substack{(n_1,n_2)\\ n_1+n_2=k-1}}\frac{\rho_1^{n_1}\rho_2^{n_2}}{n_1!n_2!}P_0]\right).$$

3.4 Policy four

This policy is similar to policy one except we consider a cutoff level *k* for low priority customers. In other words, both high and low priorities are treated equally when number of occupied servers are less than *k*. A low priority is always lost beyond the cutoff level *k*.

As we mentioned in policy three, the total serving time prior to cutoff level *k* should remain constant. Here too, we maintain this condition, but instead of having a cutoff level for high priority we have it for low priority. Suppose n_1 and n_2 are the number high and low priority customers in the system and assume $P(n_1,n_2)=$ Pr {n_1 high priority in system, n_2 low priority in the system}($n_1+n_2<k, =0$ otherwise), and P_m $=$ Pr {m servers are busy in system | n_1 servers are serving low priority customer in system}($m=k,...,C$; $n_1=0,1,...,k; =0$ otherwise).

Theorem 3.4.1

The probability distribution of the customers in the system is given by

$$P(m) = \sum_{\substack{(n_1,n_2)\\ n_1+n_2=m}}\frac{(\lambda_1/\mu_1)^{n_1}(\lambda_2/\mu_2)^{n_2}}{n_1!n_2!}P_0\ , \qquad \text{if } m<k$$

$$= \sum_{n_1=0}^{k}\frac{N'_C\ldots.N'_{m+1}}{\rho_1^{\,c-m}}\left(\frac{\rho_2^{n_1}}{n_1!}\right)P_C\ P'_0\ , \qquad \text{if } m=k,\ldots.,C$$

Where P_0 is given from $\sum_{m=0}^{C}p(m)=1$ and

$$P_C=\frac{\rho_1^{c-k-1}}{N_C{}'\ldots N'_{k+1}}\left[(\rho_1+N_k{}')\sum_{\substack{(n_1,n_2)\\ n_1+n_2=k}}\frac{(\lambda_1\mu_1)^{n_1}(\lambda_2\mu_2)^{n_2}}{n_1!n_2!}-\rho\sum_{\substack{(n_1,n_2)\\ n_1+n_2=k-1}}\frac{(\lambda_1\mu_1)^{n_1}(\lambda_2\mu_2)^{n_2}}{n_1!n_2!}\right]P_0$$

And $N'_m=n_1+(m-n_1)\dfrac{\mu_1}{\mu_2},\rho=\dfrac{\lambda_1+\lambda_2}{\mu_1},\rho_1=\dfrac{\lambda_1}{\mu_1},and\ \ \rho_2=\dfrac{\lambda_2}{\mu_1}.$

Proof: See Theorem (3.3.1).

From the above theorem we have the following observations:

1. The probability of losing a low priority customer is given by

$$\sum_{n=0}^{k} \frac{N'_C \ldots N'_{k+1}}{\rho_1^{\,c-m}} \left(\frac{\rho_2^n}{n!}\right) P_C \; P'_0 \; .$$

2. The probability of losing a high priority customer is given by

$$\sum_{n=0}^{k} N'_C \left(\frac{\rho_2^n}{n!}\right) P_C \; P'_0 \; .$$

Corollary 3.4.1

The probability distribution of the customers in the system when $\mu_1 = \mu_2$ is

Pr{m customers in the system} = $\displaystyle\sum_{n_1+n_2=m} \frac{(\lambda_1/\mu_1)^{n_1}(\lambda_2/\mu_2)^{n_2}}{n_1!\,n_2!} P_0 = \frac{\rho^m}{m!} P_0$, if $m < k$

= $\displaystyle\frac{\rho_1^{m-k}}{m!} \rho^k P_0$. if $m \geq k$

Proof:

The proof for $m \geq k$ is given below, while the proof for m< k can be easily followed from Corollary (2.3).

