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WEAKLY C-NORMAL AND Cs-NORMAL SUBGROUPS OF FINITE GROUPS *

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ABSTRACT

A subgroup H of a finite group G is weakly $c-$normal subgroup of G if there exists a subnormal subgroup N of G such that $G = HN$, and $H \cap N \leq core_G(H)$, where $core_G(H)$ denotes the core of H in G, which is the largest normal subgroup of G contained in H . If $H \cap N \leq core_{G..}(H)$, then H is c_S-normal subgroup of G, where $core_{G..}(H)$ denotes the higher core of H in G, which is the largest subnormal subgroup of G contained in H .

In this paper, we investigate some properties of weakly $c-$normal and c_S-normal subgroups of finite groups, and using the weakly $c-$normality and c_S-normality of some Sylow and maximal subgroups to determine the structure of finite groups.

1. INTRODUCTION

It is interesting to use some information on the subgroups of a finite group G to determine the structure of the group G . The normality of subgroups of a finite group plays an important role in the study of finite groups. *Wang, 1996* initiated the concepts of $c-$normal subgroups and used the $c-$normality of maximal subgroups to give some conditions for solvability and supersolvability of a finite group. *Lujin Zhu and et al, 2002* have introduced the concepts of weakly $c-$normal subgroups and they have used the weakly $c-$normality of some maximal and Sylow subgroups to determine the structure of a finite group.

Definition 1.1 [10]: Let $H \leq G$. We say that H is a subnormal subgroup of G if there is a series from H to G .

A subgroup H of a group G is called $c-$normal subgroup of G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G$, where $H_G = core_G(H)$ is the largest normal subgroup of G contained in H .

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Definition 1.2 [6]: Let G be a group. We call a subgroup H weakly $c-$normal subgroup of G if there exists a subnormal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G$, where $H_G = core_G(H)$ is the largest normal subgroup of G contained in H.

Example 1.3: Let H be a sylow $2-$subgroup of the symmetric group S_3. Then H is weakly $c-$normal subgroup of S_3.

It is easy to see that, every $c-$normal subgroup of a group G is weakly $c-$normal in G, however, the converse is not true, see [6].

2. PRELIMINARIES

In this section, we give some definitions and basic results which are essential in the sequel. Let π be a nonempty set of primes, π' the complement set of π in the set of all prime numbers. Let G be a group, we denote the set of all prime divisors of the order the group G by $\pi(G)$; the maximal normal $p-$ subgroup of G by $O_p(G)$ and the Fitting subgroup of G by $F(G)$. We introduce the following concept which is closely related with the subnormal subgroups of a group.

Definition 2.1: Let G be a group. We call a subgroup H c_S-normal subgroup of G if there exists a subnormal subgroup N of G such that $G = HN$ and $H \cap N \leq H_{G..}$, where $H_{G..}$ denotes of the higher core of H in G which is the maximal subnormal subgroup of G contained in H.

Example 2.2: Let H be a sylow $2-$subgroup of the alternating group A_4. Then H is c_S-normal subgroup of A_4.

It is easy to see that, every normal (subnormal) subgroup of a group G, is c_S-normal subgroup of G, but the converse is not true. To see this, let $G = S_3$, then the subgroup $H = \langle (12) \rangle$ of G is c_S- normal in G, but H is not normal subgroup of G.

Clearly, every $c-$normal subgroup of a group G is c_S-normal subgroup of G, also, every weakly $c-$ normal subgroup of a group G is also c_S- normal subgroup of G.

Lemma 2.3 [10]: Let G be a group with subgroups A and B such that A is an Abelian subgroup and $G = AB$. Then one of the following two conditions is satisfied: *(i)* A contains a normal subgroup C of G such that $C \neq 1$, or *(ii)* $B \cap \left(x^{-1}Ax\right) = 1$, for all $x \in G$.

The following example shows that the property of $c_S -$normality cannot imply weakly $c -$normality.

Example 2.4: Let $G = S_3 \times S_3$ be the direct product of S_3 by itself, and $K = \Delta(A_3 \times A_3) = \{1\ ,\ ((123),(123))\ ,\ ((132),(132))\}$ be the diagonal subgroup of $H = A_3 \times A_3$ with $K_G = 1$ and $K_{G_{..}} = K$.

