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NONLINEAR APPROXIMATION IN SOME SEQUENCE SPACES

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Abstract. We show that approximation of an element in ℓ^p space by finite number of terms is arbitrary slow, but if we use ℓ^q norm, with $q > p$, as a measure of the error, then the approximation is faster. Also, we use nonlinear approximation to characterize Lorentz spaces by the error of approximation of their elements.

1. Introduction

Nonlinear approximation is utilized in many numerical algorithms, it occurs in several applications. In mathematics and applications it is very important to write a function in some function space in the form

$$f = \sum_{k\in\Lambda} c_k g_k$$

where Λ is an indexed set and $\{g_k : k \in \Lambda\}$ is a set of functions. The case in which this set is obtained from a single function is very interesting. It is essential in both Gabor and wavelets decompositions. In the context of wavelet theory, let ψ be a fixed function in $L_2(\mathbb{R})$, and $I = I_{kl} = \{x \in \mathbb{R} : 2^{-k}l \le x \le 2^{-k}(l+1), k, l \in \mathbb{Z}\}$. The function ψ is called an orthonormal wavelet provided that the system $\{\psi_I(x)\} = \{2^{k/2}\psi(2^k x - l)\}$ is an orthonormal basis for $L_2(\mathbb{R})$. The wavelet decomposition of a general function $f \in L_2(\mathbb{R})$ is given by

$$f = \sum_{I dyadic} a_I \psi_I, \quad a_I = \langle f, \psi_I \rangle.$$

Wavelets allow us to characterize a large class of function spaces in terms of wavelet coefficients such as Lebesgue spaces L^p [6], Hardy spaces H^p, $(0 < p < 1)$, Besov spaces $\mathcal{B}_p^{\alpha,q}$, $\alpha \in \mathbb{R}$, and $0 < p, q \le \infty$

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and others. For more details on these characterizations, see [6] and references there.

In the context of Gabor theory where dilation is replaced by modulation, Gabor coefficients are also used to characterize a class of functions called modulation spaces $M^w_{p,q}$, $0 < p, q < \infty$ and w some suitable weight function. If $T_x f(t) = f(t - x)$ and $M_y f(t) = e^{2\pi i t y} f(t)$ denote the translation and modulation operators then any function $f \in M^w_{p,q}$ can be written as

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{n\beta} T_{k\alpha} g\rangle M_{n\beta} T_{k\alpha} \gamma \\ &= \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{n\beta} T_{k\alpha} \gamma\rangle M_{n\beta} T_{k\alpha} g \end{aligned}$$

for all $M^w_{p,q}$ if $0 < p, q \le \infty$, with unconditional convergence in $M^w_{p,q}$ if $p, q < \infty$, and with weak-star convergence in $M^{1/w}_{\infty,\infty}$ otherwise. Furthermore, there are constants $A, B > 0$ such that for all $f \in M^w_{p,q}$

$$A||f||_{M^w_{p,q}} \le \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\langle f, M_{n\beta} T_{k\alpha} g\rangle|^p w(k\alpha, n\beta)^p\Big)^{q/p}\Big)^{1/q} \le B||f||_{M^w_{p,q}}$$

Likewise, the quasi-norm equivalence

$$A'||f||_{M^w_{p,q}} \le \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\langle f, M_{n\beta} T_{k\alpha} \gamma\rangle|^p w(k\alpha, n\beta)^p\Big)^{q/p}\Big)^{1/q} \le B'||f||_{M^w_{p,q}}$$

holds on $M^w_{p,q}$. In all these characterizations the coefficients play a central role.

Another important theme of the coefficient sequence appears in nonlinear approximations. If I is any countable index set and $\Psi = \{\varphi_k, k \in I\}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$ then any $f \in \mathcal{H}$ has the orthogonal expansion $f = \sum_{k\in I} \langle f, \varphi_k\rangle \varphi_k$. If Σ_n is the subset of all finite linear combinations of n elements of Ψ and

$$\sigma_n(f)_{\mathcal{H}} = \inf_{s\in\Sigma_n} \|f - s\|_{\mathcal{H}} \tag{1}$$

is the approximation error of $f \in \mathcal{H}$. Stechkin results [11] and its generalization by Devore and temlyakove [2] can be stated as:

$$\sum_{k\in I} |\langle f, \varphi_k|^p < \infty, \ \text{ if and only if } \ \sum_{1}^{\infty} n^{-p/2} \sigma_n(f)^p_{\mathcal{H}} \tag{2}$$