Pr{*m* customers in the system} $\displaystyle = \sum_{n=0}^{k} \frac{N'_C \ldots N'_{m+1}}{\rho_1^{C-m}} \left(\frac{\rho_2^n}{n!} P'_0\right) P_C$ if $m \geq k$

Since $N'_m = m$ when $\mu_1 = \mu_2$, then

Pr {*m* customers in the system} =

$$\sum_{n=0}^{k} \left(\frac{\rho_2^n}{n!} P'_0\right) \frac{C \ldots (m+1)}{\rho_1^{C-m}} \frac{\rho_1^{C-k-1}}{C \ldots (k+1)} [(\rho_1 + k)\left(\frac{\rho^k}{k!}\right) - \frac{\rho^k}{(k-1)!}] P_0$$

$$= \sum_{n=0}^{k} \frac{\rho_1^{m-k-1}}{m \ldots (k+1)} \left(\frac{\rho_1}{k} \frac{\rho^k}{(k-1)!}\right) P_0 \left(\rho_2^n \frac{P_0'}{n!}\right)$$

$$= \frac{\rho_1^{m-k}}{m!} \rho^k P_0 \sum_{n=0}^{k} \frac{\rho_2^n}{n!} P_0'$$

$$= \frac{\rho_1^{m-k}}{m!} \rho^k P_0 \; .$$

4. EXAMPLE

This section gives an example of a particular case, in which we compare, illustrate and summarise the models and policies discussed in the previous sections.

Consider a system for which the arrival rates of high and low priorities are Poisson distributed with rates 2 and 4 respectively. Let the service time for the high and low priorities be exponentially distributed with a mean of 4. Assume that the number of servers in the system is 10. This data is not real it is just assumed data. Further, suppose that in policy two four servers are allocated to high priority customers and six to the low priority customers.

Table 1 shows the probability distribution of the number of customers (high and low priorities) in the system when policy one applied. Here the system is full 61% of the time and this indicates that the probability of losing any low priority customer is 61%, with dropping probability of 49%, while the probability of losing a customer of high priority is 12%.

In policy two the probability distribution of the customers in the low priority subsystem is given in

Table **2** and the probability distribution of the customers in the high priority subsystem is given in

Table **3**. From those tables the probability of dropping a low priority customer is 40%, while the probability of losing a high priority customer is 9%. In policy three, assume the cutoff level is 3. The probability distribution of the customers in low priority subsystem is shown in Table **4**, and the probability of losing a high priority customer is 57% and losing a low priority customer is 65%.

In policy four, we chose the cutoff level to be 8. This is the closest comparable partition to the other policies. Table **5** shows the probability distribution of the customers, where the probability of losing a high priority customer is 95% and losing a low priority customer is 100%.

Four policies are summarised in Table 6. This table shows that the when there is no division into two subsystems policy one give us a lower loss probability for both high and low priority customers. The probabilities of losing a low and high priority customer, using policy one, are 61% and 12% respectively. Further, when division is allowed policy two give us a lower loss probability for both high and low priority customers. The probabilities of losing a low and high priority customer, using policy two, are 73% and 9% respectively.

Table 1: Probability distribution (%) of customers using policy one

Priority	Number of customers in the system										
	0	1	2	3	4	5	6	7	8	9	10
High & Low	0	0	0	0	0	0	1	3	10	25	61
High	0	0	1	4	7	12	15	17	17	15	12

Table 2: Probability distribution (%) of customers using policy two

Priority	Number of customers in the system						
	0	1	2	3	4	5	6
High & Low	0	0	0	1	5	21	73
High	1	6	13	20	22	21	17

Table 3: Probability distribution (%) of customers using policy two

Priority	Number of customers in the system				
	0	1	2	3	4
High	0	3	11	29	57

Table 4: Probability distribution (%) of customers using policy three

Priority	Number of customers in the system						
	0	1	2	3	4	5	6
High & Low	0	0	0	2	8	25	65

Table 5: Probability distribution (%) of customers using policy four

Priority	Number of customers in the system										
	0	1	2	3	4	5	6	7	8	9	10
High & Low	0	0	0	0	0	0	0	0	0	5	95

Table 6: Summary

	Dropping a low priority customer	probability of losing a low priority customer	probability of losing a high priority customer
Policy one	49%	61%	12%
Policy two	40%	73%	9%
Policy three		65%	57%
Policy four		100%	95%

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