Then $K \lhd\lhd\ G$ since $K \lhd H \lhd G$ and hence K is $c_S -$normal subgroup of G . But there is no subnormal subgroup, say N of G , such that $G = NK$ and $N \cap K \leq K_G$. Suppose not, then there exists a subnormal subgroup N of G with order having three cases:

<u>Case (*i*):</u> If $|N| < 12$, then $|NK| < 36$, and therefore there is no subnormal subgroup N of G such that $G = NK$.

<u>Case (*ii*):</u> If $|N| > 12$, then by Lagrange theorem there are two situations; either, *(a)* $|N| = 18$, then $|NK| > 36$, or *(b)* $|N| = 36$, then we have $G = N$, and thus $NK = GK = G$, but $N \cap K = G \cap K = K \not\subset K_G = 1$. Therefore from the two situations there is no subnormal subgroup N of G such that $G = NK$.

<u>Case (*iii*):</u> Assume that $|N| = 12$, and suppose that $G = NK$. Since K is an Abelian subgroup, then either N is a normal subgroup of G , or one cannot find any normal subgroup K such that $G = NK$. In this case N is normal in G if it is the maximal subgroup of G .

Assume that N is normal in G , then *(ii)* of lemma 2.3 cannot be satisfied here, and hence we have $K \cap \left(x^{-1}Nx\right) = K \cap N \neq 1$, for all $x \in G$, hence $K \subset N$. Therefore $G \neq NK$. If N is not normal of G then K cannot found any subnormal subgroup such that $G = NK$. Thus K is not weakly $c -$normal subgroup of G .

3. ELEMENTARY PROPERTIES

Lemma 3.1: Let G be a group with subgroup H. Then *(i)* $H_{G..} \lhd\lhd H$, and *(ii)* If $H \leq L_1 \cap L_2$ where L_1 and L_2 are two maximal subgroups of G with $L_1 \neq L_2$, then $H_{G..} \lhd H$.

Proof: *(i)* We know that $H_{G..} \leq H$ and $H_{G..} \lhd\lhd G$ (by definition), then $H_{G..} \lhd\lhd (H \cap G)$. Thus $H_{G..} \lhd\lhd H$. *(ii)* If H is subnormal of G, then the result is obvious, so it is enough to show the case when H is not subnormal of G. By the definition of the higher core and *(i)*, there is a series of minimal length $n > 1$ that has the form $H_{G..} = M_n \lhd \ ... \ \lhd M_1 \lhd M_0 = H$, where M_i is not subnormal of G for all $i = 1, 2, \ ... \ , n$. Then there exists a unique maximal subgroup of G, say M, such that $M_i \leq M$ for all $i = 1, 2, \ ... \ , n$ which is a contradiction with $H \leq L_1 \cap L_2$. This impels that $n = 1$. Hence $H_{G..} \lhd H$.

Lemma 3.2: Let G be a group with a subgroup H. Then $H_G \leq H_{G..}$.

Lemma 3.3 [6]: Let G be a group. Then the following statements hold.

(i) If H is weakly $c-$normal subgroup of G with $H \leq K \leq G$, then H is weakly $c-$normal subgroup of K.

(ii) Let K be a normal subgroup of G with $K \leq H$. Then H is weakly $c-$normal of G iff H/K is weakly $c-$normal of G/K.

Lemma 3.4: Let G be a group. Then the following statements hold.

(i) If H is c_S-normal subgroup of G with $H \leq K \leq G$, then H is c_S-normal subgroup of K.

(ii) Let K be a normal subgroup of G with $K \leq H$. Then H is c_S-normal subgroup of G iff H/K is c_S-normal subgroup of G/K.

Proof: *(i)* Suppose that H is c_S-normal in G, then there exists a subnormal subgroup N of G such that $G = HN$ and $H \cap N \leq H_{G..}$. Then

$K = K \cap G = K \cap HN = H(K \cap N)$, and hence $(K \cap N)$ is subnormal of K, and $H \cap (N \cap K) = (H \cap N) \cap K \le H_{G_{..}} \cap K \le K_{G_{..}}$. Thus H is $c_S -$ normal in K.

(ii) Suppose that H/K is $c_S -$ normal subgroup in G/K, then there exists a subnormal subgroup N/K of G/K such that $G/K = (H/K)(N/K)$, and $(H/K) \cap (N/K) \le (H/K)_{(G/K)_{..}}$. Then we have $G = HN$ and $H \cap N \le H_{G_{..}}$. Hence H is $c_S -$ normal in G.