The proof of this result depends mainly on the corresponding sequence space. These facts shed some light on the importance of the sequence spaces. Let $\ell^p,\ 1 \le p < \infty$ be the space of all functions x on $\{1, 2, \ldots\}$ whose norm

$$\|x\|_{\ell^p} = \left\{ \sum_{n=1}^{\infty} |x(n)|^p \right\}^{1/p}$$

is finite and let $L_p,\ 1 \le p < \infty$ be the space of all measurable functions f such that

$$\|f\|_{L_p} = \left\{ \int_{\mathbb{R}} |f(t)|^p dt \right\}^{1/p} < \infty$$

The relation between the function spaces L_p and the sequence spaces ℓ^p was considered first by Banach, Pelczynski and Hsiano et al. Banach [1] proved that L_p and ℓ^p can only be isomorphic when $p = 2$ or $p = \infty$. It is very easy to show, using Haar basis, that L_2 is indeed isomorphic to ℓ^2, and Pelczynski proved that L_∞ is isomorphic to ℓ^∞. For other values of p, Hsiano and others [5] proved that $L_p,\ (1 < p < \infty)$ is isomorphic to a sequence space y, for which we have the continuous inclusions

$$\ell^{p,min(p,2)} \subset y \subset \ell^{p,max(p,2)}. \tag{3}$$

where $\ell^{p,q}$ denote the Lorentz spaces based on an infinite countable index set. In this paper we consider nonlinear approximation in ℓ^p and in $\ell^{p,q}$ then we use interpolation of Lorentz spaces to characterize L^p spaces by their best approximation.

2. Nonlinear approximation in ℓ^p

In this section we give some results in approximating ℓ^p spaces which show that approximating an element in ℓ^p by n terms is arbitrary slow.

Approximation of an element in ℓ^p by finitely many terms is slow in the sense: Let X be a Banach space, $\{X_n, \{A_n\}_{n\in\mathbb{N}}\}$ be an approximation scheme, and denote by $E(x, A_n)$ the error of best approximation of $x \in X$ with elements of $A_n, n \in \mathbb{N}$.

Let us assume that there exists a certain constant $c > 0$ and an infinite set $N_\circ \subseteq N$ such that for all $n \in N_\circ$, there exists some $x_n \in X \setminus A_n$ which satisfies

$$E(x, A_n) \le cE(x, A_{K(n)}).$$

Then for all non-increasing sequence $(\epsilon_n)_{n=0}^{\infty}$ which converges to zero for $n \to \infty$ there exists some $x \in X$ such that $E(x, A_n) \ne O(\epsilon_n)$.

We'll give simple proof in the case of ℓ^p spaces

Lemma 2.1. *If $c^{(N)}$ consists of N real numbers, then there exists $c = (c_n) \in \ell^p$ such that $c^{(N)}$ consists of the N largest entries of c and*

$$\|c - c^{(N)}\|_{\ell^p} \to 0$$

as $N \to \infty$ arbitrarily slow.

Proof. It is clear that if $c^{(N)} = (c_{n_j})_{j=1}^{N}, \ \ n_j \in \mathbb{N}$, then
$\|c - c^{(N)}\|_{\ell^p}^p \to 0$ as $N \to \infty$.

To show that the convergence is arbitrarily slow, we may assume (ϵ_N) and c are decreasing sequences (by decreasing rearrangement of their entries). Let

$$\sum_{n=N+k+1}^{\infty} |c_n|^p = \epsilon_{N+k}$$

for $k \in \mathbb{N}$ then $\epsilon_N - \epsilon_{N+1} = |c_{N+1}|^p$ and

$$\epsilon_0 - \epsilon_{k+1} = \sum_{N=0}^{N=k} \epsilon_N - \epsilon_{N+1} = \sum_{N=0}^{N=k} |c_{N+1}|^p$$

so, if we let $k \to \infty$, then we get $\sum_{N=0}^{\infty} |c_{N+1}|^p = \epsilon_0$, therefore $(c_n) \in \ell^p$ and

$$\|c - c^{(N)}\|_p = \sum_{n>N} |c_n|^p = \sum_{n=N+1}^{\infty} |c_n|^p = \epsilon_N$$

for all N. □

If we measure the error of approximation of an element in ℓ^p by the ℓ^q norm with $p < q$, then we get better estimation.

Theorem 2.2. *If $c \in \ell^p$, and $c^{(N)}$ as in lemma 2.1, then,*

$$\|c - c^{(N)}\|_{\ell^q} = \mathcal{O}(N^{\frac{1}{q}-\frac{1}{p}})(\to 0),$$

for $0 < p \leq q$.