Conversely, assume that H is $c_S -$ normal subgroup in G, then there exists a subnormal subgroup N of G such that $G = HN$ and $H \cap N \le H_{G_{..}}$. Then we have that $G/K = (H/K)(NK/K)$, and then NK is a subnormal of G, and

$$(H/K) \cap (NK/K) = (H \cap NK)/K = K(H \cap N)/K \le KH_{G_{..}}/K \le (H/K)_{(G/K)_{..}}.$$

Hence H/K is $c_S -$ normal subgroup of G/K.

Definition 3.5 [10]: For any set π of prime numbers, we denote by π' the set of all primes which do not belong to π. If $H \le G$, then H is said to be a Hall $\pi -$subgroup of G if $|H|$ is a $\pi -$number and $[G:H]$ is a $\pi' -$number.

Definition 3.6 [2]: A group G is called $\pi -$solvable if it has a subnormal series whose factors are $\pi -$groups or $\pi' -$groups and the $\pi -$factors are solvable.

Lemma 3.7 [9]: Let G be a $\pi -$solvable group. Then G has at least one solvable Hall $\pi -$subgroup G_π, and for any $\pi -$subgroup A of G, there is an element $x \in G$ such that $A^x \le G_\pi$. In particular, any two Hall $\pi -$subgroups are conjugate in G.

For the proof of this lemma, the reader can see [2] and [9].

Definition 3.8: Let G be a group. We call a group G weakly $p-$nilpotent if G has a subnormal $p-$complement in G. *i.e.*, if H is a subnormal subgroup of G and P is a sylow $p-$subgroup of G such that $G = HP$ and $H \cap P = 1$ then G is called weakly $p-$nilpotent.

Clearly, if G is $p-$nilpotent, then G is weakly $p-$nilpotent, however, the converse is not true. The following example shows that the property of weakly $p-$nilpotent cannot imply $p-$nilpotent.

4. THEOREMS

Theorem 4.1: If H is weakly $c-$normal subgroup of a group G, then H/H_G has a subnormal complement in G/H_G, *i.e.*, there exists a subnormal subgroup K/H_G of G/H_G such that G/H_G is the semidirect product of K/H_G and H/H_G. Conversely, if H is a subgroup of G such that H/H_G has a subnormal complement in G/H_G, then H is weakly $c-$normal of G.

Proof: Let H be a weakly $c-$normal subgroup of G, then there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. If $H_G = 1$, then $H \cap K = 1$. Hence K is a subnormal complement of H in G. Assume that $H_G \neq 1$, then we can construct the factor groups H/H_G and KH_G/H_G and by Dedekind's Identity, we have

$$\left(H/H_G\right) \cap \left(KH_G/H_G\right) = \left(H \cap KH_G\right)/H_G = H_G\left(H \cap K\right)/H_G \leq H_G/H_G = 1.$$

Hence KH_G/H_G is a subnormal complement of H/H_G in G/H_G.

Conversely, if F is a subgroup of G such that F/H_G is a subnormal complement of H/H_G in G/H_G, then we have that

$$G/H_G = \left(H/H_G\right)\left(F/H_G\right), \text{ and } \left(H/H_G\right)\cap\left(F/H_G\right) = 1.$$

Then $G = HF$, where F is a subnormal subgroup in G , and $H \cap F \le H_G$. Therefore H is weakly $c-$normal subgroup in G .

Corollary 4.2: Let H be a subgroup of a group G such that H/H_G has a subnormal complement in G/H_G . Then H is c_S-normal in G .

The following example shows that the converse of the above corollary is not necessarily true.

Example 4.3: Let $G = H \propto Z_p$ be the semidirect product of a subgroup H and the cyclic group Z_p with; *(i)* Z_p is a maximal subgroup in G , *(ii)* H is not normal in G with $H_G \ne 1$, *(iii)* $\left(|H|,|Z_p|\right) = 1$.

Then $Z_p \lhd G$ since p dose not divide $|H|$ (by Sylow theorem), and hence Z_p is c_S-normal in G , also H is c_S-normal in G . We claim that H/H_G has no subnormal complement in G/H_G , suppose not, *i.e.*, H/H_G has a subnormal complement in G/H_G , say K/H_G . Then $G/H_G = \left(H/H_G\right)\left(K/H_G\right)$, and $\left(H/H_G\right)\cap\left(K/H_G\right) = \left(H \cap K\right)/H_G = 1$. Then $G = HK$, and $H \cap K = H_G$. But we know that $|G| = \dfrac{|H|\;|K|}{|H \cap K|} = \dfrac{|H|\;|K|}{|H_G|}$. Since $|G| = |H|\;|Z_p| \Rightarrow$

$$|H| = \frac{|G|}{|Z_p|} = \frac{|G|}{p} \Rightarrow |G| = \frac{\left(\frac{|G|}{p}\right)\;|K|}{|H_G|} \Rightarrow |K| = p\;|H_G|.$$

If $|K| = p$, we get a contradiction with $1 < H_G \le K$. By using Cauchy Theorem we have K contains a subgroup of order p , say L , since $K \le G$, then L

will be a sylow $p-$subgroup of G since $(|H|,|Z_p|)=1$. Since Z_p is the unique sylow $p-$subgroup of G, and its normal in G, then $L=Z_p$. Then $Z_p<K\neq G$. This contradicts the maximality of Z_p in G.

Theorem 4.4: A group G has a weakly $c-$normal sylow $p-$subgroup if and only if the factor group $G/O_p(G)$ is weakly $p-$nilpotent.

Proof: Assume that G has a weakly $c-$normal sylow $p-$subgroup, say P. Then by lemma 3.3 we have P/P_G has a subnormal complement in G/P_G (notice that $P_G=O_p(G)$). Hence $G/O_p(G)$ is weakly $p-$nilpotent.

Conversely, assume that $G/O_p(G)$ is weakly $p-$nilpotent. Then there exists a subnormal subgroup $K/O_p(G)$ of $G/O_p(G)$ such that

$$G/O_p(G)=\left(K/O_p(G)\right)\left(P/O_p(G)\right), \text{ and } \left(K/O_p(G)\right)\cap\left(P/O_p(G)\right)=1.$$

Then we have $G=KP$ such that K is a subnormal subgroup in G, and $K\cap P=O_p(G)=P_G$. Therefore P is weakly $c-$normal subgroup in a group G.

Corollary 4.5: If a factor group $G/O_p(G)$ is weakly $p-$nilpotent, then G has a c_S-normal sylow $p-$subgroup.

This corollary is obvious and we omit the proof. The converse of the Corollary 4.5 is not necessarily true; regarded with Example 4.3, we have seen that Z_p is c_S-normal sylow $p-$subgroup of G such that $O_p(G)=Z_p$. Hence $G/O_p(G)=H$ has no subnormal $p-$complement.

Theorem 4.6: A group G is metanilpotent if and only if every sylow subgroup of G is weakly $c-$normal in G.

Proof: Suppose that G is metanilpotent. Let $p \in \pi(G)$ and P be a sylow $p-$subgroup of G. Since $G/F(G)$ and $F(G)$ are solvable. Then G is solvable. Moreover G is $p'-$solvable. By lemma 3.5, we can replace G_π by K, and A^x by $O_{p'}(F(G))$ to conclude that G has a solvable Hall $p'-$subgroup K. Since $O_{p'}(F(G))$ is normal in G. $O_{p'}(F(G)) \le K$ and hence $O_p(G)K = F(G)K$. Since $G/F(G)$ is nilpotent, then we have $KF(G)/F(G)$ is a normal Hall $p'-$subgroup of $G/F(G)$. Hence $H = O_p(G)K = KF(G)$ is normal subgroup in G, and consequently, we deduce that $G = PH$ and $P \cap H = P \cap O_p(G)K = O_p(G)(P \cap K) = O_p(G) = P_G$. Therefore P is weakly $c-$normal in G.

Conversely, Suppose that every sylow subgroup of G is weakly $c-$normal in G. Then, by the third isomorphism theorem and Theorem 4.4 and the fact that $F(G) = O_p(G)$,

$$G/O_p(G) \cong \left(G/O_p(G)\right) \Big/ \left(F(G)/O_p(G)\right) \cong G/F(G)$$

is $p-$nilpotent for all $p \in \pi(G)$. Therefore $G/F(G)$ is nilpotent, and hence G is metanilpotent.

Corollary 4.7: A group G is metanilpotent if and only if every sylow subgroup of G is c_S-normal in G.

This corollary is obvious and we omit the proof.

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