Proof. We prove this theorem for sequences $c \in \ell^1$ first and use it to prove the theorem for sequences in $c \in \ell^p$.

If $c \in \ell^1$, then

$$\sum_{n=1}^{\infty} |c_n| < \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

By rearrangement of (c_n), we assume that $(|c_n|)$ is non increasing, so we have

$$|c_n| \geq 0, |c_n| \searrow 0 \ \text{ and } \ \sum_{n=1}^{\infty} |c_n| < \infty$$

this implies

$$\frac{n}{2}|c_n| \le \sum_{\frac{n}{2}}^{n} |c_k|$$

Since $|c_n|$ is decreasing, we have

$$n|c_{2n-1}| \le \sum_{k=n}^{2n-1} |c_k| \text{ and } (n+1)|c_{2n}| \le \sum_{k=n}^{2n-1} |c_k| \text{ for all } n.$$

Now the condition $c \in \ell^1$ implies that $\left(\sum_{k=1}^{n} |c_k|\right)_n$ is a Cauchy sequence and $nc_n \to 0$. Hence,

$$|c_n| = \mathcal{O}(n^{-1}).$$

Now, if $c \in \ell^p$, then, by applying the previous idea, we have

$$\frac{n}{2}|c_n|^p \le \sum_{\frac{n}{2}}^{n} |c_k|^p \to 0 \text{ as } n \to \infty),$$

which implies

$$|c_n| = \mathcal{O}(n^{\frac{-1}{p}}).$$

Now, $|c_n| \le \frac{k^{\frac{1}{q}}}{n^{\frac{1}{p}}}$ for some constant k, which implies $|c_n|^q \le kn^{\frac{-q}{p}}$. To approximate the tail of $\{c_n\}$, we set

$$||c - c^{(N)}||_{\ell^q} = \sum_{n>N} |c_n|^q \le k \sum_{n>N} n^{\frac{-q}{p}} \le k \int_N^{\infty} t^{\frac{-q}{p}} dt = k_1 N^{1-\frac{q}{p}}$$

Therefore,

$$\left(\sum_{n>N} |c_n|^q\right)^{\frac{1}{q}} \le CN^{\frac{1}{q}-\frac{1}{p}}$$

which implies

$$\|c - c^{(N)}\|_{\ell^q} = \mathcal{O}(N^{\frac{1}{q}-\frac{1}{p}})(N \to \infty).$$

□

3. Lorentz Spaces

In this section, we recall the definition of Lorentz spaces and give some well-known results which will be used later,

Definition 3.1. [9] *Let* $0 < p < \infty$ *and* w *be a weight. The Lorentz space is defined by*

$$\Lambda^p(w) = \{f : R^+ \to R^+; \|f\|_{\Lambda^p(w)} < \infty\},$$

where

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty (f^*(t))^p w(t) dt \right)^{1/p} \tag{4}$$

and f^* is the decreasing rearrangement of the function f.

In general (4) does not define a norm. In fact, Lorentz [8] proved that (4) is a norm if and only if $p \geq 1$ and w is a decreasing function. For more information on Lorentz spaces see [8, 9] and references there.

Classical Lorentz spaces are obtained when power weights are considered. If $w(t) = t^{(p/q)-1}$ then $\Lambda^p(w) = L^{q,p}$, which is the Lebsegue space and actually we need the following well known definition

Definition 3.2. *Let (X, μ) be a totally $\sigma-$finite measure space and suppose that $0 < p, q \leq \infty$. The Lorenz space $L^{p,q} = L^{p,q}(X, \mu)$ consists of all $\mu-$measurable functions f for which the quantity*

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right)^{1/q}, & (0 < q < \infty) \\ \sup_{0<t<\infty} (t^{1/p} f^*(t)), & (q = \infty), \end{cases}$$

is finite.

The Lorentz space $L^{p,p}, (0 < p \leq \infty)$, coincide with the Lebseque space L^p. Also it is known that if $0 < p \leq \infty$, and $0 < q \leq r \leq \infty$. Then $L^{p,q} \subseteq L^{p,r}$ and $\|f\|_{L^{p,r}} \leq C\|f\|_{L^{p,q}}$, for some constant C and all $\mu-$measurable functions f. In the case of a measure space of all dyadic cubes equipped with counting measure, we define the descrete Lorentz spaces as follows: For $0 < p, q < \infty$, the Lorentz pace $\ell^{p,q}$ is the set of all sequences $\{c_k\}$ such that the following norm is finite

$$\|\{c_k\}\|_{\ell^{p,q}} = \begin{cases} \left(\sum_{k=1}^\infty [k^{\frac{1}{p}-\frac{1}{q}} |c_k^*|]^q \right)^{1/q} & , 0 < q < \infty \\ \sup_{k \in \mathbb{N}} k^{1/p} |c_k^*| & , q = \infty \end{cases}$$

where $\{c_k^*\}$ denotes the decreasing rearrangement of $\{c_k\}$. Let $a = \{a_m,\ m \in \mathbb{N}\}$ be the non-increasing rearrangement of the sequence $\{k^{\frac{1}{p}-\frac{1}{q}} c_k\ k \in \mathbb{N}\}$ and let $\sigma_n = (\sum_{m=n}^\infty a_m^2)^{1/2}$, then the following theorem characterizes Lorentz spaces $\ell^{p,q}$ by the order of σ_m.

Theorem 3.3. *If $0 < p \leq q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{2}$, then*

$$\sum_{n \in \mathbb{N}} (\sigma_n n^\alpha)^p \frac{1}{n} < \infty$$

if and only if a is in Lorenz space $\ell^{p,q}$

Proof. Suppose that $\sum_{n\in\mathbb{N}}(\sigma_n n^\alpha)^p \frac{1}{n} < \infty$.
Since $a_{2m-1}^2 \le a_k^2,\ k = m, m+1, \ldots, 2m-1$, we have $m a_{2m-1}^2 \le \sum_{k=m}^{2m-1} a_k^2$ which implies

$$a_{2m-1} \le m^{-\frac{1}{2}}(\sum_{k=m}^{2m-1} a_k^2)^{\frac{1}{2}} \le m^{-\frac{1}{2}}\sigma_m$$

Also, we have, $a_{2m} \le a_{2m-1} \le m^{-\frac{1}{2}}\sigma_m$. From these two inequalities, we get

$$a_{2m-1}^q \le m^{-q/2}\sigma_m^q \quad \text{and} \quad a_{2m}^q \le m^{-q/2}\sigma_m^q \tag{5}$$

so,

$$(2m)^{q/p-1}a_{2m}^q \le (2m)^{q/p-1}m^{-q/2}\sigma_m^q = 2^{q/p-1}(m^{q/p-q/2}\sigma_m^q)\frac{1}{m},$$

taking sums, we get

$$\sum_{m=1}^{\infty}(2m)^{q/p-1}a_{2m}^q \le 2^{q/p-1}\sum_{m=1}^{\infty}(m^{1/p-1/2}\sigma_m)^q\frac{1}{m}. \tag{6}$$

Also, if $\frac{q}{p} \ge 1$, then we have

$$(2m-1)^{q/p-1}a_{2m-1}^q \le (2m)^{q/p-1}a_{2m-1}^q \le (2m)^{q/p-1}(m^{-q/2}\sigma_m^q)$$

taking sums for the last inequality, we get

$$\sum_{m=1}^{\infty}(2m-1)^{q/p-1}a_{2m-1}^q \le 2^{q/p-1}\sum_{m=1}^{\infty}(m^{1/p-1/2}\sigma_m)^q\frac{1}{m}. \tag{7}$$

From the two sums (6) and (7), we get

$$\sum_{m=1}^{\infty}(m^{1/p-1/q}a_m)^q \le 2^{q/p}\sum_{m=1}^{\infty}(m^{1/p-1/2}\sigma_m)^q\frac{1}{m} \tag{8}$$

Hence,

$$\|\{a_m\}\|_{\ell^{p,q}} \le 2^{1/p}\left(\sum_{m=1}^{\infty}(m^\alpha\sigma_m)^q\frac{1}{m}\right)^{1/q}$$

for $\alpha = 1/p - 1/2$. For the other direction see [7, 10] and references therein. □

This theorem shows that if the error σ_m of approximating the sequence $a = (a_m)$ is of order $m^{-\alpha}$, then $a = (a_m) \in \ell^{p,q}$.

On the other hand, if $0 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, we can show that if a sequence $(a_m) \in \ell^{p,q}$, then the error of approximating this sequence will be of order $m^{-\alpha}$, but before proving this result, we need to use the following lemma about numerical sequences.

Lemma 3.4. [2, 4] *Let $a = \{a_k : k \in \mathbb{N}\}$ be a decreasing sequence of positive numbers. Set $\sigma_{N,q} = (\sum_{k=N}^{\infty} a_k^q)^{1/q}$. Then, for $0 < p \le q < \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, we have*

$$2^{-1/p}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p} \le \left(\sum_{m=1}^{\infty}[m^{\alpha}\sigma_{m,q}]^p \frac{1}{m}\right)^{1/p} \le c\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}$$

with a constant $c > 0$ depending only on p.

Let $\mathcal{E} = \{e_k : k \in \mathbb{N}\}$ be the canonical basis for $\mathbb{C}^{\infty}$, and Σ_n be the subset of all finite linear combinations of n elements of $\mathcal{E}$, more precisely,

$$\Sigma_n = \left\{\sum_{k\in F} d_k e_k : d_k \in \mathbb{C}, |F| \le n\right\}.$$

If σ_n is defined as in (1), then we have the following theorem:

Theorem 3.5. *For $0 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, if the sequence $c = (c_k)_{k\in\mathbb{N}} \in \ell^{p,q}$, then*

$$\sum_{n=1}^{\infty}\left(n^{\alpha}\sigma_n(c)\right)^p \frac{1}{n} < \infty.$$

Proof. Given any sequence $c = (c_k)_{k\in\mathbb{N}} \in \ell^{p,q}$, the error of approximating $(c_k)_{k\in\mathbb{N}}$ measured by the $\ell^{p,q}$-norm is

$$\begin{aligned}
\sigma_n(c)_{\ell^{p,q}} &= \inf_{s\in\Sigma_n} \|c - s\|_{\ell^{p,q}} \\
&= \inf_{\substack{d_k \\ F:|F|\le n}} \Big\|\sum_{k\in\mathbb{N}} c_k e_k - \sum_{k\in F} d_k e_k\Big\|_{\ell^{p,q}} \\
&= \inf_{\substack{d_k \\ F:|F|\le n}} \Big\|\sum_{k\in F}(c_k - d_k)e_k + \sum_{k\notin F} c_k e_k\Big\|_{\ell^{p,q}} \\
&\le \Big\|\sum_{k\notin F} c_k e_k\Big\|_{\ell^{p,q}} \\
&\equiv \Big\|\sum_{i=n+1}^{\infty} c_{k_i} e_{k_i}\Big\|_{\ell^{p,q}} \\
&= \|(c_{k_{n+1}}, c_{k_{n+2}}, c_{k_{n+3}}, \ldots.)\|_{\ell^{p,q}} \\
&= \left(\sum_{i=n+1}^{\infty}\left(k_i^{\frac{1}{p}-\frac{1}{q}}|c^*_{k_i}|\right)^q\right)^{1/q}
\end{aligned}$$

where $\{c^*_{k_i}\}$ denotes the non-increasing rearrangement of $\{c_{k_i}\}$. Again, let $a = \{a_i\}$ be the non-increasing rearrangement of $\{k_i^{\alpha}|c^*_{k_i}|\}_{i=n+1}^{\infty}$,

$\alpha = \frac{1}{p} - \frac{1}{q}$. Moreover, set $\sigma_{n,q} = \left(\sum_{i=n}^{\infty} a_i^q\right)^{1/q}$, then we have

$$\sigma_n(c)_{\ell^{p,q}} \leq \left(\sum_{i=n}^{\infty} a_i^q\right)^{1/q} = \sigma_{n,q}$$

and

$$\left(n^{\alpha}\sigma_n(c)_{\ell^{p,q}}\right)^p \frac{1}{n} \leq \left(n^{\alpha}\sigma_{n,q}\right)^p \frac{1}{n}, \quad for\ all\ n \in \mathbb{N}$$

so,

$$\begin{aligned} \sum_{n=1}^{\infty}\left(n^{\alpha}\sigma_n(c)_{\ell^{p,q}}\right)^p \frac{1}{n} &\leq \sum_{n=1}^{\infty}\left(n^{\alpha}\sigma_{n,q}\right)^p \frac{1}{n} \\ &\leq C.\|a\|_{\ell^p}^p. \end{aligned}$$

using the previous lemma. Therefore, if $c \in \ell^{p,q}$, then we have

$$\sum_{n=1}^{\infty}\left(n^{\alpha}\sigma_n(c)_{\ell^{p,q}}\right)^p \frac{1}{n} < \infty$$

. □

The following theorem indicates the relation between a sequence space that is isomorphic to L^p and the Lorentz spaces $\ell^{p,q}$.

Theorem 3.6. [5] *Let* $1 < p < \infty$*. Suppose* y_p *is a rearrangement invariant space based on a countably infinite measure space, normalized so that the mass of each point is one. If* L^p *is isomorphic to* y_p*, then*

$$\ell^{p,min(p,2)} \subset y_p \subset \ell^{p,max(p,2)}.$$

By interpolation we characterize the space L^p, using the isomorphic space y_p and theorem 3.3. This will appear in other articles.

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